Generic Uniqueness in Polarization Tomography

Sean Holman

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Sean Holman

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examining committee have been made.

Chair of the Supervisory Committee:

Gunther Uhlmann

Reading Committee:

Gunther Uhlmann

Robin Graham

Yu Yuan

Date:

University of Washington

Abstract

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Sean Holman

Chair of the Supervisory Committee:
Professor Gunther Uhlmann
Department of Mathematics

The problem of polarization tomography is considered on a Riemannian manifold. This problem comes from the physical problem of recovering the anisotropic part of the dielectric permittivity tensor of a quasi-isotropic medium from polarization measurements made around the boundary, but is more general. In greater than three dimensions local uniqueness and stability are established for generic background metrics, and near generic tensor fields through the study of a related linear inverse problem. The same results are established on a natural subspace of tensor fields in dimension three.

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DEDICATION

Dedicated to Elizabeth and Katie.

Chapter 1

INTRODUCTION

Suppose we have high-frequency monochromatic light passing through an anisotropic medium. In general, light emerging from such a medium will have a different polarization from that entering, and by measuring the polarization of the emerging light we might hope to determine some information about the anisotropy. This inverse problem is known as *polarization tomography* ([15], [18], [20]). There exists a significant amount of work in the optics literature on the practical aspects of this problem, and in particular much work has been done in the context of the photoelastic effect, where one well known technique of inversion is *integrated photoelasticity* ([1], [2], [3], [8], [7]). In the case that the anisotropy is sufficiently weak we may hope to apply the methods of geometric optics to reduce the problem to examining the behavior along "rays," and indeed study how the polarization evolves as light travels along a given ray. To model such a weak anisotropy we assume that the medium is not magnetic, and that the dieletric permittivity in the interior is given by

$$\epsilon^j_l(x) = \epsilon(x)\delta^j_l + \frac{1}{k}\chi^j_l(x). \tag{1.1}$$

Here k is the wave number, and because of the presence of the factor $1/k$ in front of the anisotropic part χ^j_l the equation of the zero approximation of geometric optics is the same as that of the isotropic medium $\epsilon^j_l = \epsilon(x)\delta^j_l$. Such a medium is called *quasi-isotropic* and was originally proposed for study by Kravstov ([12], [13]). Physical media with sufficiently weak anisotropies such as plasmas and weakly stressed elastic media may be well modeled by quasi-isotropic media (see [13]). In such physical media χ^j_l may be complex valued, but is Hermitian.

The inverse problem of determining χ^j_l, the anisotropic part of ϵ^j_l from polarization measurements of light with wave number k on the boundary of a given region $\Omega \subset \mathbb{R}^3$ containing a quasi-isotropic medium is considered in [18] and [20]. The same problem with phase information is also considered in both [15] and [18]. Through the method of geometric optics this problem, with phase information, is changed to the geometric problem on a Riemannian manifold, possibly of dimension greater than 3, described below, and it is this geometric problem that we will consider in the majority of the present work. For details of this conversion, and also further details of all the objects defined below, see [18]. The *polarization vector* η introduced below corresponds physically to the zero approximation electrical field normalized to unit length in the background isotropic metric $\epsilon(x)e$ where e is the Euclidean metric. It is measurements of this vector that we will consider here, although it should be noted that this vector actually contains both phase and polarization information.

The following theorem is the main result of the present work.

Theorem 1 *Assume that (M, g) is a real-analytic simple manifold of dimension greater than 3 with real-analytic metric g. If $\hat{f} \in \tau_1^1(M)$ is real-analytic, then there exists an $\epsilon > 0$ such that whenever $g' \in S_2(M)$ is another metric on M and f_1, $f_2 \in \tau_1^1(M)$ are such that*

$$\|g - g'\|_{C^4 S_2(M)} < \epsilon, \quad \text{and} \quad \|\hat{f} - f_i\|_{C^3 \tau_1^1(M)} < \epsilon \quad \text{for } i = 1 \text{ and } 2,$$

if the polarization data of f_1 and f_2 with respect to the metric g' are the same then $f_1 = f_2$. Furthermore, there is a stability estimate for such f_1 and f_2

$$\|f_1 - f_2\|_{L^2 \tau_1^1(M)} \leq C \|U_2' - U_1'\|_{H^1 \beta_1^1((\partial_+ \Omega^{\mathbb{R}})' M)}$$

for some constant $C > 0$. If the dimension is 3, then the statement of local injectivity still holds if we also assume that $\mathrm{d}_\beta(f_1 - f_2) = 0$ and that $f_1 - f_2$ satisfies the tangential boundary condition with respect to the metric g. The stability estimate also holds if f_1 and f_2 are further restricted to have support within a given compact set $K \Subset M$ where the constant C in (4.16) may then depend on the set K.

