0F2 - Vectors (written by Dr Mike Simon)

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Note: this course is also known as MATH19842.

Timetable:

Lectures are in Crawford House TH 1 on Thursdays at 9am and Renold C16 on Fridays at 4pm.

Tutorials start in week 2 and are on Wednesdays at 9am; the tutorial room list is yet to be circulated.

Assessment:

Coursework test 1 on this topic will take place in week 6 (worth 10%), and half of the exam at the end of the course will be on this topic.

Syllabus:

1. Introduction
2. Basic concepts of vectors
3. Addition of vectors
4. Position vectors
5. Cartesian coordinates
6. Scalar product (or dot product)
7. Lines and planes
8. Further 3-D geometry
9. Vector product (or cross product)

1. Introduction

1.1 What is a vector?

Most quantities in Engineering can be divided into two types:
1) **scalars** and 2) **vectors**.

A **scalar** is a quantity which can be described by a single number (positive, negative or zero). If we ignore the sign, that gives us the **magnitude**, which is positive (or zero) and shows us how big the scalar is.

An example of a physical quantity which is a scalar is the volume of tea that my mug will hold.

A **vector** is a quantity which has both a **magnitude** and also a specific **direction** in space. The magnitude (also known as the *modulus* or *length*) of a vector is positive (or zero).

An example of a physical quantity which is a vector is the force needed to hold this pen stationary in the air.

Note: Often the vectors we are interested in are physical quantities which can be thought of in **2 or 3 dimensions**. However there are occasions where we can generalise the results and imagine vectors in n-dimensional space.

1.2 When do we use vectors?

If we talk about the **speed** of a car (*e.g.* 60 miles per hour) this is just a number (positive or zero) in appropriate units; this is a **scalar**.

However, it may be important to consider also the direction the car is travelling. In that case we talk about the **velocity** of the car (*e.g.* 60 miles per hour heading due west); velocity is a **vector** which includes the direction as well as the speed.

There are many other examples in both engineering and the sciences of the use of vectors.

2. Basic concepts of vectors

A vector can be represented by drawing a line with a direction arrow on it, referred to as a *directed line segment*. The *directed line segment* has two important characteristics:-

1. The length of the line represents the magnitude of the vector *given some appropriate scale.*
2. The direction of the line (arrow) represents the direction of the vector *given some appropriate orientation.*

2.1 General Notation

More generally an arbitrary vector quantity is represented as follows as a directed line between two points A and B:-
(Draw in vector between A and B)

It is **extremely** important that vectors are written in such a way as to distinguish them from scalars. Various notations are used:-

1. For a vector between two endpoints A and B in the direction from A to B, the notation $\overrightarrow{AB}$ is used.
2. Often in textbooks, vectors are indicated by using a bold typeface such as $\mathbf{a}$.
3. In handwritten text, vector quantities are often underlined such as $\underline{a}$.

In this, the three notations are alternatives for the same vector, and so

$$\overrightarrow{AB} = \mathbf{a} = \underline{a}.$$

2.2 Two fundamental ideas

The following ideas are essential:-

1. Two vectors **a** and **c** are *equal* if they have the same magnitude and direction, regardless of where they are.
   (Draw $\mathbf{a} = \mathbf{c}$ *and* $\overrightarrow{AB} = \overrightarrow{CD}$*.)*

2. A vector which has the same magnitude as the vector **a** but has the *opposite* direction is denoted by $-\mathbf{a}$.
   (Draw $\mathbf{a}$ *and* $-\mathbf{a}$ *as well as* $\overrightarrow{AB}$ *and* $\overrightarrow{BA}$*.)*

Note that the above idea of the equality of vectors does not depend upon *location in space*, and therefore these vectors are sometimes referred to as *free* vectors, meaning free from a specific location.
For example, we have the same vector if we travel 40 miles north from Manchester as if we travel 40 miles north from London.

2.3 Magnitude only

If we wish to refer to just the magnitude (length) of a vector we write this using modulus signs such as $|\overrightarrow{AB}|$ or $|\underline{a}|$ or $|\mathbf{a}|$.

Later on it will also be useful to have vectors which we think of as being only a direction!

3. Addition of vectors

Example:

Suppose an aircraft starts at point A and flies 200 miles due East to point B, and then turns and flies 250 miles North-East to point C. The end result is equivalent to flying in a straight line from point A to point C; this is most easily seen geometrically.

(Put in triangle with points A, B and C.)

Therefore vectors are said to satisfy the *triangle law of addition*. That is, since travelling from A to B and then B to C is equivalent to travelling directly from A to C, we write

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

3.1 Definition of vector addition

In the general situation if we want to add the vector $\underline{b}$ to $\underline{a}$, we copy the vector $\underline{b}$ in the place where its *tail* coincides with the *head* of $\underline{a}$. Then $\underline{a}$ and $\underline{b}$ form two sides of a triangle; the third side is the vector sum $\underline{a} + \underline{b}$.

Note: If you think of the vector as an *arrow*, the *head* is the sharp forward end and the *tail* has the feathers; so for a vector, the head is the end to which the arrow points and tail is the opposite end.)

Also note: this particular sort of vector copying can be thought of a moving it in a special way where the start and end points change but the direction and length are unchanged - we talk about this as a *translation*.

(Put in diagrams illustrating the above.)

3.2 Question:
Is $\underline{b} + \underline{a}$ equal to $\underline{a} + \underline{b}$?

(Put in parallelogram to illustrate.)

Answer: Yes!

There is a mathematical word for the fact that you can change the order of the vectors without changing the sum - we say that vector addition is *commutative.* This means that

$$\begin{aligned} \overrightarrow{OP} + \overrightarrow{PQ} &= \overrightarrow{PQ} + \overrightarrow{OP}, \\ \underline{c} + \underline{d} &= \underline{d} + \underline{c}, \qquad \text{and} \\ \mathbf{b} + \mathbf{a} &= \mathbf{a} + \mathbf{b}. \end{aligned}$$

3.3 Another Question:
Is $\underline{a} + (\underline{b} + \underline{c})$ equal to $(\underline{a} + \underline{b}) + \underline{c}$?
There is an example sheet question on this for you to do.

Answer: Yes!

The mathematical word for the fact that the way in which you add three vectors does not matter is to say that vector addition is *associative.*

This means that if I wish to add together $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and $\overrightarrow{RS}$, I can add $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ first to get $\overrightarrow{PR}$, and then add that to $\overrightarrow{RS}$; **or** I can add $\overrightarrow{QR}$ and $\overrightarrow{RS}$ first to get $\overrightarrow{QS}$, and then add that to $\overrightarrow{PQ}$.

3.4 Subtraction of a vector

This is simply done by adding the negative of the vector.

$$\begin{aligned} \underline{a} - \underline{b} &= \underline{a} + (\underline{-b}), \qquad \text{and} \\ \mathbf{c} - \mathbf{d} &= \mathbf{c} + (-\mathbf{d}). \end{aligned}$$

(Put in diagrams illustrating the above.)

A special case is where you subtract a vector from itself:-

$$\underline{a} - \underline{a} = \underline{0}.$$

Note 1: The right-hand side is **not** just 0 (which is a scalar); it is the **zero vector** which is a vector but has no magnitude. In textbooks it is often written **0** in bold (so that $\mathbf{a} - \mathbf{a} = \mathbf{0}$).

Note 2: The zero vector **0** is such that $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ for any vector **a**.

3.5 Parallelograms

Parallelogram have been used in previous examples.

A parallelogram $ABCD$ is a special *quadrilateral* where opposite sides (*e.g.* AB and CD) are parallel and of equal length (and the same will apply to AC and BD).

(Put in parallelogram diagram.)

As a result, the vectors for opposite sides of the parallelogram are equal. That is, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$.

3.6 Multiplying a vector by a scalar

If we add $\underline{a}$ to itself we get a vector which is in the same direction as $\underline{a}$ but twice as long; we call this vector $2\underline{a}$.

Similarly, for any scalar k and vector $\underline{a}$ we can form the vector $k\underline{a}$, which is in the same direction as $\underline{a}$ but has magnitude k times bigger ($i.e.$ $k \times |\underline{a}|$).

The vector $k\underline{a}$ is said to be a *scalar multiple* of the vector $\underline{a}$.

Rules for scalar multiplication:

$$\begin{aligned} k(\underline{a}+\underline{b}) &= k\underline{a}+k\underline{b}, \\ (k+l)\underline{a} &= k\underline{a}+l\underline{a}, \qquad \text{and} \\ k(l\underline{a}) &= (kl)\underline{a}. \end{aligned}$$

If we want to *divide* a vector by a scalar k we simply multiply by $\frac{1}{k}$ so that $\underline{a}/k = \frac{1}{k}\underline{a}$.

(Put in diagram with some examples.)

3.7 Unit vectors

A vector which has a magnitude 1 is said to be a *unit vector*.

Recall that a vector $\underline{a}$ has magnitude $|\underline{a}|$; we can therefore create a unit vector in the direction of $\underline{a}$ by dividing $\underline{a}$ by the scalar $|\underline{a}|$. This unit vector is often denoted by $\hat{\underline{a}}$. Therefore

$$\hat{\underline{a}} = \frac{1}{|\underline{a}|} \times \underline{a} = \frac{\underline{a}}{|\underline{a}|}.$$

(This is a special case of scalar multiplication.)

4. Position Vectors

Free vectors have no specific location. However, some vectors represent a specific position. The letter O is generally used to denote a fixed *origin* in space.

Then the vector $\overrightarrow{OA}$ is called the **position vector** $\mathbf{r} = \overrightarrow{OA}$ of A relative to O.

This *displacement* is unique and **cannot** be represented by any other line of equal length and direction. In other words, for position vectors we have specified a starting point in space as well as direction and magnitude.

Note that in some problems you are told where to put the origin; however, there are situations where you can *choose* where to put the origin, and sometimes certain choices will make the solution easier than other choices.

