Lecture 19

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10131 - Calculus and Vectors

Test revision + Cross Product

Lecture 19

1. Test revision
2. Cross product as determinant
3. Scalar triple products and volume of parallelepiped

Cross product

We can write the cross product as

$$\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where

$$\begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \overrightarrow{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \overrightarrow{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \overrightarrow{\mathbf{k}}$$

Example
Find the cross product of two vectors

$$\overrightarrow{\mathbf{a}} = (3,\ 2,\ -1) \quad \textit{and} \quad \overrightarrow{\mathbf{b}} = (1,\ -1,\ 1)$$

Scalar triple product and volume of parallelepiped

Scalar triple product of the three vectors $\overrightarrow{\mathbf{a}}$, $\overrightarrow{\mathbf{b}}$ and $\overrightarrow{\mathbf{c}}$ is

$$\overrightarrow{\mathbf{a}} \cdot (\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

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The volume V of the parallelepiped determined by three vectors $\overrightarrow{\mathbf{a}}$, $\overrightarrow{\mathbf{b}}$ and $\overrightarrow{\mathbf{c}}$ is the magnitude of their scalar triple product $V = \left|\overrightarrow{\mathbf{a}} \cdot (\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{c}})\right|$.

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Area of triangle ABC:

$$\frac{1}{2} \left| \overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}) \right| = \frac{1}{2} |\overrightarrow{\mathbf{AB}}||\overrightarrow{\mathbf{AC}}| \sin\theta,$$

where $\theta = \angle CAB$

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Example. Find the area of the triangle with vertices at $\overrightarrow{\mathbf{A}} = (3, 3, 3)$, $\overrightarrow{\mathbf{B}} = (1, 1, 1)$ and $\overrightarrow{\mathbf{C}} = (5, 4, 3)$.