Roughly speaking this theorem says that there is local uniqueness and stability for this inverse problem near a generic set of tensor fields in a natural subspace of L^2, and for a generic set of background metrics including real-analytic ones. Of course some of the terms in this statement have not been defined yet. The *polarization data* of the tensor fields f_1 and f_2 is defined in section 1.2, and corresponds to polarization and phase measurements of rays passing through a medium with anisotropy corresponding to either f_1 or f_2 respectively. The kernel of the differential operator d_β, which is defined in section 1.3, gives the "natural subspace of L^2."

In section 1.1 we will briefly review the problem and results in the case where the background metric is Euclidean (when $\epsilon(x) = 1$) and we work on a region of $\mathbb{R}^3$. Section 1.2 then introduces the more general problem on a Riemannian manifold that will be the subject of the rest of the work, and discusses the main results in that case. In section 1.3 we discuss a natural non-uniqueness in the inverse problem that occurs in dimension 3. Finally in section 1.4 we derive the main identity which provides the linearization that we will use to analyze the general problem.

1.1 Polarization tomography in the Euclidean case

If in (1.1) we assume that $\epsilon(x) = 1$ is constant, then we are in the case where the background metric is Euclidean, and the rays, or geodesics, are simply straight lines. In this case we can consider the evolution of the length normalized complex electrical field vector in the zero approximation of geometrical optics along each ray. As mentioned above, we will call this the *polarization vector*, and will consistently refer to it with the notation η. If we choose Cartesian coordinates on $\mathbb{R}^3$ so that the ray in question is given by $\{(x^1, x^2, x^3) \in \mathbb{R}^3 \,|\, x^1 = a,\, x^2 = b\}$ where a and b are constants, then, as shown in [18], the vector $\eta(x^3)$ is always perpendicular to the ray, and evolves according to the system of differential equations

$$\begin{pmatrix} \frac{\partial \eta^1}{\partial x^3} \\ \frac{\partial \eta^2}{\partial x^3} \end{pmatrix} = \frac{i}{2} \begin{pmatrix} \chi^1_1(a,b,x^3)\,\eta^1(x^3) + \chi^1_2(a,b,x^3)\,\eta^2(x^3) \\ \chi^2_1(a,b,x^3)\,\eta^1(x^3) + \chi^2_2(a,b,x^3)\,\eta^2(x^3) \end{pmatrix}.$$

This equation may also be written in a simpler form using the notation $P_{e_3}\chi = \pi_{e_3} \circ \chi \circ \pi_{e_3}$ where π_{e_3} is the orthogonal projection onto the plane perpendicular to the x^3 axis. With this the previous equation becomes

$$\frac{\partial \eta}{\partial x^3}(x^3) = (P_{e_3}\chi(a,b,x^3))\,\eta(x^3). \tag{1.2}$$

It is in this form that we present the equation in the more general case below (see (1.6)).

The physical field vector (at least in the geometric optics approximation) at a given point along the ray is actually given by

$$\xi(t, x^3) = \mathrm{Re}\left[\eta(x^3) e^{i(kx^3 - \omega t + \phi_0)}\right]$$

where $\phi_0 \in \mathbb{R}$ is arbitrarily chosen to set the initial phase. The tip of the vector $\xi(t)$ traverses the so called *polarization ellipse* as time progresses. Measurement of the physical parameters of this ellipse together with the phase ϕ_0 precisely gives the vector $\eta(x^3)$, and we assume that we can make these measurements. Thus, if χ is compactly supported in $\mathbb{R}^3$, then the inverse problem is to recover χ based on knowing the solution of (1.2) along any line and for any initial data outside the support of χ, evaluated at any point after the ray has passed through this support.