(Put in an example.)

5. Cartesian Coordinates

Usually we want to define vectors in 2-dimensional $(2-D)$ or in 3-dimensional $(3-D)$ space. Therefore it is natural to use Cartesian co-ordinates, where we have axes at right-angles to each other.

5.1 Coordinates in 2-D

Any point P in the plane $(2-D$ space) can be defined in terms of its x and y coordinates.

Note: In writing points in this way we have already decided on the position of the origin (which becomes the unique point where $x = y = 0$) and on the x and y directions.

Often there are obvious, sensible choices for these, (*such as*) along the horizontal for x and vertically upwards for y.

(Put in picture of $x - y$ plane with point P marked.)

5.2a Basis vectors

Cartesian coordinates make use of unit vectors in the directions parallel to the x- and y-axes, known as **basis vectors**. (We need two of them in 2-D.)

A unit vector in the x-direction is usually denoted by $\mathbf{i}$ and a unit vector in the y-direction is usually denoted by $\mathbf{j}$.

Note: These are both of unit length and so $|\mathbf{i}| = |\mathbf{j}| = 1$.

Then $a\mathbf{i}$ and $b\mathbf{j}$ denote vectors of length a and b along the x and y directions, respectively.

If a is negative then $a\mathbf{i}$ is a vector of length $|a|$ in the direction of the unit vector $-\mathbf{i}$.

(Put in picture of $\mathbf{i}$ and $\mathbf{j}$ and multiples.)

Example:

Draw the vectors $-3\mathbf{i}$ and $2\mathbf{j}$. Then using the triangle law of addition we find $-3\mathbf{i} + \mathbf{2j}$.

Similarly we could draw the vector $\mathbf{i} + 7\mathbf{j}$ by first drawing the two separate vectors $\mathbf{i}$ and $7\mathbf{j}$ and then using the triangle law to add them together.

(Put in diagram with other examples.)

5.2b Expressing vectors in terms of basis vectors

Consider the vector $\mathbf{r} = \overrightarrow{AB}$. This can be regarded as coming from the sum of two vectors; one in the x direction and the other in the y direction.

In particular, we can add in a third point C such that $\overrightarrow{AC} = a\mathbf{i}$ and $\overrightarrow{CB} = b\mathbf{j}$. Then from the triangle law of vector addition

$$\mathbf{r} = \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = a\mathbf{i} + b\mathbf{j}.$$

In this same way, <u>any</u> vector in the $x - y$ plane can be expressed as a **linear combination of basis vectors** in the x and y directions.

(Put in diagram.)

5.3a Row and Column vector notation

This is a useful alternative way to write vectors which are expressed in Cartesian coordinates. Thus the vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ can instead be written as

$$(\, a \, , \; b \,) \qquad \text{or as} \qquad \begin{pmatrix} a \\ b \end{pmatrix}$$

in row or column vector notation respectively.

Examples:

a) The vector $\mathbf{t} = -\mathbf{i} - 7\mathbf{j}$ can be written as

$$\begin{pmatrix} -1 \\ -7 \end{pmatrix}$$

in column vector notation.

b) The vector $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j}$ can be written as

$$(\, -3 \, , \; 2 \,)$$

in row vector notation.

c) The vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$ can be written as

$$(\, 2 \, , \; -4 \,) \qquad \text{or as} \qquad \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

in row or column vector notation respectively.

5.3b Adding/subtracting 2-D vectors in this notation

The addition and subtraction of vectors in Cartesian coordinates is very straightforward and simply involves adding (or subtracting) the respective $\mathbf{i}$ and $\mathbf{j}$ components.

For example, if $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{s} = f\mathbf{i} + g\mathbf{j}$, then

$$\begin{aligned} \mathbf{r} + \mathbf{s} &= (a+f)\mathbf{i} + (b+g)\mathbf{j} \qquad \text{and} \\ \mathbf{r} - \mathbf{s} &= (a-f)\mathbf{i} + (b-g)\mathbf{j}. \end{aligned}$$

It is equally easy if the vectors are expressed in row or column vector notation; for example

$$\mathbf{r} + \mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} a+f \\ b+g \end{pmatrix}.$$

in column vector notation, and

$$\mathbf{r} - \mathbf{s} = (\, a\,,\;\, b\,) - (\, f\,,\;\, g\,) = (\, a-f\,,\;\, b-g\,)$$

in row vector notation.

Examples:

Calculate $\mathbf{r} + \mathbf{s}$ and $\mathbf{r} - \mathbf{s}$ in the following cases.

1. Let $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{s} = -4\mathbf{i} + 3\mathbf{j}$. Then

$$\begin{aligned} \mathbf{r} + \mathbf{s} &= (2 + (-4))\mathbf{i} + (1+3)\mathbf{j} = -2\mathbf{i} + 4\mathbf{j} \qquad \text{and} \\ \mathbf{r} - \mathbf{s} &= (2 - (-4))\mathbf{i} + (1-3)\mathbf{j} = 6\mathbf{i} - 2\mathbf{j}. \end{aligned}$$

2. Let

$$\mathbf{r} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 2 \\ 4 \end{pmatrix};$$

then

$$\mathbf{r} + \mathbf{s} = \begin{pmatrix} 1+2 \\ (-3)+4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} - \mathbf{s} = \begin{pmatrix} 1-2 \\ (-3)-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}.$$

5.4 Position vectors in 2-D

When we set up coordinates in the $2 - D$ plane, we choose the *origin*, denoted O, which becomes the point $(0, 0)$. Every vector in the plane is then taken to start at O. Then the vector from O to the point $P = (a, b)$ is known as the *position vector* of P.

The position vector of P with coordinates (a, b) is $\mathbf{r} = \overrightarrow{OP} = a\mathbf{i} + b\mathbf{j}$. As mentioned before position vectors are fixed vectors in that their origin (starting point) is specified.

(Put in diagram.)

Examples:

State the position vectors of the points P, Q, R S with respective coordinates, $(-3, 1), (2, 2), (0, 5)$ and $(-1, -1)$.

Answers:

$\overrightarrow{OP} = (-3)\mathbf{i} + 1\mathbf{j} = -3\mathbf{i} + \mathbf{j}$,

$\overrightarrow{OQ} = 2\mathbf{i} + 2\mathbf{j}$,

$\overrightarrow{OR} = 0\mathbf{i} + 5\mathbf{j} = 5\mathbf{j}$ and

$\overrightarrow{OS} = (-1)\mathbf{i} + (-1)\mathbf{j} = -(\mathbf{i} + \mathbf{j})$.

(Put in diagram.)

Question:

From the above points, find the vector $\overrightarrow{PQ}$.

a) Directly from the diagram $\overrightarrow{PQ} = 5\mathbf{i} + \mathbf{j}$.

b) Use the triangle law of addition. First note that $\overrightarrow{PO} = -\overrightarrow{OP} = -(-3\mathbf{i} + \mathbf{j}) = 3\mathbf{i} - \mathbf{j}$. Then

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = (3\mathbf{i} - \mathbf{j}) + (2\mathbf{i} + 2\mathbf{j}) = (3 + 2)\mathbf{i} + (-1 + 2)\mathbf{j} = 5\mathbf{i} + \mathbf{j}.$$

5.5 The modulus of a 2-D vector

The modulus of a vector is simply its length. Pythagoras' theorem shows that, for $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$,

$$|\mathbf{r}| = \sqrt{a^2 + b^2}.$$

(Put in diagram.)

Examples:
Find the modulus of the vectors
$\mathbf{s} = \mathbf{i} + 2\mathbf{j}, \mathbf{t} = 5\mathbf{i} - 3\mathbf{j}$ and $\mathbf{u} = -2\mathbf{i} + 7\mathbf{j}$.
Answers:
$|\mathbf{s}| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$,
$|\mathbf{t}| = \sqrt{5^2 + (-3)^2} = \sqrt{25+9} = \sqrt{34}$, and
$|\mathbf{u}| = \sqrt{(-2)^2 + 7^2} = \sqrt{4+49} = \sqrt{53}$.

Note 1: Resist the temptation to press the buttons on your calculator at this stage - anything your calculator produces as $\sqrt{5}$ will only be an approximation whereas $\sqrt{5}$ is the exact result for $|\mathbf{s}|$.

Note 2: The modulus is positive unless the vector itself is $\mathbf{0}$ - only the zero vector has zero modulus (and therefore has components zero in both the $\mathbf{i}$ and $\mathbf{j}$ directions).

Note 3: Different vectors can have the same modulus; for instance, all the vectors $6\mathbf{i} + 7\mathbf{j}$, $-6\mathbf{i} + 7\mathbf{j}$, $6\mathbf{i} - 7\mathbf{j}$, $-6\mathbf{i} - 7\mathbf{j}$, $7\mathbf{i} + 6\mathbf{j}$, $7\mathbf{i} - 6\mathbf{j}$, $7\mathbf{i} + 6\mathbf{j}$, $-7\mathbf{i} - 6\mathbf{j}$, $9\mathbf{i} + 2\mathbf{j}$, $9\mathbf{i} - 2\mathbf{j}$, $-9\mathbf{i} + 2\mathbf{j}$, $-9\mathbf{i} - 2\mathbf{j}$, $2\mathbf{i} + 9\mathbf{j}$,
$2\mathbf{i} - 9\mathbf{j}$, $-2\mathbf{i} + 9\mathbf{j}$ and $-2\mathbf{i} - 9\mathbf{j}$
have the same modulus $\sqrt{85}$. This means that all the points represented by these position vectors are the same distance from the origin, and that means that they must all lie on a circle.

5.6a Cartesian coordinates in 3-D

Any point in three-dimensional space can be defined in terms of its x, y and z coordinates.

The three axes in the x, y and z directions are mutually perpendicular; that is, there is a right angle ($i.e.$ 90^o or $\frac{\pi}{2}$ radians) between any two of the axes.

