Lecture 31

Lecturer: Prof. Sergei Fedotov

10131 - Calculus and Vectors

Green's Theorem

1. Green's theorem

2. Independence of path

Line integral

We define the line integral of $\overrightarrow{\mathbf{f}}$ along a smooth curve C as

$$\int_C \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}} = \int_a^b \overrightarrow{\mathbf{f}}\left(\overrightarrow{\mathbf{r}}(t)\right) \cdot d\overrightarrow{\mathbf{r}}'(t)dt.$$

Suppose that the vector field $\overrightarrow{\mathbf{f}}$ on $\mathbb{R}^2$ is given in component form by the equation

$$\overrightarrow{\mathbf{f}} = P(x,y)\overrightarrow{\mathbf{i}} + Q(x,y)\overrightarrow{\mathbf{j}},$$

then

$$\int_C \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}} = \int_C P(x,y)dx + Q(x,y)dy.$$

Green's Theorem

Green's Theorem. Let C be a positively oriented simple closed curve in the plane. Let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ are continuous on D, then

$$\int_C P(x,y)dx + Q(x,y)dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Note that positive orientation refers to counterclockwise traversal of C.

Green's Theorem. Let C be a positively oriented simple closed curve in the plane. Let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ are continuous on D, then

$$\int_C P(x,y)dx + Q(x,y)dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Note that positive orientation refers to counterclockwise traversal of C.

Example. Find

$$\int_C \overrightarrow{\mathbf{f}} \cdot d\,\overrightarrow{\mathbf{r}}$$

for the vector field

$$\overrightarrow{\mathbf{f}} = (-y, x),$$

where C is the circle of radius a.

Independence of path

We say that the line integral $\int_C \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}}$ is independent of path if

$$\int_{C_1} \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}} = \int_{C_2} \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}}$$

for any two paths C_1 and C_2 that have the same initial and terminal points.

Independence of path

We say that the line integral $\int_C \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}}$ is independent of path if

$$\int_{C_1} \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}} = \int_{C_2} \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}}$$

for any two paths C_1 and C_2 that have the same initial and terminal points.

Theorem. Let C be a plane curve given by the parametric equation $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}(t)$, $a \leq t \leq b$. Let $\varphi(x, y)$ be a differentiable function whose gradient $\nabla\varphi$ is continuous on C. Then

$$\int_C \nabla\varphi \cdot d\overrightarrow{\mathbf{r}} = \varphi\left(\overrightarrow{\mathbf{r}}(b)\right) - \varphi\left(\overrightarrow{\mathbf{r}}(b)\right).$$

Independence of path

We say that the line integral $\int_C \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}}$ is independent of path if

$$\int_{C_1} \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}} = \int_{C_2} \overrightarrow{\mathbf{f}} \cdot d\overrightarrow{\mathbf{r}}$$

for any two paths C_1 and C_2 that have the same initial and terminal points.

Theorem. Let C be a plane curve given by the parametric equation $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}(t)$, $a \leq t \leq b$. Let $\varphi(x, y)$ be a differentiable function whose gradient $\nabla\varphi$ is continuous on C. Then

$$\int_C \nabla\varphi \cdot d\overrightarrow{\mathbf{r}} = \varphi\left(\overrightarrow{\mathbf{r}}(b)\right) - \varphi\left(\overrightarrow{\mathbf{r}}(b)\right).$$

Proof: Using the definition of the line integral, we write

$$\int_C \nabla\varphi \cdot d\overrightarrow{\mathbf{r}} = \int_a^b \nabla\varphi\left(\overrightarrow{\mathbf{r}}(t)\right) \cdot d\overrightarrow{\mathbf{r}}'(t)dt$$