This inverse problem is nonlinear, and so we would like to study a linearization. In order to do this let us consider the fundamental matrix for (1.2) along any line. For $\xi \in \mathbb{S}^2$ we will write $U(x, \xi)$ for the solution of

$$\xi \cdot \nabla_x U(x, \xi) = \frac{i}{2} \left(P_\xi \chi(x)\right) U(x, \xi) \tag{1.3}$$

and $U(x, \xi) = \mathrm{Id}$ for $x \cdot \xi$ sufficiently negative so that x is not in the support of χ. As above, $P_\xi \chi = \pi_\xi \circ \chi \circ \pi_\xi$ where π_ξ is the orthogonal projection onto the plane perpendicular to ξ. This matrix $U(x, \xi)$ is indeed the fundamental matrix for (1.2) along each ray. By integrating (1.3) we have

$$\int_{-\infty}^{t} \frac{i}{2} \left(P_\xi \chi(x + s\xi)\right) U(x + s\xi, \xi) \,\mathrm{d}s = U(x + t\xi, \xi) - \mathrm{Id}. \tag{1.4}$$

For t sufficiently large, the right hand side of (1.4) is precisely the data of our inverse problem. We can rewrite (1.4) as

$$\int_{-\infty}^{t} \frac{i}{2} \left(P_\xi \chi(x + s\xi)\right) \left(U(x + s\xi, \xi) - \mathrm{Id}\right) \mathrm{d}s + \int_{-\infty}^{t} \frac{i}{2} \left(P_\xi \chi(x + s\xi)\right) \mathrm{d}s = U(x + t\xi, \xi) - \mathrm{Id}.$$

If we assume a priori that χ is sufficiently small, then it is possible to show that $\|U - \mathrm{Id}\| \approx \|\chi\|$, and so the previous identity looks like

$$\int_{-\infty}^{t} \frac{i}{2} \left(P_\xi \chi(x + s\xi)\right) \mathrm{d}s + \mathcal{O}(\|\chi\|^2) = U(x + t\xi, \xi) - \mathrm{Id}.$$

Of course I am being imprecise here, but the details will be given below. From the previous formula we see that a linearization of the inverse problem is the inversion of the integral transform which takes χ to

$$J[\chi](x,\xi) = \int_{-\infty}^{\infty} \frac{i}{2} \left(P_\xi \chi(x+s\xi)\right) \mathrm{d}s. \tag{1.5}$$

This linearization is studied in [18] and [15] where it is referred to as the *transverse ray transform*. The problem may be studied in dimensions greater than 3 as well, and in such dimensions the transform is invertible, and there is a stability estimate. In dimension 3 the transform has a kernel which may be explicitly identified (see [15]), but is invertible on the orthogonal complement of this kernel, and there is a stability estimate there. Furthermore, local uniqueness and stability for the nonlinear problem have been established previously in [15] (actually the results in that work are more general- see below). In the present work we will study a similar linearization, but we will not just linearize the problem near the zero tensor field as we have done here. In fact we will study the linearized problem near arbitrary tensor fields on generic simple Riemannian manifolds, and establish stability in those cases (Theorems 12 and 13) and uniqueness when the linearization is done near a generic class of fields (Theorem 15). Here generic means sufficiently close to a real-analytic field or metric. Using these results we will also show local uniqueness in the nonlinear problem for χ a priori sufficiently close to some real-analytic reference tensor field.

We now continue to present the same problem in the more general context of a Riemannian manifold.

1.2 Polarization tomography on a Riemannian manifold

The problem of polarization tomography that we will consider here is an inverse problem defined on a certain class of Riemannian manifolds called *compact non-trapping.*

Definition 1 *A compact, connected Riemannian manifold (M, g) with nonempty, strictly convex boundary such that no geodesic has infinite length will be called compact non-trapping, or simply CNT.*

The quantities involved in the physical problem of polarization tomography generally are complex valued, and so we will be considering the complexifications of most of the various vector bundles discussed below. To reduce notation we will omit the customary notation of $\mathbb{C}$ for complexification, and for the remainder of this work it will simply be understood that any real vector bundle is complexified unless a notation such as $T^{\mathbb{R}}M$ is used. Also, the notation $\langle \cdot, \cdot \rangle_g$ will refer to the sesquilinear inner product on TM (which, in accordance with the comments above, is really $T^{\mathbb{C}}M$) defined by

$$\langle \eta, \zeta \rangle_g = \eta^i g_{ij} \overline{\zeta^j}$$

in coordinates.

