Lecture 12

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10131 - Calculus and Vectors

Indefinite Integrals and Techniques of Integration

1. Indefinite integrals
2. Elementary integrals
3. Techniques of integration

Indefinite integrals

The notation $\int f(x)dx$ is used for an antiderivative of f and is called an indefinite integral

$$\int f(x)dx = F(x) \qquad \textit{means} \qquad F'(x) = f(x)$$

Example:

$$\int \frac{1}{x}dx = \ln x + C \qquad \textit{means} \qquad \frac{d}{dx}(\ln x + C) = \frac{1}{x}$$

Table of indefinite integrals.

Some basic integrals:

$f(x)$	$\int f(x)\,\mathrm{d}x$
e^x	$e^x + C$
x^n for $n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$1/x$ for $x \neq 0$	$\ln\|x\| + C$
a^x or $e^{x\ln(a)}$ for $a \neq 1,\ a > 0$	$\frac{a^x}{\ln(a)} + C$
e^{ax} for $a \neq 0$	$\frac{e^{ax}}{a} + C$
$\cos(ax)$ for $a \neq 0$	$\frac{1}{a}\sin(ax) + C$
$\sin(ax)$ for $a \neq 0$	$-\frac{1}{a}\cos(ax) + C$
$\frac{1}{x^2+a^2}$ for $a \neq 0$	$\frac{1}{a}\tan^{-1}(x/a) + C$
$\frac{1}{a^2-x^2}$ for $\|x\| < \|a\|,\ a \neq 0$	$\frac{1}{a}\tanh^{-1}(x/a) + C$
$\frac{1}{x^2-a^2}$ for $\|x\| > \|a\|,\ a \neq 0$	$-\frac{1}{a}\coth^{-1}(x/a) + C$
$\frac{1}{x^2-a^2}$ for $\|x\| \neq \|a\|,\ a \neq 0$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| + C$
$\frac{1}{\sqrt{x^2+a^2}}$ for $a \neq 0$	$\sinh^{-1}(x/a) + C$
$\frac{1}{\sqrt{a^2-x^2}}$ for $\|x\| < a,\ a > 0$	$\sin^{-1}(x/a) + C$
$\frac{1}{\sqrt{x^2-a^2}}$ for $x > a,\ a > 0$	$\cosh^{-1}(x/a) + C$
$\frac{1}{\sqrt{x^2-a^2}}$ for $x < -a,\ a > 0$	$-\cosh^{-1}(-x/a) + C$

Techniques of integration

The Substitution Rule. Let $u = g(x)$ be a differentiable function with range D and f be continuous on D, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

This is the integral version of the Chain Rule.

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Proof: Let F be an antiderivative of f: $F'(x) = f(x)$, then

$$\int F'(g(x))g'(x)dx = F(g(x)) + C.$$

This is because of the Chain Rule:

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x).$$

Let us make the change of variables or substitution as $u = g(x)$, then....

The Substitution Rule Examples

The main idea behind the Substitution Rule is to replace a relatively complicated integral by a simple integral by using the change of variables $x \to u$. The new variable u is a function of x.

Example: Find

$$\int 6x^5 \cos\left(x^6 + 34\right) dx.$$

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Example: Find

$$\int \sqrt{1 + e^x} e^x dx.$$

Let $u = 1 + e^x$, then...

Substitution Rule for Definite Integrals

The Substitution Rule for Definite Integrals . Let $g'(x)$ be a continuous function on the closed interval $[a, b]$ and $f(x)$ be continuous function on the range of $u = g(x)$, then

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Example: Find

$$\int_0^4 \sqrt{2x+1} dx.$$