Lecture 18

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10131 - Calculus and Vectors

Cross Product

Lecture 18

1. Cross product
2. Basis vectors
3. Cross product as determinant

Definition

For two vectors

$$\overrightarrow{\mathbf{a}} = (a_1, a_2, a_3) \quad and \quad \overrightarrow{\mathbf{b}} = (b_1, b_2, b_3),$$

we define the cross product $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$ as

$$\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1).$$

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The $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$ is orthogonal to both vectors $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$.

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If θ is angle between vectors $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$, then

$$|\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}| = |\overrightarrow{\mathbf{a}}||\overrightarrow{\mathbf{b}}| \sin\theta.$$

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Geometric interpretation: the magnitude of $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$ is the area of the parallelogram having $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$ as sides.

Basis vectors

It is convenient to introduce the basis vectors:

$$\overrightarrow{\mathbf{i}} = (1, 0, 0), \quad \overrightarrow{\mathbf{j}} = (0, 1, 0), \quad \overrightarrow{\mathbf{k}} = (0, 0, 1).$$

They have length 1 and point in directions of the positive x, y, z axes.

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If $\overrightarrow{\mathbf{a}} = (a_1, a_2, a_3)$, then we write

$$\overrightarrow{\mathbf{a}} = a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}.$$

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Example: Find the unit vector in the direction of the vector $3\overrightarrow{\mathbf{i}} + 4\overrightarrow{\mathbf{k}}$.

Determinants

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Determinant of order 3:

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Cross product

We can write the cross product as

$$\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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Example
Find the cross product of two vectors

$$\overrightarrow{\mathbf{a}} = (3,\ 2,\ -1) \quad \textit{and} \quad \overrightarrow{\mathbf{b}} = (1,\ -1,\ 1)$$