TWO AND A HALF HOURS

THE UNIVERSITY OF MANCHESTER

CALCULUS AND VECTORS

DATE: 19 January 2018

TIME: 09:45 - 12:15

Answer ALL FIFTEEN questions

Electronic calculators or formula tables are not permitted.

This examination makes up 75% of the overall assessment for this course unit.

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1. Sketch the graphs of the following real-valued functions of x satisfying

(a) $f(x) = 3^{-|x|}$ (b) $f(x) = \ln(4 - x^2)$. [4]

2. Express the following complex numbers in the standard form $x + iy$

(a) $\frac{2-i}{3+2i}$ (b) $e^{-\frac{i\pi}{2}}$. [4]

3. For the function $f(x) = 1 - 3/x^4$, with domain $x > 0$, find an explicit formula for the inverse function f^{-1}. Sketch the curves of $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ all on the same graph. [4]

4. By using the definition of a derivative, find the derivative of the following function

$$f(x) = \frac{x}{x+1}.$$

[4]

5. Find the limits

(a) $\lim_{x\to\infty} e^{-x} \ln x$ (b) $\lim_{x\to 0} \dfrac{\cos(mx) - \cos(nx)}{x^2}$. [4]

6. Find the equation for the tangent line to the curve defined by

$$y^2(y^2 - 4) = x^2(x^2 - 5)$$

at the point $(0, -2)$. [6]

7. Evaluate the definite integrals

(a) $\int_0^1 x^2 \exp\left(3x^3\right) dx$ (b) $\int_{-2}^{5/2} \dfrac{x+1}{\sqrt{5-2x}} dx$. [6]

8. Sketch the curve with the polar equation $r = 2(1 - \sin\theta)$. [4]

9. Distance ℓ of the point $\vec{P}$ from the line $\vec{r} = \vec{r}_0 + t\,\vec{v}$ can be calculated as

$$\ell = \frac{|(\vec{P} - \vec{r}_0) \times \vec{v}|}{|\vec{v}|}.$$

Find the distance between the point $(2, 2, 1)$ and the line that passes through $(2, 3, 0)$ parallel to the line $\vec{r} = (t, 1 + 2t, -4t)$, where $t \in R$. [8]

10. What is the volume of the parallelepiped with three of its edges given by the vectors $\vec{a} = (1, 4, -7)$, $\vec{b} = (2, -1, 4)$ and $\vec{c} = (0, -9, 18)$. [4]

11. Find the equation of the tangent plane to the surface

$$x^2 + 2y^2 + z^2 - 4 = 0$$

at the point $(1, -1, 1)$. [6]

12. Use polar coordinates to evaluate the following integral

$$\int\int_D dxdy,$$

where D is the interior of the circle $(x - 2)^2 + y^2 = 4$. [6]

13. Find an equation of the plane that passes through the following points $A(1, 3, 2)$, $B(3, -1, 6)$ and $C(5, 2, 0)$. [4]

14. Find the directional derivative of the function $f(x, y) = \ln(x^2 + y^2)$ at the point $(2, 1)$ in the direction of vector $\vec{v} = (-1, 2)$. [4]

15. By using polar coordinates, or otherwise, find the volume of the solid bounded by the paraboloid $10 - 3x^2 - 3y^2 - z = 0$ and the plane $z = 4$. [7]

END OF EXAMINATION PAPER