The unit vector in the z direction is denoted by $\mathbf{k}$. Therefore in the natural extension of the $2-D$ case, the point P having coordinates (a, b, c) has position vector

$$\overrightarrow{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

5.6b The modulus of a 3-D vector

For a vector expressed as $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, the modulus of $\mathbf{r}$ can again be found using Pythagoras' theorem.

$$|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}.$$

Example:

Points A, B and C have coordinates $(1, -3, 2), (-3, -2, -1)$ and $(4, 0, 2)$, respectively.

$a)$ Find the position vectors of A, B and C.

$b)$ Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$c)$ Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$.

Answer:

a) $\overrightarrow{OA} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
$\overrightarrow{OB} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
$\overrightarrow{OC} = 4\mathbf{i} + 0\mathbf{j} + 2\mathbf{k} = 4\mathbf{i} + 2\mathbf{k}.$

b)

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + (-3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \\ &= \big(-1 + (-3)\big)\mathbf{i} + \big(-(-3) + (-2)\big)\mathbf{j} + \big(-2 + (-1)\big)\mathbf{k} \\ &= -4\mathbf{i} + \mathbf{j} - 3\mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} = -\overrightarrow{OB} + \overrightarrow{OC} \\ &= -(-3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + 4\mathbf{i} + 0\mathbf{j} + 2\mathbf{k} \\ &= \big(-(-3) + 4\big)\mathbf{i} + \big(-(-2) + 0\big)\mathbf{j} + \big(-(-1) + 2\big)\mathbf{k} \\ &= 7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\end{aligned}$$

c) $|\overrightarrow{AB}| = \sqrt{(-4)^2 + 1^2 + (-3)^2} = \sqrt{16 + 1 + 9} = \sqrt{26}$
and $|\overrightarrow{BC}| = \sqrt{7^2 + 2^2 + 3^2} = \sqrt{49 + 4 + 9} = \sqrt{62}.$

Another Example:
Let $\underline{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and let B be the point $B(-2, 3, 1)$; find the point A such that

$$\overrightarrow{AB} = \underline{u}.$$

Answer:
We know that $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Then

$$\begin{aligned}\overrightarrow{OA} &= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - \overrightarrow{AB} \\ &= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - \mathbf{u} \qquad \textit{by assumption} \\ &= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - \big(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}\big) \\ &= -4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.\end{aligned}$$

5.7 Equal vectors and parallel vectors

Equal vectors

Suppose that the vectors $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{s} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ are equal (*i.e.* $\mathbf{r} = \mathbf{s}$). That means that

$$\mathbf{0} = \mathbf{r} - \mathbf{s} = (a-f)\mathbf{i} + (b-g)\mathbf{j} + (c-h)\mathbf{k}.$$

Now remember that the components of the zero vector $\mathbf{0}$ are all zero, which therefore means that $a - f = b - g = c - h = 0$.
In this way we see that

$$\mathbf{r} = \mathbf{s} \quad \underline{\text{if and only if}} \quad a = f, b = g \text{ and } c = h.$$

Parallel vectors

Two vectors $\mathbf{r}$ and $\mathbf{s}$ are parallel if there exists a scalar m such that

$$\mathbf{r} = m\mathbf{s}$$

(*Put in diagram*)
If we have $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{s} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ as before, this means they will be parallel (*i.e.* the vectors (a, b, c) and (f, g, h) will be parallel) when

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{r} = m\mathbf{s} = mf\mathbf{i} + mg\mathbf{j} + mh\mathbf{k}.$$

This therefore requires $a = mf, b = mg$ and $c = mh$, and so

$$\frac{a}{f} = \frac{b}{g} = \frac{c}{h} \quad (= m).$$

Example:
$\mathbf{r} = (2, 4, 8)$ and $\mathbf{s} = (3, 6, 12)$ are parallel because $\mathbf{r} = \frac{2}{3}\mathbf{s}$:-

$$\frac{a}{f} = \frac{2}{3} = \frac{b}{g} = \frac{4}{6} = \frac{c}{h} = \frac{8}{12}.$$

5.8 Unit vectors

We already have unit vectors denoted by $\mathbf{i}$ in the x direction, $\mathbf{j}$ in the y direction and $\mathbf{k}$ in the z direction.

Sometimes we adopt the notation where a 'hat' over a vector signifies that it is a unit vector; thus we sometimes call our three basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$.

In general we might want to *create* a unit vector in the direction of a given vector $\mathbf{v}$. As already mentioned (section 3.7) we do this by making $\hat{\mathbf{v}} = k\mathbf{v}$, with k chosen so that $|\hat{\mathbf{v}}| = 1$.
Now as $|\hat{\mathbf{v}}| = |k\mathbf{v}| = k|\mathbf{v}|$, this therefore requires $k = 1/|\mathbf{v}|$ and so we see that the unit vector in the direction of $\mathbf{v}$ is $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$.

Hence a unit vector $\underline{\hat{r}} = \hat{\mathbf{r}}$, in the direction of $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, is given by

$$\begin{aligned}\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} &= \frac{1}{\sqrt{a^2+b^2+c^2}}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \\ &= \frac{a}{\sqrt{a^2+b^2+c^2}}\mathbf{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\mathbf{k}.\end{aligned}$$

6. Scalar Product

There are two types of vector multiplication:-

1. Scalar (or dot) product – the answer is a scalar (a number);

2. Vector (or cross) product – the answer is another vector.

These are completely different, with different definitions and uses; they are <u>both</u> important in all the sciences.

We first consider the scalar product:-

6.1 Definition

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$. Place the vectors such that their <u>tails</u> are touching.

Let θ denote the angle between the two vectors. Note that we could measure the angle in two ways (as illustrated below) but θ is chosen to lie between 0 and $180^o = \pi$ radians.

(Put in diagram to show angle θ.)

Then the scalar product is denoted by $\mathbf{a} \cdot \mathbf{b}$ and it is a scalar defined by the product formula

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta \qquad .$$

Note that a dot ($\cdot$) has been used for multiplication in the formula. This is very important and why the scalar product is often referred to as the *dot product*. The $\times$ sign for multiplying vectors is reserved for the vector product, and so, $\underline{a} \times \underline{b} \neq \underline{a} \cdot \underline{b}$.

Note: <u>Be careful!</u>

The scalar product is *commutative*. That is,

$$\underline{b}\cdot\underline{a} = |\underline{b}|\,|\underline{a}|\cos\theta = |\underline{a}|\,|\underline{b}|\cos\theta = \underline{a}\cdot\underline{b}.$$

Example:
Vector $\mathbf{r}$ has modulus 5 and $\mathbf{s}$ has modulus 4 and the angle between them is $\frac{\pi}{6}$. Calculate $\mathbf{r}\cdot\mathbf{s}$.

Put in diagram to show the vectors and angle.

Answer: $\mathbf{r}\cdot\mathbf{s} = |\mathbf{r}|\,|\mathbf{s}|\cos(\pi/6) = 5\times 4\times\frac{\sqrt{3}}{2} = 10\sqrt{3}$

6.2 What is $\mathbf{r}\cdot\mathbf{r}$?

The angle between a vector and itself is 0^o. Therefore

$$\mathbf{r}\cdot\mathbf{r} = |\mathbf{r}|\,|\mathbf{r}|\cos(0) = |\mathbf{r}|^2\times 1 = |\mathbf{r}|^2.$$

6.3 The basis vectors and the scalar product

We know that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$; so, from the previous calculation, we find that

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \quad .$$

What about $\mathbf{i}\cdot\mathbf{j}$?

The angle between $\mathbf{i}$ and $\mathbf{j}$ is $\frac{\pi}{2}$ (or 90^o) since the x and y-axes are perpendicular. Therefore

$$\begin{aligned}\mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{j} &= |\mathbf{i}|\,|\mathbf{j}|\cos(\pi/2)\\ &= 1\times 1\times 0 = 0.\end{aligned}$$

The same is true for $\mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i}$ and $\mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j}$. Thus

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{j} = 0 \quad .$$

6.4 Other properties of the scalar product

More generally, for any two vectors **a** and **b** that are perpendicular (that is, the angle between the vectors is $\pi/2 = 90^o =$ a right-angle),

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\pi/2) = 0.$$

In addition, it is taken as understood that the scalar product follows the same rules for brackets as we have for ordinary multiplication. That is

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d} \quad .$$

Notice also that, although $|\mathbf{a}|$ and $|\mathbf{b}|$ are both positive (or zero), $\cos\theta$ can be negative (if θ is an obtuse angle) and so the scalar product can be negative.

Now we return to working out the scalar product.

Example:

Find the scalar product of $\underline{a} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\underline{b} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Answer:

For the scalar product formula we need

$|\underline{a}| = \sqrt{3^2 + (-1)^2 + 3^2} = \sqrt{19}$,

$|\underline{b}| = \sqrt{2^2 + 4^2 + (-1)^2} = \sqrt{21}$

and the angle between $\underline{a}$ and $\underline{b}$; how do we know what this angle is?

This is in general difficult to find, and so we derive a different formula for calculating the dot product.

6.5 Scalar Product in terms of coordinates

First, in two dimensions

Suppose we have two vectors $\mathbf{a}$ and $\mathbf{b}$ in the $2-D$ plane, given by $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$. Then, using the result in 6.4,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) \\ &= a_1\mathbf{i} \cdot (b_1\mathbf{i} + b_2\mathbf{j}) + a_2\mathbf{j} \cdot (b_1\mathbf{i} + b_2\mathbf{j}) \\ &= a_1b_1\,(\mathbf{i} \cdot \mathbf{i}) + a_1b_2\,(\mathbf{i} \cdot \mathbf{j}) + a_2b_1\,(\mathbf{j} \cdot \mathbf{i}) + a_2b_2\,(\mathbf{j} \cdot \mathbf{j}) \end{aligned}$$

Now using the results for unit vectors in section 6.3 we find

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1(1) + a_1b_2(0) + a_2b_1(0) + a_2b_2(1) = a_1b_1 + a_2b_2 \quad .$$

Now, in three dimensions

Suppose our vectors $\mathbf{a}$ and $\mathbf{b}$ are now given by $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \quad .$$

(Again this uses the result about brackets in section 6.4 and the results for unit vectors in section 6.3)

After section 6.4 we had the vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$; we now see that their scalar product is

$$\mathbf{a} \cdot \mathbf{b} = (3 \times 2) + (-1 \times 4) + (3 \times -1) = 6 - 4 - 3 = -1.$$

and so the angle between them is given by $\cos\theta = -1/(\sqrt{19}\sqrt{21})$.

