Lecture 22

Lecturer: Prof. Sergei Fedotov

10131 - Calculus and Vectors

Functions of two variables

Lecture 22

1. Real valued functions of two variables
2. Limits and continuity
3. Partial derivatives

Real valued functions of two variables and continuity

Real valued function of two variables $f: \ D \to \mathbb{R}$ is a rule which assigns to each point $(x, y) \in D \subseteq \mathbb{R}^2$ a unique $z \in \mathbb{R}$ denoted by

$$z = f(x, y).$$

The set D is the domain of f and its range is the set

$$\left\{ f(x, y) \, |_{(x,y) \in D} \right\}.$$

Real valued functions of two variables and continuity

Real valued function of two variables $f: \ D \to \mathbb{R}$ is a rule which assigns to each point $(x, y) \in D \subseteq \mathbb{R}^2$ a unique $z \in \mathbb{R}$ denoted by

$$z = f(x, y).$$

The set D is the domain of f and its range is the set

$$\left\{ f(x, y) \,|_{(x,y)\in D} \right\}.$$

If f is a function of two variables with domain D, then the graph of f is the surface in $\mathbb{R}^3$

$$S = \left\{ (x, y, z) \,|_{z=f(x,y) \text{ and } (x,y)\in D} \right\}.$$

Real valued functions of two variables and continuity

Real valued function of two variables $f: \; D \to \mathbb{R}$ is a rule which assigns to each point $(x, y) \in D \subseteq \mathbb{R}^2$ a unique $z \in \mathbb{R}$ denoted by

$$z = f(x, y).$$

The set D is the domain of f and its range is the set

$$\left\{ f(x, y) \, |_{(x,y) \in D} \right\}.$$

If f is a function of two variables with domain D, then the graph of f is the surface in $\mathbb{R}^3$

$$S = \left\{ (x, y, z) \, |_{z = f(x,y) \text{ and } (x,y) \in D} \right\}.$$

Examples:

1. Let $f: \; D \to \mathbb{R}$ defined by $f(x, y) = x \ln(y^4 - x)$. Find the domain D.
2. Let $f: \; D \to \mathbb{R}$ defined by $f(x, y) = \sqrt{1 - x^2 - y^2}$. Find the domain D.

A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} = f(a, b).$$

Limits and continuity

A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} = f(a, b).$$

Example: Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Partial derivatives

If f is a function of two variables, its partial derivatives are

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Partial derivatives

If f is a function of two variables, its partial derivatives are

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Notations:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\Big|_y = \frac{\partial}{\partial x} f(x, y) = f_x = D_x f = \frac{\partial z}{\partial x}.$$

Partial derivatives

If f is a function of two variables, its partial derivatives are

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Notations:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}|_y = \frac{\partial}{\partial x} f(x,y) = f_x = D_x f = \frac{\partial z}{\partial x}.$$

Example: Find the partial derivatives for

$$f(x,y) = x^2 y + y^3 x$$

and

$$f(x,y) = \sin\left(xy + y^2\right).$$

Partial derivatives

If f is a function of two variables, its partial derivatives are

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Notations:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}|_y = \frac{\partial}{\partial x} f(x,y) = f_x = D_x f = \frac{\partial z}{\partial x}.$$

Example: Find the partial derivatives for

$$f(x,y) = x^2 y + y^3 x$$

and

$$f(x,y) = \sin\left(xy + y^2\right).$$

Find the second partial derivatives: $f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}$, $f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}$ and $f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y}$