Lecture 25

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10131 - Calculus and Vectors

Taylor series and directional derivative

Lecture 25

1. Taylor series expansion of $f(x, y)$
2. Directional derivative of $f(x, y)$

Taylor series

We consider only the quadratic approximation to $f(x, y)$ at (a, b). If $f(x, y)$ has continuous second-order partial derivatives at (a, b), then the Taylor series of f at (a, b) is

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) +$$

$$\frac{1}{2!} f_{xx}(a, b)(x - a)^2 + \frac{1}{2!} f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b) + ...$$

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Example: Find the Taylor series of

$$f(x, y) = \sin(xy)$$

at the point $\left(1, \frac{\pi}{2}\right)$ (only quadratic approximation).

Directional derivative

The purpose is to find the rate of change of $f(x, y)$ in the direction of unit vector $\overrightarrow{\mathbf{u}} = (u_1, u_2)$.

Definition.

The directional derivative of f at (x, y) in the direction of unit vector $\overrightarrow{\mathbf{u}} = (u_1, u_2)$ is

$$D_{\overrightarrow{\mathbf{u}}} f = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

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Theorem.

If f is a differentiable function of x and y, then the directional derivative of f at (x, y) is

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Directional Derivative and Gradient

If we introduce the Gradient Vector

$$\nabla f = (f_x(x,y), f_y(x,y)),$$

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Example:
Find the directional derivative $D_{\overrightarrow{\mathbf{u}}} f$ of the function defined by

$$f(x,y) = x^2 + y^4$$

in the direction of the vector

$$\overrightarrow{\mathbf{v}} = (1,2).$$