Let (M, g) be a CNT manifold of dimension $n \geq 3$. Let $i : \partial M \to M$ be the inclusion mapping, and first consider $i^*(T^{\mathbb{R}}M \bigoplus TM)$, the direct sum bundle over ∂M. Define the two subsets of this bundle

$$\Psi_{\pm}(\partial M) = \Big\{ (\xi, \eta) \in i^*(T^{\mathbb{R}}M \bigoplus TM) \ : \ \|\xi\|_g = 1 \ , \ \pm\langle \xi, \nu \rangle_g \geq 0 \Big\}$$

where ν is the outward pointing normal vector to ∂M. Now, take any $f \in \tau_1^1(M)$, the space of smooth (1,1) tensor fields on M. The problem we will consider is that of determining f given g and the map $\mathcal{F}[f] : \Psi_-(\partial M) \to \Psi_+(\partial M)$ defined as follows. For $(\xi, \eta_0) \in \Psi_-(\partial M)$ let γ_ξ be the unit speed geodesic in (M, g) with initial data ξ. Assume this geodesic has length l so that $\gamma_\xi(l) \in \partial M$. Now consider the vector field η along γ_ξ that solves the initial value problem

$$\frac{D\eta}{\mathrm{ds}} = (P_{\dot{\gamma}_\xi} f)\eta \quad , \quad \eta(\gamma_\xi(0)) = \eta_0. \tag{1.6}$$

Here $D\eta/\mathrm{ds}$ is the covariant derivative of η along γ_ξ, and $P_{\dot{\gamma}_\xi} f = \pi_{\dot{\gamma}_\xi} f \pi_{\dot{\gamma}_\xi}$ where $\pi_{\dot{\gamma}_\xi}$ is the orthogonal projection onto the subspace of $T_{\gamma_\xi}M$ perpendicular to $\dot{\gamma}_\xi$. This formula corresponds to (1.2) in the Euclidean case. Now we define $\mathcal{F}[f](\xi, \eta_0) = (\dot{\gamma}_\xi(l), \eta(\gamma_\xi(l)))$.

Note that, just as in the Euclidean case, from (1.6) we can see that this inverse problem is nonlinear. In [15] and [18] the authors show that a linearization of the problem is given by the inversion of the so called *transverse ray transformation*. If we choose a geodesic γ of g with length l, and a parallel orthonormal frame $\{e_1(t), \ \dots \ , e_n(t)\}$ along the geodesic γ with $e_n(t) = \dot{\gamma}(t)$, then in the coordinates given by this frame the transverse ray transformation is

$$J[f](\gamma; \xi, \eta) = \int_0^l \sum_{i,j=1}^{n-1} f_{ij}(\gamma(t)) \xi^i \eta^j \ \mathrm{d}t \tag{1.7}$$

where $\xi = \xi^i e_i(t)$ and $\eta = \eta^j e_j(t)$ are vector fields parallel along γ. In the Euclidean case this transform is precisely (1.5). This linearization is actually a special case of the linearized problems that we will consider in this work. In fact, (1.7) in some sense corresponds to our linearization near $f = 0$. Indeed, our method will produce an integral that appears similar to the transverse ray transform with the addition of weight functions depending on f in a nonlinear way. We will then fix these weight functions at particular values, and consider this as a linearization. Using this we will then be able to find results for the nonlinear problem.

When the dimension n is 3, this inverse problem corresponds to the physical problem described above of determining the anisotropic part of the dielectric permittivity of a quasi-isotropic medium from polarization and phase data recorded around the boundary. In this setting the geodesics are considered to be light rays, and the vector field η along γ_ξ given by (1.6) is the polarization vector mentioned above. The tensor f and the anisotropic part of the dielectric permittivity are related by the formula

$$f = \frac{i}{2\epsilon}\chi. \tag{1.8}$$

We imagine that we can set the initial polarization vector, η_0, as any light ray enters the medium, and then measure the polarization vector, $\eta(l)$, as that ray leaves the medium. In [15] uniqueness and stability of this inverse problem are established under curvature assumptions on g and a priori smallness assumptions on f. In the present work we establish local uniqueness and stability results for this inverse problem for generic metrics g, and for f a priori close to an analytic tensor field. An inverse problem corresponding to (1.6) without the projection $P_{\dot\gamma_\xi}$ is also considered in [31].

Next we reformulate the geometric inverse problem presented above. First we must introduce some new notation and terminology.