Example:
Find the scalar product of the vectors $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{s} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, and hence find the angle θ between them.

Answer:

$$\mathbf{r} \cdot \mathbf{s} = (2 \times 3) + (-1 \times 3) + (3 \times -1) = 0.$$

Thus $|\mathbf{r}|\,|\mathbf{s}|\cos\theta = 0$; as $|\mathbf{r}|$ and $|\mathbf{s}|$ are clearly not zero, this must mean that $\cos\theta = 0$ and so $\theta = \pi/2$.

Therefore $\mathbf{r}$ and $\mathbf{s}$ are perpendicular.

6.6 Finding the angle between vectors

We have two formulas to work out the scalar product of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

1. $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

2. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ (as well as $|\mathbf{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$ and similarly for $|\mathbf{b}|$.)

We can clearly re-arrange to derive

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}|\,|\mathbf{b}|}$$

which enables us to find the angle θ.

Example:

Find the angle θ between the following pairs of vectors:-

1. $\underline{a} = 3\mathbf{i} + 2\mathbf{j}$ and $\underline{b} = 2\mathbf{i} + 4\mathbf{j}$;
2. $\underline{a} = -2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ and $\underline{b} = 7\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

Answer:

1. $|\underline{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$
   $|\underline{b}| = \sqrt{2^2 + 4^2} = \sqrt{20}$
   $\underline{a} \cdot \underline{b} = (3 \times 2) + (2 \times 4) = 14$

   Therefore
   $$\cos\theta = \frac{14}{\sqrt{13} \times \sqrt{20}} = 0.868$$
   giving
   $$\theta = \cos^{-1}(0.868) = 0.519$$
   $\theta = 0.519\,\text{radians}\ (= 29.7^o)$

   Put in diagram to illustrate the vectors and angle.

2. $|\underline{a}| = \sqrt{(-2)^2 + 5^2 + 3^2} = \sqrt{38}$
   $|\underline{b}| = \sqrt{7^2 + 1^2 + (-4)^2} = \sqrt{66}$
   $\underline{a} \cdot \underline{b} = (-2 \times 7) + (5 \times 1) + (3 \times -4) = -21$

   Therefore
   $$\cos\theta = \frac{-21}{\sqrt{38} \times \sqrt{66}} = -0.419$$
   giving
   $$\theta = \cos^{-1}(-0.419) = 2.004\,\text{radians}\ (= 114.8^o)$$

6.7 Resolving one vector along the direction of another

Quite often we want to work out the component of one vector $\mathbf{a}$ in a direction which is parallel to another given vector $\mathbf{b}$. We talk about this action as "resolving $\mathbf{a}$ in the direction of $\mathbf{b}$".
(*Put in diagram showing* $\mathbf{a}, \mathbf{b}$ *and the angle* θ *between them.*)

From the diagram we see that $\mathbf{a} = \mathbf{c} + \mathbf{d}$ where $\mathbf{d}$ is a vector perpendicular to $\mathbf{b}$ and $\mathbf{c}$ is a vector parallel to, *or* opposite to, $\mathbf{b}$. *Put in diagram.*

We talk about $\mathbf{c}$ as "*the projection of* $\mathbf{a}$ *onto* $\mathbf{b}$".

We also talk about "*the component of* $\mathbf{a}$ *in the direction of* $\mathbf{b}$", which is the scalar $|\mathbf{a}|\cos\theta$; this is essentially the length of $\mathbf{c}$, but can be positive or negative. It will be positive if $\mathbf{c}$ is in the same direction as $\mathbf{b}$, but will be negative if $\mathbf{c}$ is opposite to $\mathbf{b}$.

This use of the word "*component*" agrees with the use when we express vectors in terms of basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$.

Now as $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ we have

$$|\mathbf{a}|\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \cdot \hat{\mathbf{b}},$$

where $\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$.

Hence if we want the component of $\mathbf{a}$ in the direction of $\mathbf{b}$, all we need to do is to form the dot product of $\mathbf{a}$ with the unit vector $\hat{\mathbf{b}}$.

Notice also that

$$\mathbf{c} = |\mathbf{c}|\hat{\mathbf{b}} = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} = \left(\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2},$$

so that

$$\mathbf{d} = \mathbf{a} - \mathbf{c} = \mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} = \frac{(\mathbf{b}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}.$$

Exercise for you:- check that $\mathbf{d}$ *is perpendicular to* $\mathbf{b}$*.*

Example of dot product:

What is $|\underline{u}+\underline{v}|^2$?

Put in vector diagram

Recall that $|\underline{a}|^2 = \underline{a}\cdot\underline{a}$; so

$$\begin{aligned}
|\underline{u}+\underline{v}|^2 &= (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) = \underline{u}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v} \\
&= |\underline{u}|^2 + 2\underline{u}\cdot\underline{v} + |\underline{v}|^2 \\
&= |\underline{u}|^2 + 2|\underline{u}|\,|\underline{v}|\cos\theta + |\underline{v}|^2; \\
\textit{i.e.}\ |\underline{u}+\underline{v}|^2 &= |\underline{u}|^2 - 2|\underline{u}|\,|\underline{v}|\cos\theta_c + |\underline{v}|^2
\end{aligned}$$

where θ_c is the complement of the angle θ (*see diagram*) and you should recognise this result as the Cosine Rule.

7. Lines and Planes

7.1a The vector equation of a line

Consider the points A and B in $3-D$ space given by the position vectors $\mathbf{a}$ and $\mathbf{b}$. There exists a (straight) line which passes through the points A and B. Let P be a general point on the line through A and B and suppose that P has position vector $\mathbf{r}$.

Put in diagram to illustrate.

Note that $\overrightarrow{AB}$ and $\overrightarrow{AP}$ are parallel. Therefore $\overrightarrow{AP}$ is simply a scalar multiple of $\overrightarrow{AB}$, that is, $\overrightarrow{AP} = m\overrightarrow{AB}$ for some number m.

Using the triangle law of addition:-

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}.$$

Hence

$$\overrightarrow{AP} = m\overrightarrow{AB} = m(\mathbf{b} - \mathbf{a})$$

and so the general point P has position vector

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + m(\mathbf{b} - \mathbf{a}) = (1 - m)\mathbf{a} + m\mathbf{b}.$$

This equation $\left(\mathbf{r} = \mathbf{a} + m(\mathbf{b} - \mathbf{a})\right)$ is **the required form of the equation of the line** through the points with position vectors $\mathbf{a}$ and $\mathbf{b}$.

It becomes clear that m gives a measure of how far along the line the point P is from A, towards B:-

If $m = 0$, $\mathbf{r} = \mathbf{a}$ $(P = A)$ and if $m = 1$, $\mathbf{r} = \mathbf{b}$ $(P = B)$.
For $0 < m < 1$, the point P lies on the line between A and B;
for $m > 1$, the point lies on the line beyond B (from A) and
for $m < 0$, the point lies on the line beyond A (from B).

You could equally well re-write the vector equation of the same line through the same points A and B as

$$\mathbf{r} = \mathbf{b} + m'(\mathbf{a} - \mathbf{b})$$

with $m' = 1 - m$ (and so m' gives a measure of the distance along the line measured <u>from B towards A</u>).

Examples:
Write down the vector equation of the line which passes through the points with position vectors:-

Line 1 through: $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$;

Line 2 through: $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

Answers:
For Line 1 we have
$\mathbf{b} - \mathbf{a} = \big((2\mathbf{i} + 3\mathbf{j}) - (3\mathbf{i} - \mathbf{j})\big) = \big(-\mathbf{i} + 4\mathbf{j}\big)$,
and so Line 1 is given by $\mathbf{r} = (3\mathbf{i} - \mathbf{j}) + m\big(-\mathbf{i} + 4\mathbf{j}\big)$;

and for Line 2,
$\mathbf{b} - \mathbf{a} = \big((2\mathbf{i} + 5\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\big) = \big(\mathbf{i} + 2\mathbf{j} - \mathbf{k}\big)$
and so Line 2 is given by $\mathbf{r} = (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + m\big(\mathbf{i} + 2\mathbf{j} - \mathbf{k}\big)$.

7.1b The mid-point of two points

As the value of m gives a measure of how far the point P is along the line, one particular point of interest is the mid-point M, which is half way from A to B.

With $m = \frac{1}{2}$, we have

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Notice that we can read this as follows:-

To get the position vector of M, we need the vector from the origin O to the point M. We find this by first going from O to A, and then going from A to M, and the vector from A to M is half the vector from A to B.

Notice also that the position vector of M is not simply $\frac{1}{2}(\mathbf{b} - \mathbf{a})$; this vector is half the vector from A to B but that is not the same as the position vector (which is by definition a vector from O).

In the same way as deriving the position vector of the mid-point, we could find a point which divides AB into any given ratio; for instance the two 'trisection' points (which are one third of the way along AB from one end or the other) are given by

$$\mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{(2\mathbf{a} + \mathbf{b})}{3} \quad \text{and} \quad \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a}) = \frac{(\mathbf{a} + 2\mathbf{b})}{3}.$$

7.1c The Cartesian form of the equation of a line

It is sometimes useful to work with the Cartesian form of the vector equation of a line.

