TWO HOURS

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO FINANCIAL MATHEMATICS

DATE: ...May, 2008

TIME:

Answer THREE questions out of four

If you attempt more than 3 questions [total: 80 marks]
your marks will be counted from your 3 best answers

No prepared notes of any kind are to be brought into the examination room, the numerical table for $N(x)$ is provided

This examination makes up 80% of the overall assessment for this course unit;

1

(**a**) (3 marks) Write down the boundary conditions for the American put option $P(S,t)$ at $S = S_f(t)$, where $S_f(t)$ is the exercise boundary.

(b) The Black-Scholes equation has the explicit solution for the European call

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

where

$$N(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy, \quad d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T-t)}{\sigma(T-t)^{\frac{1}{2}}}, \quad d_2 = d_1 - \sigma(T-t)^{\frac{1}{2}}$$

and E is the exercise price, σ is the volatility, r is the continuous interest rate, T is the expiry date.

(i) (6 marks) Using the explicit expression for the European call, find the limits

$$\lim_{E\to 0} C(S,t), \qquad \lim_{\sigma\to 0} C(S,t).$$

(ii) (6 marks) Using the table for $N(x)$ and $N(-x) = 1 - N(x)$, find the value of a three-month European call option on a stock with a exercise price of \$100 when the current stock price is \$100 and $\sigma = 1$. The risk-free interest rate is 0% p.a.

(c) (5 marks) A zero coupon bond, B, issued at $t = 0$, is worth 2 at $t = 2\pi$. Find the bond price $B(t)$ at time $t < 2\pi$ and $B(0)$, when the continuous interest rate is

$$r(t) = 1 - \sin t.$$

2

(a) (5 marks) Sketch the graphs of the payoff diagrams for the following portfolios:

(i) short one European call and short four European puts, all with exercise price E.

(ii) short one share, long one European call and long two European puts, all with exercise price E.

(b) (5 marks) Derive the put-call parity relationship between the value of C of a European call option and the value of P of a European put option, with the same strike price E and expiry date T, when the interest rate is

$$r(t) = r_0 + r_1 t,$$

where r_0 and r_1 are constants.

(c) (5 marks) (i) By using the put-call parity relationship, show that a lower bound for the European call option with exercise price \$30 when the stock price is \$40, the time to maturity is six months, and the constant rate of interest is 2% p.a. is greater than \$10.

(ii) Consider the situation where the European call option is \$10. Show that there exists an arbitrage opportunity.

(d) (5 marks) By using the stochastic differential equation (SDE) for a stock price $S(t)$

$$dS = \mu S dt + \sigma S dW,$$

where where $W(t)$ is the standard Wiener process; μ and σ are constants, and *Ito's Lemma*

$$df = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW,$$

find the explicit expression for the stock price $S(t)$ when $S(t_0) = S_1$ and $W(t_0) = W_1$.

3

(**a**) (3 marks) Sketch the graphs of $\Delta = \frac{\partial P}{\partial S}$ and $\Delta = \frac{\partial C}{\partial S}$ as functions of S prior to expiry, where $P(S,t)$ and $C(S,t)$ are the European put option and European call option correspondingly.

(b) (7 marks) By using an one-step binomial model and the replicating portfolio $\Pi = \Delta S - NB$, where Δ is the number of shares, and N is the number of bonds, show that the value of a call option is

$$C_0 = e^{-rT}\left[pC_u + (1-p)C_d\right],$$

where r is the continuous interest rate, T is the maturity time, C_u is the payoff from the option if the stock price moves up, C_d is the payoff from the option if the stock price moves down.

Find the value of p.

(c) (10 marks) By using the explicit solution for the European call option

$$C(S,t) = SN\left(d_1\right) - Ee^{-r(T-t)}N\left(d_2\right),$$

where

$$N\left(x\right) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy,\ \ d_1 = \frac{\ln\left(S/E\right) + \left(r + \sigma^2/2\right)\left(T-t\right)}{\sigma(T-t)^{\frac{1}{2}}},\ d_2 = d_1 - \sigma(T-t)^{\frac{1}{2}}$$

and E is the exercise price, σ is the volatility, r is the continuous interest rate, T is the expiry date,

show that the vega, $\frac{\partial C}{\partial \sigma}$, can be written as

$$\frac{\partial C}{\partial \sigma} = f(t,S)N'\left(d_1\right).$$

Find (i) the function $f(t,S)$ and (ii) the value of σ at which $\frac{\partial C}{\partial \sigma}$ attains its maximum.

4

(a) (6 marks) By using the portfolio $\Pi = \Delta S - V$, where Δ is the number of shares, show that the option price $V(S,t)$ satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where σ is the volatility, r is the continuous interest rate.

(b) (5 marks) Verify by substitution that

$$V(S,t) = \frac{\exp\left[\left(\sigma^2 - 2r\right)(T-t)\right]}{S}.$$

is an exact solution of the Black-Scholes equation; here T is the expiry date.

(c) (9 marks) If $V(S,t)$ satisfies the Black-Scholes equation, find the equation for the function $U(Z,t)$ defined by

$$U(Z,t) = V\left(e^Z, t\right).$$