Lecture 15

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10131 - Calculus and Vectors

Length of a Curve and Parametric Equations

Lecture 15

1. Length of a curve
2. Curves defined by parametric equations
3. Calculus with parametric curves

Length of a curve

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Example: Find the length of the arc of $y = x^{\frac{3}{2}}$ between the points $(1, 1)$ and $\left(2, 2\sqrt{2}\right)$.

Curves defined by parametric equations

The functions

$$x = x(t), \quad y = y(t)$$

are called the parametric equations for the curve. Each value of t (the parameter) determines a point (x, y). As t varies, the point $(x(t), y(t))$ varies and traces out a curve C, which is called a parametric curve.

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Example: Sketch the curve with parametric equations $x = \sin t$ and $y = \sin^8 t$.

Calculus with parametric curves

Theorem. Let C be a curve described by the parametric equations $x = x(t)$, $y = y(t)$, $\alpha \le t \le \beta$, where $x'(t)$ and $y'(t)$ are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

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