Lecture 9

Lecturer: Prof. Sergei Fedotov

10131 - Calculus and Vectors

Taylor and Maclaurin series.

Lecture 9

1. Power series
2. Taylor and Maclaurin series

Power series

A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ...$$

is called a power series centered at a. Here x is a variable and c_n are constants called the coefficients of the series. There is a positive number R (radius of convergence)) such that if $|x-a| < R$ then the series converges, if $|x-a| > R$ the series diverges.

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Example: Geometric series

$$\sum_{n=0}^{\infty} x^n, \quad a = 0 \quad |x| < 1$$

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Example: Geometric series

$$\sum_{n=0}^{\infty} x^n, \quad a = 0 \quad |x| < 1$$

Example:

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{n},$$

$a = 4$ and $|x-4| < 1$ that is $R = 1$.

Power series expansion

Theorem. If f has a power series expansion at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R$$

then its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

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Example:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

Example:

$$\frac{1}{1+x^6} = \frac{1}{1-(-x^6)} = 1 - x^6 + x^{12} - x^{24} + ..$$

Taylor and Maclaurin series

Taylor series of the function f at a :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

For the special case $a = 0$, the Taylor series becomes Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

or

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

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Example: Show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example: Find the Taylor series for e^x at $a = 5$.