Suppose that we write

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ and } \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then $\mathbf{r} = \mathbf{a} + m(\mathbf{b} - \mathbf{a})$ (= the equation of a line) can be rewritten

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + m \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} = \begin{pmatrix} a_1 + m(b_1 - a_1) \\ a_2 + m(b_2 - a_2) \\ a_3 + m(b_3 - a_3) \end{pmatrix}.$$

Now look at the separate x, y and z components; these give:-

$$\begin{aligned} x &= a_1 + m(b_1 - a_1) \\ y &= a_2 + m(b_2 - a_2) \\ z &= a_3 + m(b_3 - a_3) \end{aligned}$$

As the value of m is the same in each case we see that therefore $m =$

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}.$$

This last set of equations is the *Cartesian form* of the equation of the straight line which passes through the points with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3).

There is a potential problem with the above if $b_1 = a_1$, since then $b_1 - a_1 = 0$. However, if this is so then any point on the straight line through A and B must also have $x - a_1 = 0$, so that $x = a_1$. The same idea applies to the y and z coordinates.

Example:

Write down the equation of the line L through the points $\mathbf{a} = (3, 5, -4)$ and $\mathbf{b} = (2, -1, 3)$ in

A. Cartesian form, and in

B. Vector form.

Do either of the points $C(0, -13, 17)$ or $D(2, -1, 8)$ lie on this line?

Answer:

A. In Cartesian form we write L as

$$\frac{x-2}{3-2} = \frac{y-(-1)}{5-(-1)} = \frac{z-3}{-4-3}$$

giving

$$\frac{x-2}{1} = \frac{y+1}{6} = \frac{z-3}{-7} \qquad (*).$$

B. In vector form the equation of the line L is

$$\begin{aligned} \mathbf{r} = \mathbf{a} + m(\mathbf{b} - \mathbf{a}) &= \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} + m \left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} \right\} \\ &= \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} + m \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix} \qquad (**). \end{aligned}$$

The point C will lie on the line L if the three ratios in $(*)$ are the same, and if an appropriate value of m can be found in $(**)$. As C is the point where $x = 0, y = -13$ and $z = 17$, we have

$$\frac{x-2}{1} = -2, \; \frac{y+1}{6} = \frac{-12}{6} = -2, \; \frac{z-3}{-7} = \frac{14}{-7} = -2,$$

and so the point C does indeed lie on the line L.

Alternatively, it is not hard to see that $(**)$ is satisfied at C with the value $m = +3$.

For the point D on the other hand, we need to find a value of m to satisfy

$$\overrightarrow{OD} = \begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} + m \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix},$$

which requires

$$\begin{aligned} 2 &= +3 - m, \\ -1 &= +5 - 6m, \text{ and} \\ 8 &= -4 + 7m. \end{aligned}$$

Although the first two of these both give $m = 1$, the third equation gives a different value of m and so this point does not lie on the line in question. (There is no single value of m which does the job.)

If we instead consider the ratios in $(*)$, we find that the first two are both zero but the third is not; again this shows that this point is not on the line L.

Up to now we have defined a line in terms of two points through which it passes; it is also possible to define a line in terms of 'one point + a direction'.

Example:
Given the point A with position vector $\mathbf{a}$ and the vector $\mathbf{u}$ find the parametric equation of the line passing through A and parallel to $\mathbf{u}$.
Answer:
The position vector $\mathbf{r}$ of any point P on the line is given by

$$\mathbf{r} = \mathbf{a} + m\mathbf{u}$$

for some scalar multiple m.

NB. To obtain the previous form of equation, set $\mathbf{u} = \mathbf{b} - \mathbf{a}$.

7.2 Vector equation of a plane

Consider a plane and suppose that A is a known point on the plane. Let A have position vector $\mathbf{a}$. Let P be an general point in the plane with position vector $\mathbf{r}$. Then clearly $\overrightarrow{AP}$ lies in the plane.

Let $\mathbf{n}$ denote a vector which is perpendicular to the plane. The vector $\mathbf{n}$ is called a "the normal vector to the plane". For example, $\mathbf{k}$ is perpendicular to (so is a normal vector to) the xy-plane.

Sometimes it is convenient if $\mathbf{n}$ is a unit vector, and we describe it as a "unit normal" (such as $\mathbf{k}$ above).

Put in diagram.

Question:

How are $\mathbf{n}$ and the vector $\overrightarrow{AP}$ related?

Answer:

Let $\mathbf{b} = \overrightarrow{AP}$. Then since $\mathbf{b}$ lies in the plane and $\mathbf{n}$ is perpendicular to the plane, $\mathbf{n}$ and $\mathbf{b}$ must be perpendicular, so the scalar product gives

$$\mathbf{n} \cdot \mathbf{b} = 0;$$

we can also write the position vector of the general point P as
$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \mathbf{b}$. Hence

$$\begin{aligned} \mathbf{n} \cdot \mathbf{r} &= \mathbf{n} \cdot (\mathbf{a} + \mathbf{b}) \\ &= \mathbf{n} \cdot \mathbf{a} + \mathbf{n} \cdot \mathbf{b} \\ &= \mathbf{n} \cdot \mathbf{a} \qquad . \end{aligned}$$

We can therefore arrive at a **Definition:-**

A plane passing through the point with position vector **a** and perpendicular to the vector **n** has equation

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \qquad .$$

That is, all points whose position vectors satisfy the above equation lie on the plane through A which is perpendicular to **n**. This is **the required vector form of the equation of a plane.**

The above formula does not depend upon on the modulus of **n**. However, if we have a unit normal $\hat{\mathbf{n}}$, then the dot product $\mathbf{a} \cdot \hat{\mathbf{n}}$ represents the perpendicular (*i.e.* shortest) distance from the origin O to the plane. This distance is usually denoted d. Thus for any point P on the plane with position vector **r**, we have

$$\mathbf{r} \cdot \hat{\mathbf{n}} = d \qquad .$$

Put in diagram.

7.3 The Cartesian equation of the plane

From the equation $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ we can find the equation of a plane in Cartesian coordinates.

Question:

Suppose we know that a plane is perpendicular to the vector $\mathbf{n} = (n_1, n_2, n_3)$ and contains the point A with position vector $\mathbf{a} = (a_1, a_2, a_3)$. Which other points $\mathbf{r} = (x, y, z)$ lie on the plane?

Answer:
We know that any point P with position vector $\mathbf{r}$ on the plane satisfies

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

Therefore the general Cartesian equation of this plane is

$$n_1 x + n_2 y + n_3 z = n_1 a_1 + n_2 a_2 + n_3 a_3.$$

Example:
Find the vector equation of the plane which
a) passes through the point with position vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and which is also
b) perpendicular to $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Then give the Cartesian equation of this plane.

Answer:
With $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, the vector equation $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ becomes

$$\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) = (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2 + (-1) + (-4) = -3.$$

Then writing $\mathbf{r} = (x, y, z)$ gives

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) = -3$$

and so the Cartesian equation is

$$x + y - z = -3.$$

Another Example:
(a) Find the equation of the plane which is perpendicular to $\mathbf{i} + \mathbf{k}$ and which passes through the point A with position vector $3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$.
(b) What is the shortest distance from the origin O to the plane?

Answer:
(a) Let P be a general point in the plane, and let P have position vector $\mathbf{r} = (x, y, z)$. As the vector $\overrightarrow{AP}$ lies in the plane, it is at right-angles to the normal $\mathbf{n} = (1, 0, 1)$; thus we have $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$, giving the vector equation of the plane

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

where $\mathbf{a} = (3, 2, 5)$. Therefore we have $\mathbf{r} \cdot \mathbf{n} = (x, y, z) \cdot (1, 0, 1) = x + z$; also $\mathbf{a} \cdot \mathbf{n} = (3, 2, 5) \cdot (1, 0, 1) = 3 + 5 = 8$.

So the Cartesian equation of this plane is simply

$$x + z = 8.$$

(b) We find the distance d from O to the plane by dividing the vector equation by $|\mathbf{n}| = \sqrt{1^2 + 1^2} = \sqrt{2}$, and thus re-writing the equation as $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = d$, where $\hat{\mathbf{n}}$ is a <u>unit</u> normal to the plane, i.e.

$$\hat{\mathbf{n}} = \mathbf{n}/|\mathbf{n}| = \frac{1}{\sqrt{2}}(1, 0, 1).$$

Then

$$d = \mathbf{a} \cdot \hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(3 + 5) = \frac{8}{\sqrt{2}} = 4\sqrt{2}.$$

8. Further 3-D geometry

8.1 Intersection of lines

In $2-D$, two lines are either parallel or they intersect at exactly one point in space.

In $3-D$, two lines can be (i) parallel, be (ii) non-parallel and intersect at exactly one point in space or be (iii) non-parallel and non-intersecting.

(If either of the first two applies, then there must be some plane which contains both lines.)

Question:

Does the line through the points $A = (-1, 2, 3)$ and $B = (2, 4, -1)$ intersect the line through the points $C = (5, -2, 2)$ and $D = (3, 2, 0)$?

Answer:

Let $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ denote the position vectors of A, B, C and D, respectively.

A point P with position vector $\underline{r}$ will lie on the line through A and B if, for some l,

$$\underline{r} = \underline{a} + l(\underline{b} - \underline{a}) \tag{1}$$

and will lie on the line through C and D if, for some m,

$$\underline{r} = \underline{c} + m(\underline{d} - \underline{c}). \tag{2}$$

If the two lines intersect, there will be a point P common to both lines and so we must be able to find values of l and m such that the two position vectors in (1) and (2) are the same (and this means *they must have the same coordinates*).

The way we do this is to find l and m which makes $\underline{r}$ have the same x and y coordinates in the two equations (1) and (2); once this is done we compare the z coordinates.