Definition 2 *A semi-basic (m,n) tensor field on M is a tensor field U on $T^{\mathbb{R}}M$ whose representation in any natural coordinates (x,ξ) is of the form*

$$U = u^{j_1 \ldots j_m}_{i_1 \ldots i_n}(x,\xi)\,\frac{\partial}{\partial \xi^{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{j_m}} \otimes \mathrm{d}x^{i_1} \otimes \ldots \otimes \mathrm{d}x^{i_n}.$$

The bundle of (m,n) semi-basic tensors over $T^{\mathbb{R}}M$ is naturally isomorphic to the pullback bundle $\pi^*(T^m_n M)$ where $\pi : T^{\mathbb{R}}M \to M$ is the projection mapping. Thus an intuitive way to understand semi-basic tensor fields is as tensor fields on M that are also allowed to depend on the fiber variables ξ. We will most commonly use $(1,1)$ semi-basic tensors, which at a given point $(x,\xi) \in T^{\mathbb{R}}M$ may be identified with linear maps on T_xM. The vector space of (m,n) semi-basic tensors at a given point $(x,\xi) \in T^{\mathbb{R}}M$ will be denoted by $(B^m_n)_{(x,\xi)}(T^{\mathbb{R}}M)$, the vector bundle over $T^{\mathbb{R}}M$ of (m,n) semi-basic tensors will be $B^m_n(T^{\mathbb{R}}M)$, and the set of (m,n) semi-basic tensor fields will be $\beta^m_n(T^{\mathbb{R}}M)$. If $\mathcal{G}$ is a submanifold of $T^{\mathbb{R}}M$, then we will denote the set of (m,n) semi-basic tensor fields restricted to $\mathcal{G}$ as $\beta^m_n(\mathcal{G})$, which is the set of smooth sections of the vector bundle $B^m_n(\mathcal{G})$. In particular, we will commonly work with the set of (1,1) semi-basic tensor fields restricted to the unit sphere bundle $\Omega^{\mathbb{R}}M$, which is $\beta^1_1(\Omega^{\mathbb{R}}M)$.

At this point I would like to make a remark on the notation (x,ξ) for points in TM. Unless we are working in a set of local coordinates, ξ is interpreted as the point in TM, and x is merely a label for $\pi(\xi)$ used so that we can avoid having to write $\pi(\xi)$ when we wish to refer to the base point. Thus the notation $(x,\xi) \in TM$ is actually an abuse that means $\xi \in TM$ and $x = \pi(\xi)$. This makes the notation in the next paragraph consistent with that of the previous paragraph. When we work in coordinates we may also write (x,ξ) for a $2n$-tuple where x refers to the n coordinates on the base manifold, and ξ refers to the fiber coordinates.

Given any tensor field $f \in \tau^1_1(M)$ consider the following equation for $U \in \beta^1_1(\Omega^{\mathbb{R}}M \setminus T\partial M)$ where H is differentiation with respect to the geodesic flow (see [18]), $\partial_-\Omega^{\mathbb{R}}M$ is the space of inward pointing unit vectors on ∂M, and E is the identity.

$$HU(\xi) = [(P_\xi f)(x)]U(\xi) \quad \text{on } \Omega^{\mathbb{R}}M \setminus T\partial M, \quad U|_{\partial_-\Omega^{\mathbb{R}}M} = E. \tag{1.9}$$

U can be thought of as the “fundamental matrix” for (1.6). In fact problems (1.6) and (1.9) are related as follows.

$$\eta(\gamma_\xi(t)) = U(\dot\gamma_\xi(t))\, \mathcal{I}^{\gamma_\xi}_{0,t}\, \eta_0. \tag{1.10}$$

Here $\mathcal{I}^{\gamma_\xi}_{0,t}$ is parallel translation along γ_ξ. In the context of (1.9), the inverse problem is to determine f from the metric g and $U|_{\partial_+\Omega^{\mathbb{R}}M}$ (where $\partial_+\Omega^{\mathbb{R}}M$ is the space of outward pointing unit vectors). We will call $U|_{\partial_+\Omega^{\mathbb{R}}M}$ the *polarization data* corresponding to f (recall the use of this term in the statement of Theorem 1). The main problem in which we will be interested is that of establishing uniqueness for this nonlinear inverse problem, or equivalently showing that the map $f \mapsto U|_{\partial_+\Omega^{\mathbb{R}}M}$ is injective. The main results are given in Theorem 1.

Before proceeding further we prove one small lemma which gives a few properties of U.

