Lecture 5

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10131 - Calculus and Vectors

The limit of a function

Lecture 5

1. The limit of the function:

$$\lim_{x \longrightarrow a} f(x)$$

2. Continuity

Limit

We define

$$\lim_{x \to a} f(x) = A$$

The basic idea of the limit is that we can make the values of $f(x)$ arbitrary close to A with values of x approaching a. Note that $x \neq a$.

Example: Find the limits

$$\lim_{x \to 1} (x^2 - 4), \quad \lim_{x \to 3} (x^2 - 4).$$

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Example: Find the limit $\lim_{x \to 0} f(x)$ for the following functions

$$f(x) = \begin{cases} 2x + 1 & \text{for} \quad x \leq 0 \\ x^2 - x & \text{for} \quad x > 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{5x + |x|}{x} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

Limits

We define

$$\lim_{x \to a^-} f(x) = A$$

and say the left-hand limit of $f(x)$. The symbol $x \to a^-$ means that x approaches a and we consider only $x < a$.

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$$\lim_{x \to a^+} f(x) = A$$

and say the right-hand limit of $f(x)$.

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Infinite limits: $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} f(x) = -\infty$

Example. Find the limits:

$$\lim_{x \to 0} x^{-2} \qquad \lim_{x \to 4^+} \frac{3x}{x-4} \qquad \lim_{x \to 4^-} \frac{3x}{x-4}$$

Limits at infinity:

$$\lim_{x \to \infty} f(x) = A \quad \lim_{x \to -\infty} f(x) = A.$$

The line $y = A$ is called a horizontal asymptote of the curve $y = f(x)$.

Example: Find

$$\lim_{x \to \infty} \frac{3x^4 + x + 5}{2x^4 + 45x + 56}.$$

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Limit Laws. Suppose the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a} [f(x) \times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$

Continuity

Definition.

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

This definition implicitly requires

1) $f(a)$ is defined (a is in the domain of f)

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

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Example: Find the value A for which the following function is continuous for all $x \in \mathbb{R}$

$$f(x) = \begin{cases} \frac{4}{4+x^4} & \text{if} \quad x \neq 0 \\ A & \text{if} \quad x = 0 \end{cases}$$