If the two equations (1) and (2) are both true we have

$$\begin{aligned} \underline{a} + l(\underline{b} - \underline{a}) &= \underline{c} + m(\underline{d} - \underline{c}) \qquad \text{which means} \\ (-1, 2, 3) + l(3, 2, -4) &= (5, -2, 2) + m(-2, 4, -2) \end{aligned}$$

Equating the x and y coordinates gives

$$\begin{aligned} -1 + 3l &= 5 - 2m \qquad \text{and} \\ 2 + 2l &= -2 + 4m, \end{aligned}$$

which we rearrange to give a pair of simultaneous equations

$$3l + 2m = 6 \tag{3}$$

$$2l - 4m = -4. \tag{4}$$

Multiply equation (3) by 2 and add equation (4); this gives $6l + 2l = 12 - 4$, *i.e.* $8l = 8$ and so $l = 1$; when we use this value in either (3) or (4) we find $m = 3/2$.

Now we compare the z coordinates. On the line through A and B we have $z = 3 - 4l$, which gives $z = -1$ at the point we have found; on the other line we have $z = 2 - 2m$ in general, which again gives $z = -1$ at the point we have found.

As the values of z we have found are the same, we deduce that the lines do intersect.

Another Question:

The line L passes through the point A with coordinates $(2, -1, 3)$ in the direction $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + 6\mathbf{k}$.

The line L' passes through the point B with coordinates $(6, -1, 4)$ in the direction $\mathbf{v} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$.

Do the lines intersect each other? If so, where?

Solution:

The equation of L is

$$\mathbf{r} = (2, -1, 3) + l(3, -1, 6)$$

and the equation of L' is

$$\mathbf{r} = (6, -1, 4) + m(1, 1, -5).$$

At their intersection we would have

$$\begin{aligned} 2 + 3l &= 6 + m, & (5) \\ -1 - l &= -1 + m \quad \text{and} & (6) \\ 3 + 6l &= 4 - 5m. & (7) \end{aligned}$$

representing 3 equations in 2 unknowns.

From equations (5) and (6) we find $l = 1$ and $m = -1$, giving $x = 5$ and $y = -2$.

Substituting these values into (7) we find that $lhs = rhs = 9$. Hence the equation (7) is satisfied by these values of l and m, and we conclude that L and L' intersect at the point $(5, -2, 9)$.

8.2 Lines intersecting planes

In $3-D$, a line is either parallel to a plane or intersects the plane at exactly one point.

Example:

A plane P is perpendicular to the vector $\mathbf{n} = (1,1,1)$ and contains the point A with position vector $\mathbf{a} = (1,0,-3)$.

A line L is parallel to the direction $\mathbf{d} = (2,-1,3)$ and passes through the point B with position vector $\mathbf{b} = (1,2,3)$.

Find the point where the line L meets the plane P.

Answer:

Let Q, with position vector $\mathbf{q} = (x,y,z)$, denote the point at which the line and plane meet. What do we know about $\mathbf{q}$?

1. It lies on the plane P with normal vector $\mathbf{n}$, and P contains the point A. Therefore $\mathbf{q}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ which means
$$x+y+z = 1+0-3 = -2. \tag{8}$$
2. It also lies on the line L. So there is a value λ such that $\mathbf{q} = \mathbf{b} + \lambda\mathbf{d}$ which means
$$(x,y,z) = (1,2,3) + \lambda(2,-1,3) = (1+2\lambda, 2-\lambda, 3+3\lambda). \tag{9}$$

Therefore equations (8) and (9) together give
$$-2 = x+y+z = (1+2\lambda) + (2-\lambda) + (3+3\lambda) = 6+4\lambda,$$
and so $4\lambda = -8$ which gives $\lambda = -2$.

We now put this value into (9), showing that
$$(x,y,z) = \big(1+2(-2), 2-(-2), 3+3(-2)\big) = (-3,4,-3)$$
is the point at which the plane P and line L meet.

We can now do the same thing more generally:-

Question:

When does a line L, given by $\mathbf{r} = \mathbf{b} + \lambda\mathbf{d} = (b_1, b_2, b_3) + \lambda(d_1, d_2, d_3)$, intersect the plane P which passes through the point $\mathbf{a} = (a_1, a_2, a_3)$ and has normal vector $\mathbf{n} = (n_1, n_2, n_3)$?

Answer:

Suppose as before there is a point of intersection Q with position vector $\mathbf{q} = (x, y, z)$. Then as before we know that:-

1. $\mathbf{q}$ lies on the plane P through $\mathbf{a}$ with normal vector $\mathbf{n}$, and so $\mathbf{q} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, which means
$$n_1x + n_2y + n_3z = n_1a_1 + n_2a_2 + n_3a_3. \tag{10}$$

2. $\mathbf{q}$ also lies on the line L, and so there is a value λ such that $\mathbf{q} = \mathbf{b} + \lambda\mathbf{d}$ which means that
$$(x, y, z) = (b_1 + \lambda d_1, b_2 + \lambda d_2, b_3 + \lambda d_3). \tag{11}$$

Substituting (11) into (10) gives

$$\begin{aligned} n_1(b_1 + \lambda d_1) + n_2(b_2 + \lambda d_2) + n_3(b_3 + \lambda d_3) &= n_1a_1 + n_2a_2 + n_3a_3 \\ (n_1b_1 + n_2b_2 + n_3b_3) + \lambda(n_1d_1 + n_2d_2 + n_3d_3) &= n_1a_1 + n_2a_2 + n_3a_3 \\ \text{which is} \qquad \mathbf{b} \cdot \mathbf{n} + \lambda(\mathbf{d} \cdot \mathbf{n}) &= \mathbf{a} \cdot \mathbf{n}. \end{aligned}$$

Therefore we have an equation of the form

$$B + \lambda D = A. \tag{12}$$

where $A = \mathbf{a} \cdot \mathbf{n}$, $B = \mathbf{b} \cdot \mathbf{n}$ and $D = \mathbf{d} \cdot \mathbf{n}$.

(Notice also that (12) could be achieved more directly by substituting $\mathbf{q} = \mathbf{b} + \lambda\mathbf{d}$ into $\mathbf{q} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.)

Solutions to equation (12)

There are different possibilities for the values of A, B and D, leading to different types of solution.

1. $D \neq 0$

If $D \neq 0$, there is one value of λ which solves equation (12), namely

$$\lambda = \frac{A - B}{D},$$

and there is therefore one point of intersection given by $(x, y, z) = (b_1 + \lambda d_1, b_2 + \lambda d_2, b_3 + \lambda d_3)$.

2. $D = 0$

Notice that $D = \mathbf{d} \cdot \mathbf{n}$ relates the direction $\mathbf{d}$ of the line to the direction $\mathbf{n}$ of the normal to the plane; if $D = 0$ then these directions are perpendicular, and so the line is parallel to the plane. Therefore the condition for the line L to be parallel to the plane P is that $D = \mathbf{d} \cdot \mathbf{n} = 0$ *i.e.* (d_1, d_2, d_3) is perpendicular to (n_1, n_2, n_3).

This case $D = 0$ then gives rise to two possibilities:-

1. $A \neq B$. Then no value of λ solves $B + \lambda D = A$. In this case the line is parallel to the plane, and they do not intersect.

2. $A = B$. Then any value of λ solves $B + \lambda D = A$. In this case the line lies in the plane (which is a special case of being parallel.)

9. The Vector (or cross) Product

9.1 Right-hand screw rule

Consider two vectors **a** and **b** in the plane of the paper (or board). Imagine you are holding a screwdriver in your right hand and screwing a screw into some wood, in the process of which you are turning your hand from **a** to **b**.

As you are turning the screw, it is rotating but also going into the wood, travelling in a direction at right angles to the plane of the paper (or board), and therefore perpendicular to both **a** and **b**; let this direction be given by the unit vector $\hat{\mathbf{e}}$. You can think of $\hat{\mathbf{e}}$ as along the shaft of the screwdriver.

We define the vector product of **a** and **b** as being a vector of magnitude

$$|\mathbf{a}|\,|\mathbf{b}|\,\sin(\theta)$$

in the direction of $\hat{\mathbf{e}}$.

We denote this vector by $\mathbf{a} \times \mathbf{b}$ and read this as "**a** cross **b**". This is the reason why we often refer to this as the "cross product".

Thus the vector product is defined by

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin(\theta)\;\hat{\mathbf{e}}$$

where $\hat{\mathbf{e}}$ is the direction defined by the "right-hand screw rule" above.

9.2 Properties of the vector product

1. As $\mathbf{b} \times \mathbf{a}$ creates a normal vector $\hat{\mathbf{e}}$ in the opposite direction to that of $\mathbf{a} \times \mathbf{b}$ we have

$$\mathbf{b} \times \mathbf{a} = -\,\mathbf{a} \times \mathbf{b}.$$

2. We have $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$, because the angle θ between any vector and itself is zero, which gives $\sin\theta = 0$. [Another way to see the same result is to put $\mathbf{a} = \mathbf{b}$ in Property 1; then we see that

$$\mathbf{a} \times \mathbf{a} = -\mathbf{a} \times \mathbf{a},$$

and so both sides must be $\mathbf{0}$.]

In particular

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

for the unit vectors along the x, y and z directions.

3. The relationships

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

follow from $\sin\frac{\pi}{2} = 1$ and the fact that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is a right-handed set of vectors. It also follows from Property 1 that

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

9.3 General formula for the vector product

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \qquad (\star)\end{aligned}$$

This expression is proved by expanding out the brackets and then repeatedly applying Properties 2 & 3. For example

$$\begin{aligned}a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) &= a_1b_1\mathbf{i}\times\mathbf{i} + a_1b_2\mathbf{i}\times\mathbf{j} + a_1b_3\mathbf{i}\times\mathbf{k} \\ &= a_1b_1\mathbf{0} + a_1b_2\mathbf{k} - a_1b_3\mathbf{j} \quad etc. \qquad (\star\star)\end{aligned}$$

Using the expansion of a 3×3 determinant, we obtain the alternative formula

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \qquad (\star\star\star)$$

Although $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are vectors, this does not affect the expansion of the determinant.