Lemma 1 *If U is the solution of (1.9) for some $f \in \tau_1^1(M)$, then for every $v \in \Omega^{\mathbb{R}}M \backslash T\partial M$ $U(v) : T_{\pi(v)}M \to T_{\pi(v)}M$ is invertible and*

$$U(v)|_{\mathrm{span}(v)} = Id \quad \textit{and} \quad U(v) : \mathrm{span}(v)^{\perp} \to \mathrm{span}(v)^{\perp}. \tag{1.11}$$

Furthermore, if we assume that f is of the form (1.8) where ϵ is real valued and χ is Hermitian (which is the physical case), then U is unitary.

Proof: Suppose $v = \dot{\gamma}_\xi(t)$, which is always true for some $\xi \in \partial_-\Omega^{\mathbb{R}}M$ and $t \in \mathbb{R}$, and let $\{e_1(s), \dots, e_n(s)\}$ be a parallel set of vector fields along γ_ξ which are orthonormal for every s, and such that $e_n(s) = \dot{\gamma}_\xi(s)$ for every s. It is always possible to find such a set of vector fields by taking a frame at the point $\gamma_\xi(t)$, and then using parallel translation along γ_ξ. If η is a solution of (1.6) and $\eta(s) = \eta^i(s)\, e_i(s)$, then the differential equation in (1.6) becomes the system

$$\frac{\partial \eta^i}{\partial s}(s) = \sum_{j=1}^{n-1} f^i_j(\gamma(s))\eta^j(s) \quad \text{for } i = 1, \dots, n-1$$

and

$$\frac{\partial \eta^n}{\partial s}(s) = 0.$$

Therefore η^n is constant, and so $\eta^n(s) = (\eta_0)^n$, and in light of (1.10) and the uniqueness of solutions to initial value problems the first part of the lemma is proven.

To prove the second statement, assume that f has the form (1.8) with ϵ real valued and χ Hermitian. Then we can easily see that f is skew-Hermitian. Therefore

$$H(U^*U) = U^*[P_\xi f]^*U + U^*[P_\xi f]U = -U^*[P_\xi f]U + U^*[P_\xi f]U = 0,$$

and so $U^*U = E$ everywhere.

□

The last statement of this lemma is actually of little interest in the present work since all of the results found here apply to general $f \in \tau_1^1(M)$.

1.3 Non-uniqueness in the three dimensional case

In dimension $n = 3$ there is a natural non-uniqueness to the inverse problem which we will now describe. This is a reformulated and slightly expanded version of what can be found in [15]. We first of all make the additional assumption that (M, g) is oriented in which case we may define the Hodge star operator $* : \Lambda^k T^*M \to \Lambda^{3-k}T^*M$ from k forms to $3-k$ forms on M (see for example [16]), and also the adjoint of the exterior derivative $\delta = *d*$ where d is the exterior derivative. Loosely speaking, the non-uniqueness now results from the fact that, as we will see below, for any "coexact" tensor field, $f \approx \delta h$, the polarization data of f is entirely determined by the values of the potential h at the boundary. More precisely, we have the following theorem.

Theorem 2 *If a CNT manifold M^3 is orientable and $h_1,\ h_2 \in C^\infty(M)$ are such that $h_1|_{\partial M} = h_2|_{\partial M}$, then the polarization data for $f_i = (\delta(h_i\, \mathrm{d}v_g))^\#$ $(i = 1, 2)$ are the same. The # indicates that the first index of $\delta(h\, \mathrm{d}v_g)$ has been raised, and $\mathrm{d}v_g$ is the Riemannian volume form.*

Remark 1: The orientability hypothesis here arises because of the need to globally define δ. However, it is possible to find a similar non-uniqueness result in the case where M is not orientable by applying this theorem on a three dimensional orientable submanifold of M with boundary.

Proof: Let γ be any unit speed geodesic in M starting at $x \in \partial M$ with length l. As in the proof of Lemma 1, we can choose a set of vector fields $\{\partial_{x_1},\ \partial_{x_2},\ \partial_{x_3}\}$ that is a parallel orthonormal frame along the entire length of γ. These vector fields may not actually come from a single coordinate system, but using cylindrical coordinates along γ we can see that they can be chosen to be coordinate vectors at least locally. If h_i is as in the statement of the theorem for $i = 1$ or 2, then we can calculate $f_i = (\delta(h_i\ \mathrm{d}v_g))^\#$ at points on γ as follows. Letting $h = h_1$ or h_2 we have

$$\begin{aligned}
\delta(h\, \mathrm{d}v_g) &= *d(h) = *(\partial_{x^1} h\, dx^1 + \partial_{x^2} h\, dx^2 + \partial_{x^3} h\, dx^3) \\
&= \partial_{x^1} h\, dx^2 \wedge dx^3 - \partial_{x^2} h\, dx^1 \wedge dx^3 + \partial_{x^3} h\, dx^1 \wedge dx^2.
\end{aligned}$$

