Lecture 24

Lecturer: Prof. Sergei Fedotov

10131 - Calculus and Vectors

Maxima and minima of functions of two variables

Lecture 24

1. Chain rule (case 2)
2. Maxima and minima of functions of two variables
3. Critical points and second derivative test

Definition

Chain rule (case 2). If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Note that s, t are independent variables and x, y are intermediate variables

Definition

Chain rule (case 2). If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Note that s, t are independent variables and x, y are intermediate variables

Example: Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ for

$$z = x^2 + y^2,$$

where $x = r\cos\theta$ and $y = r\sin\theta$.

Solution: θ, r are independent variables and x, y are intermediate variables

Maximum and minimum

Definition 1. A function of two variables has a local maximum at point (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a local maximum value. If $f(x, y) \geq f(a, b)$ for all (x, y) in such disk, $f(a, b)$ is a local minimum value.

Definition 2. A function of two variables has a absolute maximum at point (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in domain of function f.

Definition 3. A function of two variables has a absolute minimum at point (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in domain of function f.

Maximum and minimum

Definition 1. A function of two variables has a local maximum at point (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a local maximum value. If $f(x, y) \geq f(a, b)$ for all (x, y) in such disk, $f(a, b)$ is a local minimum value.

Definition 2. A function of two variables has a absolute maximum at point (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in domain of function f.

Definition 3. A function of two variables has a absolute minimum at point (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in domain of function f.

Definition 4. The point $P(x_0, y_0)$ called the critical point of the function f if

$$f_x(x_0, y_0) = 0, \quad f_y(x_0, y_0) = 0.$$

Theorem. Suppose that (a, b) is the point of local maximum or minimum of the function $f(x, y)$ that has continuous first order derivatives f_x and f_y. Assume in addition that (a, b) is interior point of domain of the function f. Then

$$f_x(a, b) = 0, \quad f_y(a, b) = 0.$$

Local maximum or minimum

Theorem. Suppose that (a, b) is the point of local maximum or minimum of the function $f(x, y)$ that has continuous first order derivatives f_x and f_y. Assume in addition that (a, b) is interior point of domain of the function f. Then

$$f_x(a, b) = 0, \quad f_y(a, b) = 0.$$

Examples:

$$z = 2 + x^2 + y^2, \quad z = 1 - x^2 - y^2, \quad z = -x^2 + y^2.$$

Let f be a function of two variables with continuous second-order derivatives in some circle centered at the critical point (a, b) and let

$$D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b).$$

If $D > 0$, $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

If $D > 0$, $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

If $D < 0$ then f has a saddle point at (a, b)

Second derivative test

Let f be a function of two variables with continuous second-order derivatives in some circle centered at the critical point (a, b) and let

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b).$$

If $D > 0$, $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

If $D > 0$, $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

If $D < 0$ then f has a saddle point at (a, b)

Example: Find the local maximum and minimum values and saddle point of

$$f(x,y) = \frac{x^4}{4} + \frac{y^4}{4} - xy + 5.$$