Note: Some of you may already have seen determinants, and may be reminded of their properties in a few weeks' time in another lecture course. However, it is not essential to use the formula $(\star\star\star)$, which is really only a shorthand; you can simply learn to use the formula $(\star)$ instead.

Alternatively you can use $(\star\star)$ three times, doing $a_1\mathbf{i}\times\mathbf{b}$, then $a_2\mathbf{j}\times\mathbf{b}$ and finally $a_3\mathbf{k}\times\mathbf{b}$ and adding the three results together. This is a safer method which you should probably use while you are getting used to vector products, particularly if you haven't learned about determinants.

Example:

Let $\mathbf{a} = (3, -2, 5)$ and $\mathbf{b} = (7, 4, -8)$ then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 5 \\ 7 & 4 & -8 \end{vmatrix}.$$

That is

$$\mathbf{a} \times \mathbf{b} = (16 - 20)\mathbf{i} - (-24 - 35)\mathbf{j} + (12 + 14)\mathbf{k} = (-4, 59, 26).$$

As it is very easy to make a slip in working out the components of the vector $\mathbf{a} \times \mathbf{b}$, it is important to check your calculation before doing anything else.

We can check the result by making sure that it is perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$ we started with, which can be done by looking at the scalar product of $\mathbf{a} \times \mathbf{b}$ with each of $\mathbf{a}$ and $\mathbf{b}$ in turn.

In our present example we have

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= (-4, 59, 26) \cdot (3, -2, 5) \\ &= (-4)(3) + (-59)(2) + (26)(5) \\ &= -12 - 118 + 130 = 0, \end{aligned}$$

which means that $\mathbf{a} \times \mathbf{b}$ is at right-angles to $\mathbf{a}$; and

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} &= (-4, 59, 26) \cdot (7, 4, -8) \\ &= (-4)(7) + (59)(4) + (-26)(8) \\ &= -28 + 236 - 208 = 0 \end{aligned}$$

and so $\mathbf{a} \times \mathbf{b}$ is also at right-angles to $\mathbf{b}$.

9.4 Applications of the vector product

1. The area A of a triangle PQR is given by

$$\begin{aligned} A &= \frac{1}{2}\,(base)\,(height) \\ &= \frac{1}{2}\,(PQ)\,(PR)\,\sin(\angle QPR) \\ &= \frac{1}{2}\,\left|\overrightarrow{PQ}\right|\,\left|\overrightarrow{PR}\right|\,\sin(\angle QPR) \\ &= \frac{1}{2}\,\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| \end{aligned}$$

2. The moment or *torque* of the force $\mathbf{F}$ about a point O is a vector denoted by $\mathbf{M}_O$ which is a measure of the turning effect of that force about that point. It can be shown to be

$$\mathbf{M}_O = \mathbf{r}\times\mathbf{F} \qquad (\star\star\star\star)$$

where $\mathbf{r}$ is a position vector from O to *any* point P on the line of action of $\mathbf{F}$.

Proof

Imagine the plane (= the board or paper) which contains the line OP and the line of action (PQ, say) of the force $\mathbf{F}$, and let these two lines have an angle θ between them.

Then the shortest distance between O and the line PQ is $d = |\mathbf{r}|\,\sin\theta$, and the magnitude of the moment of the force $\mathbf{F}$ is

$$|\mathbf{M}_O| = d\times|\mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}|\,\sin\theta = |\mathbf{r}\times\mathbf{F}|$$

We also use the convention that for rotational motion and for moments (= rotational forces) the *direction* of the vector is that of the *axis* of rotation; then $\mathbf{M}_O$ has the direction into or out of the board (or paper) giving the result $(\star\star\star\star)$.

Examples

1) Find the area of the triangle ABC when the coordinates of the vertices are

$$A(1,2,3), \quad B(4,-3,2), \quad C(8,1,5)$$

Solution

$$\overrightarrow{AB} = (3,-5,-1) \text{ and } \overrightarrow{AC} = (7,-1,2),$$

and so

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= (-10-1, -7-6, -3+35) \\ &= (-11, -13, +32). \end{aligned}$$

Hence the area of ABC is

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{121+169+1024} = \frac{1}{2}\sqrt{1314} = \frac{3}{2}\sqrt{146}$$

2) Now find the equation of the plane $\mathcal{P}$ which contains the same three points.

Solution

As A, B and C lie in the plane, so do the directions $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Their vector product is therefore perpendicular to the plane, and so this is the normal direction $\mathbf{n}$ that we need in the equation of the plane. Hence $\mathbf{n} = (-11, -13, +32)$ and the plane $\mathcal{P}$ (which contains point A) has the equation $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, which is

$$-11x - 13y + 32z = (1,2,3) \cdot (-11,-13,+32) = -11 - 26 + 96 = +59.$$

Exercise for you: check that you get the same result if you use B or C to work out the number on the right-hand side.

Example:

Given a vector $\mathbf{a}$, find another vector $\mathbf{b}$ that is perpendicular to it.

For instance, let us consider $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. The single condition we need to satisfy is

$$\mathbf{a} \cdot \mathbf{b} = 0;$$

equivalently this says that the point with position vector $\mathbf{r} = \mathbf{b}$ must lie on a plane which is perpendicular to $\mathbf{a}$ and which passes through the origin. This obviously gives us a huge amount of choice - what scientists call "two degrees of freedom".

If we write $\mathbf{b}$ in components as $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ this condition becomes

$$2b_1 - 3b_2 + 4b_3 = 0,$$

and all we have to do is to spot <u>one</u> solution.

Because of all the choice, we can opt to look for solutions which have a zero in one component; *e.g.* if $b_2 = 0$ then $b_1 + 2b_3 = 0$ so we see one solution is $b_1 = 2, b_3 = -1$, giving $\mathbf{b} = 2\mathbf{i} + 0\mathbf{j} - 1\mathbf{k} = 2\mathbf{i} - \mathbf{k}$.

<u>Another</u> way to do this is to remember that the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$; so whatever $\mathbf{v}$ we choose the vector $\mathbf{a} \times \mathbf{v}$ is perpendicular to $\mathbf{a}$ (and so $\mathbf{b} = \mathbf{a} \times \mathbf{v}$ will solve our problem).

We could therefore choose, *for example*, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$, so that

$$\mathbf{b} = \mathbf{a} \times \mathbf{v} = -8\mathbf{i} + 20\mathbf{j} + 19\mathbf{k};$$

if we want a solution which involves the <u>easiest working out</u> then a choice such as $\mathbf{v} = \mathbf{j}$ (giving $\mathbf{b} = -4\mathbf{i} + 2\mathbf{k}$) would suffice.

Another Example

Find the moment, about the point $P(2, -2, 1)$, of the force $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ which acts at the point $Q(14, -3, 6)$.

Solution

We are interested in a moment <u>about P</u> so we need to put the relative position $\mathbf{r} = \overrightarrow{PQ}$ into the result $(\star\star\star\star)$.

In the example $\overrightarrow{PQ} = (12, -1, 5) = 12\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, which gives the moment as

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (12\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}).$$

Now this cross product can be written as

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -1 & 5 \\ 3 & 4 & 5 \end{vmatrix},$$

and evaluated as $-25\mathbf{i} - 45\mathbf{j} + 51\mathbf{k}$, and so

$$\mathbf{M}_O = (-25, -45, +51)$$

(in appropriate units).

N.B. If we have worked out the determinant correctly, the result $\mathbf{M}_O$ *should be perpendicular to* **both** *the vectors* $\overrightarrow{PQ}$ *and* $\mathbf{F}$*; check this using the dot product.*

9.5 Yet more 3-D geometry

Now that we have the equations of a line and a plane, and we know how to deal with the intersection of a line with a line or a plane, there are some more geometric questions we can ask:-

a) What happens when two planes intersect?
b) What is the shortest distance from a general point to a plane?
c) What is the shortest distance between two non-intersecting lines?
d) What is the shortest distance from a general point to a line?

a) We start with two planes, given by $\mathbf{r}\cdot\mathbf{n}_1 = \mathbf{a}\cdot\mathbf{n}_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = \mathbf{b}\cdot\mathbf{n}_2$; we see that the respective normals to the two planes are $\mathbf{n}_1$ and $\mathbf{n}_2$.

The two planes will intersect unless they are parallel, which is unless $\mathbf{n}_1$ is parallel to $\mathbf{n}_2$; that is unless $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$.

So we consider $\mathbf{n}_1 \times \mathbf{n}_2 \neq \mathbf{0}$, which means the planes do intersect, and a little thought shows the intersection to be a straight line. Let $\mathbf{u}$ to be the direction of that line. Then as $\mathbf{u}$ lies in each plane, it must be perpendicular to each normal, and the only direction that is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$ is $\mathbf{n}_1 \times \mathbf{n}_2$, which gives

$$\mathbf{u} = \mathbf{n}_1 \times \mathbf{n}_2$$

(or any scalar multiple of this).