Thus at points on γ

$$[P_{\dot\gamma}((\delta(h\,\mathrm{d}v_g))^{\#})] = \frac{\partial_{x^3}h}{2}\,(\partial_{x^1}\otimes dx^2 - \partial_{x^2}\otimes dx^1).$$

Note that here and in the remainder of this work we use the convention that $\mathrm{d}x^i\wedge\mathrm{d}x^j = (\mathrm{d}x^i\otimes\mathrm{d}x^j - \mathrm{d}x^j\otimes\mathrm{d}x^i)/2$. Now take any $\eta_0\in T_xM$ as an initial vector. We must solve the following system of equations along γ

$$\frac{\partial\eta}{\partial t} = \frac{\partial_{x^3}h}{2}\,(\partial_{x^1}\otimes dx^2 - \partial_{x^2}\otimes dx^1)\,\eta \quad , \quad \eta(0)=\eta_0.$$

It is not hard to see that the solution to this system is given with respect to the frame $\{\partial_{x^1},\,\partial_{x^2},\,\partial_{x^3}\}$ by

$$\eta(t) = \begin{pmatrix} \cos((h(\gamma(t))-h(x))/2)\,\eta_0^1 + \sin((h(\gamma(t))-h(x))/2)\,\eta_0^2 \\ \cos((h(\gamma(t))-h(x))/2)\eta_0^2 - \sin((h(\gamma(t))-h(x))/2)\,\eta_0^1 \\ \eta_0^3 \end{pmatrix}.$$

Finally, let $\gamma(l)=y$, and note that $x,\ y\in\partial M$. Then we have

$$\eta(l) = \begin{pmatrix} \cos((h(y)-h(x))/2)\,\eta_0^1 + \sin((h(y)-h(x))/2)\,\eta_0^2 \\ \cos((h(y)-h(x))/2)\eta_0^2 - \sin((h(y)-h(x))/2)\,\eta_0^1 \\ \eta_0^3 \end{pmatrix}.$$

Now we can see that since $h_1|_{\partial M} = h_2|_{\partial M}$, in fact the polarization data for f_1 and f_2 as defined in the statement of the theorem are the same.

□

In view of the previous theorem we make the following definition. A tensor field $f\in\tau_1^1(M)$ will be called *coexact* if $f=(\delta(h\,\mathrm{d}v_g))^{\#}$ for some $h\in C^\infty(M)$. We now consider how to determine a subspace of $\tau_1^1(M)$ that is complementary to the space of coexact tensor fields.

First we define the *tangential component* of $f\in\tau_1^1(M)$ to be the section of the vector bundle $i^*(T_1^1M)$ over ∂M given by $\mathbf{t}f = P_\nu(f)$, and the *normal component* of f to be (with a small abuse of notation) $\mathbf{n}f = f-\mathbf{t}f$. Recall that $i:\partial M\to M$ is the inclusion map, and ν is the outward pointing unit normal to ∂M. The tangential and normal parts of

2-forms can also be defined in a similar way. Indeed, if $h \in \tau_2(M)$ is a $(0,2)$-tensor and ξ, $\eta \in i^*(TM)$, then $\mathbf{t}h(\xi,\eta) = h(\pi_\nu(\xi), \pi_\nu(\eta))$ and $\mathbf{n}h = h - \mathbf{t}h$. With these definitions the following formulas hold for $f \in \tau_1^1(M)$.

$$(\mathbf{t}f)^\flat = \mathbf{t}f^\flat \quad \text{and} \quad (\mathbf{n}f)^\flat = \mathbf{n}f^\flat. \tag{1.12}$$

