Lecture 13

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10131 - Calculus and Vectors

Integration by Parts and Integration of Rational Functions

1. Formula for integration by parts
2. Integration of rational functions by partial fractions
3. Trigonometric substitutions

Integration by parts

The Rule for Integration by Parts corresponds to the Product Rule:

$$\int u dv = uv - \int v du$$

or

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

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Integration by parts for definite integrals:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx.$$

Integration of rational functions by partial fractions (Problem Sheet 6, 5)

A rational function is by definition the quotient of two polynomial functions. Example: $\frac{3x-1}{x^2-x-6}$.

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by using the partial fraction decomposition of the integrand.
Solution:

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$$\frac{3x-1}{x^2-x-6} = \frac{3x-1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}.$$

Then $A+B=3, -3A+2B=-1$ and $A=\frac{7}{5}$, $B=\frac{8}{5}$

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Thus

$$\int \frac{3x-1}{x^2-x-6}dx = \frac{7}{5}\int \frac{dx}{x+2} + \frac{8}{5}\int \frac{dx}{x-3} = ...$$

Completing the square

Integrals with quadratic expressions in denominator can often be reduced to standard forms by completing the square (Problem Sheet 5, 4 (a), (b))

Example:

$$\int \frac{7}{x^2 - 6x + 25} dx = 7 \int \frac{1}{(x-3)^2 + 16} dx =$$

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Example:

$$\int \frac{7}{x^2 - 6x + 25} dx = 7 \int \frac{1}{(x-3)^2 + 16} dx = \frac{7}{4} \tan^{-1}\left(\frac{x-3}{4}\right) + C.$$

Trigonometric substitutions

Let $x = g(u)$ be one-to-one function. Inverse substitution:

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Example: Find

$$\int \frac{dx}{\sqrt{a^2 - x^2}} \qquad \int \frac{dx}{a^2 + x^2}$$