All that is needed now is to find one point $\mathbf{c}$ on the line, and therefore on both planes. It can be shown that the point given by

$$|\mathbf{u}|^2\mathbf{c} = (\mathbf{a}\cdot\mathbf{n}_1)\,\mathbf{n}_2 \times \mathbf{u} - (\mathbf{b}\cdot\mathbf{n}_2)\,\mathbf{n}_1 \times \mathbf{u}$$

satisfies both the equations $\mathbf{c}\cdot\mathbf{n}_1 = \mathbf{a}\cdot\mathbf{n}_1$ and $\mathbf{c}\cdot\mathbf{n}_2 = \mathbf{b}\cdot\mathbf{n}_2$, and so does indeed lie on both planes. (To show this, use the results

$$\mathbf{n}_1\cdot(\mathbf{n}_2 \times \mathbf{u}) = (\mathbf{n}_1 \times \mathbf{n}_2)\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 = -\mathbf{n}_2\cdot(\mathbf{n}_1 \times \mathbf{u})$$

which rely on the properties of a Scalar Triple Product - see later.)

b) <u>The shortest distance from a general point to a plane.</u>

We suppose that the plane is given by

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}, \qquad \text{or} \quad \mathbf{r}\cdot\hat{\mathbf{n}} = \mathbf{a}\cdot\hat{\mathbf{n}}$$

and that we want to find the (shortest) distance to this plane from the point $\mathbf{r}_0$.

We achieve this by 'moving our origin to $\mathbf{r}_0$', which means writing each position relative to $\mathbf{r}_0$; this means that each position vector changes by $-\mathbf{r}_0$. (To see this, draw a triangle with the original origin, $\mathbf{r}_0$ and $\mathbf{r}$.) Then the equation of the plane is re-written

$$(\mathbf{r}-\mathbf{r}_0)\cdot\hat{\mathbf{n}} = (\mathbf{a}-\mathbf{r}_0)\cdot\hat{\mathbf{n}}$$

and we know that the right-hand side is the shortest distance from the (new) origin to the plane, which is what we require.

Thus the shortest distance from the point $\mathbf{r}_0$ to the plane given by $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ is

$$(\mathbf{a}-\mathbf{r}_0)\cdot\hat{\mathbf{n}} = (\mathbf{a}-\mathbf{r}_0)\cdot\mathbf{n}/|\mathbf{n}| \quad .$$

If we go back to section 8.2, we know that in general a line will always intersect a plane unless they are parallel ($D=0$); the question of the distance of a line from a plane therefore makes no sense <u>unless</u> they are parallel, in which case the distance between the line and the plane is just the distance between the plane and any one point on the line.

c) The shortest distance between two non-intersecting lines.

Suppose the two non-intersecting lines are L_1 and L_2, given by

$$\mathbf{r}_1 = \mathbf{a} + s\mathbf{u} \qquad \text{and} \qquad \mathbf{r}_2 = \mathbf{b} + t\mathbf{v}$$

respectively, where s and t are real parameters.

Imagine the lines L_1 and L_2 as infinite rigid rods of metal, and let there be a piece of strong elastic between them which is free to slide along each rod. When released, the elastic will move until it is at its shortest; once it is, it must be perpendicular to each rod, and any movement of one or other end will be trying to stretch the elastic.

Hence the shortest $\mathbf{r}_1 - \mathbf{r}_2$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, and so parallel to $\mathbf{u} \times \mathbf{v}$. Thus

$$\mathbf{a} + s\mathbf{u} - \mathbf{b} - t\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_2 = k\ \mathbf{u} \times \mathbf{v} \qquad \text{for some } k.$$

We now use the fact that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$ (again using the properties of a Scalar Triple Product - see later) to give

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v}) = k\ (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = k\ |\mathbf{u} \times \mathbf{v}|^2;$$

this gives the value of the constant k. So the required shortest distance is

$$|\mathbf{r}_1 - \mathbf{r}_2|_{min} = |k|\ |\mathbf{u} \times \mathbf{v}| = \frac{|(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.$$

Notes:

1. If the two lines do actually intersect, this distance is zero.

2. If the two lines are parallel then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and the method above will not work. As the required distance is the component of $(\mathbf{b} - \mathbf{a})$ perpendicular to $\mathbf{u}$ in this case, a little thought shows that this is

$$|\mathbf{r}_1 - \mathbf{r}_2|_{min} = \frac{|(\mathbf{a} - \mathbf{b}) \times \mathbf{u}|}{|\mathbf{u}|}.$$

d) The shortest distance from a general point to a line.

Suppose the straight line L under consideration is $\mathbf{r} = \mathbf{a} + s\mathbf{u}$, where s is a real parameter, and we wish to find the distance to the line from the point with position vector $\mathbf{b}$.

The easiest way to derive the correct formula for the shortest distance is to draw a straight line through $\mathbf{b}$ parallel to L; we then arrive at exactly the situation at the end of the previous page, and so the shortest distance is

$$|\mathbf{r} - \mathbf{b}|_{min} = \frac{|(\mathbf{a} - \mathbf{b}) \times \mathbf{u}|}{|\mathbf{u}|}.$$

Note that this can be zero, which occurs if $(\mathbf{a} - \mathbf{b}) \times \mathbf{u} = \mathbf{0}$. However this only happens if $(\mathbf{a} - \mathbf{b})$ is parallel to $\mathbf{u}$, which means that $\mathbf{b}$ must be $\mathbf{a} + m\mathbf{u}$ for some value of m, and so $\mathbf{b}$ is indeed on the line L (and therefore no distance away from it).

We now detail some material which will not be examined in this course, but may be of interest and of use for the future.

Triple Products - non-examinable

If we start with two vectors $\mathbf{a}$ and $\mathbf{b}$, we can form the vector product $\mathbf{a} \times \mathbf{b}$ and, as this is a vector, we can then go on to form a scalar or vector product of this vector with yet another vector $\mathbf{c}$.

There are therefore two 'triple products' to consider:-

Scalar Triple Product

For this we form the *scalar* product of $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$, therefore written $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, and ask whether this number has any special properties.

Vector Triple Product

For this we form the *vector* product of $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$, therefore written $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, and see how the direction of this vector is related to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

Scalar Triple Product

It can be shown that the scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is the volume of the *parallelepiped* with sides parallel to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

To show this result, recall that $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{e}}$ in our usual notation, where $\hat{\mathbf{e}}$ is the unit vector pointing perpendicular to $\mathbf{a}$ and to $\mathbf{b}$ and following the right-hand rule.

We also know that $|\mathbf{a}||\mathbf{b}| \sin\theta$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.

Now imagine the parallelepiped as a solid object (like a crystal), and place it so that the edges $\mathbf{a}$ and $\mathbf{b}$ of one face are on the table; the formula for the volume is then

Volume = 'Base area' times 'height'
= 'Area of the parallelogram face on the table' times by 'the height'.

Also, as $\hat{\mathbf{e}}$ is perpendicular to the table, the height is given by $\hat{\mathbf{e}} \cdot \mathbf{c}$. We now put all these results together so that

$$\text{Volume} = (|\mathbf{a}||\mathbf{b}| \sin\theta)(\hat{\mathbf{e}} \cdot \mathbf{c}) = (|\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{e}}) \cdot \mathbf{c} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \quad .$$

Note: It can happen that, when you form $\mathbf{a} \times \mathbf{b}$, the vector $\hat{\mathbf{e}}$ points in the *opposite* direction to $\mathbf{c}$; in that case the 'height' $\hat{\mathbf{e}} \cdot \mathbf{c}$ is negative and so is the 'Volume'.

This only arises if the three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form a left-handed set. This is not a problem, as then these vectors in the order $\mathbf{b}, \mathbf{a}$ and $\mathbf{c}$ will form a right-handed set and the Volume $\big(= (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}\big)$ will be positive as is usually required.

Scalar Triple Product - continued

Once we have the formula

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

for the scalar triple product as the volume of the parallelepiped with sides parallel to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we can easily see that it stays the same if we cyclically switch the labelling on the three vectors. That is, if we re-name $\mathbf{a}$ as $\mathbf{b}$, $\mathbf{b}$ as $\mathbf{c}$ and $\mathbf{c}$ as $\mathbf{a}$, it is still the same parallelepiped and therefore has the same volume. But with this re-naming, the volume becomes

$$(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$$

instead. Similarly, we could re-name again and see that this volume must also be

$$(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.$$

Now recall that the scalar product is commutative and so (for instance) the top result could instead be written

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

and if we compare these last two results we see that all that has happened is that "the dot and the cross have swapped places".

A little thought shows that there are two further expressions for the volume, giving altogether

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}).$$

Because all these six expressions give the same result, we no longer need to write in the brackets and we can "interchange the dot and the cross" if we want. (Even without the brackets, the cross product must be done first and *then* the dot product, as the other order would make no sense.)

Scalar Triple Product - continued again

As already stated, we can write a vector product as a determinant; for example

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \qquad (\star\star\star)$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

Hence "dotting $(\star\star\star)$ with $\mathbf{c} = (c_1, c_2, c_3)$" gives

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \qquad (\star\star\star\star\star)$$

It is now possible to see the symmetry properties on the previous page also as properties of the determinant.

Vector Triple Product

Once we have formed the vector $\mathbf{a} \times \mathbf{b}$, we can then 'cross again' to form the Vector Triple Product

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

Can we say anything about its direction, and is there another way to express this Vector Triple Product in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$?

We know that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$; hence the Vector Triple Product $\mathbf{v} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to both $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$.

Therefore $\mathbf{v}$ is perpendicular to a vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$; this means $\mathbf{v}$ must lie in the same plane as $\mathbf{a}$ and $\mathbf{b}$ and so can be expressed as

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{for some} \quad \alpha, \beta.$$

Now recall that $\mathbf{v}$ is also perpendicular to $\mathbf{c}$, so that $\mathbf{v} \cdot \mathbf{c} = 0$; this gives $\alpha(\mathbf{a} \cdot \mathbf{c}) + \beta(\mathbf{b} \cdot \mathbf{c}) = 0$, which means we can write

$\beta = (\mathbf{a} \cdot \mathbf{c})\gamma$ and $\alpha = -(\mathbf{b} \cdot \mathbf{c})\gamma$ for some γ, and so

$$\mathbf{v} = \gamma\Big\{(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}\Big\} \qquad \text{for some } \gamma.$$

By considering special cases of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ it is possible to show that $\gamma \equiv 1$, which leads to the important result that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \quad .$$

(Unfortunately for you, at some stage you will have to learn this result!)