Now consider the Helmholtz decomposition (see [16]) which says that any $h \in \Lambda^2(M)$ can be uniquely written as the sum of a coexact form with zero normal part, and a closed form. We can identify antisymmetric tensors in $\tau_1^1(M)$ with elements of $\Lambda^2(M)$ through the metric, and so in fact for any $f \in \tau_1^1(M)$ we can decompose f as

$$f = f^s + (\alpha)^\sharp + (*\mathrm{d}\,\beta)^\sharp \tag{1.13}$$

where $f^s = (f + f^t)/2$ is the symmetric part of f, $\alpha \in \Lambda^2(M)$ is closed, $\beta \in C^\infty(M)$, and $*\mathrm{d}\,\beta$ has zero normal component. This last property is equivalent to the property that β is constant on the boundary. The decomposition is also unique up to possibly changing β by a constant, and so if we add the requirement that $\beta|_{\partial M} = 0$, then the decomposition (1.13) is unique. The transpose of f, f^t, used to define f^s is the transpose with respect to the non-sesquilinear inner product corresponding to g. In coordinates this is given by

$$(f^t)_i^j = g_{ik} f_s^k g^{sj}. \tag{1.14}$$

Note that $(*\mathrm{d}\,\beta) = (\delta * \beta)$, and so according to theorem 2 we cannot expect to recover $(*\mathrm{d}\,\beta)^\sharp$ from the polarization data. However, we expect to be able to find at least the normal part of f at the boundary since this part does not depend on $(*\mathrm{d}\,\beta)^\#$. Furthermore, we might expect to be able to recover f fully when the coexact part of f, $(*\mathrm{d}\,\beta)^\sharp$, is zero. In light of this, we now record the condition on f which will guarantee that $*\mathrm{d}\,\beta$ is zero. To this end, let $\mathrm{Alt} : \tau_2(M) \to \Lambda^2(M)$ denote the projection onto the alternating tensor fields, which is given in coordinates by

$$\mathrm{Alt}(f_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j) = \sum_{j>i} (f_{ij} - f_{ji})\,\mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Now define $\mathrm{d}_\beta : \tau_1^1(M) \to C^\infty(M)$ by

$$\mathrm{d}_\beta(f) = *\,\mathrm{d}\,\mathrm{Alt}(f^\flat). \tag{1.15}$$

Since α is closed, from (1.13) we see that

$$\Delta_g \beta = \mathrm{d}_\beta f. \tag{1.16}$$

If we also require that $\beta|_{\partial M} = 0$, then $*\mathrm{d}\,\beta$ will have normal part zero, and so β is given by the solution of the Dirichlet problem corresponding to (1.16). By uniqueness of solutions to the Dirichlet problem it is clear that $*\mathrm{d}\,\beta = 0$ if and only if $\mathrm{d}_\beta f = 0$.

1.4 Main identity

The proofs in this work are based upon an integral identity along the geodesics of (M, g). The technique is inspired by the work of Stefanov and Uhlmann [22], and Uhlmann and Wang [30] on the boundary rigidity problem. Our formula can also be obtained by integrating the identity used by Novikov and Sharafutdinov in [15] along geodesics. The identity is obtained as follows. Let (M, g) be as above and $f_1,\ f_2 \in \tau_1^1(M)$. Let U_1 and U_2 be the solutions to (1.9) corresponding to f_1 and f_2 respectively. Now fix any unit speed geodesic γ between points $x,\ y \in \partial M$. In particular, assume that $\gamma(0) = x$, and $\gamma(l) = y$. Also, let us define $\dot\gamma(t) = \xi(t) \in T_{\gamma(t)}M$. Now the idea is to start with any polarization vector $\eta_0 \in T_x M$, evolve this vector according to f_1 along γ for some amount of time s, and then evolve the resulting vector along γ for some amount of time according to f_2.

Following the strategy outlined above we first note that by (1.10) when we evolve η_0 according to f_1 along γ for time s the resulting vector is

$$U_1(\xi(s))\,\mathcal{I}^\gamma_{0,s}\,\eta_0 \in T_{\gamma(s)}M.$$

Now when we evolve this vector by f_2 along the remainder of γ, once again using (1.10) we obtain

$$U_2(\xi(l))\,\mathcal{I}^\gamma_{s,l}\,U_2^{-1}(\xi(s))\,U_1(\xi(s))\,\mathcal{I}^\gamma_{0,s}\,\eta_0 \in T_y M.$$

Now choose any t_1 and t_2 with $0 \le t_1 \le t_2 \le l$, and let $\zeta \in T_y M$. Then by the fundamental theorem of calculus

$$\begin{aligned}\Big\langle U_2(\xi(l))\Big(\mathcal{I}^\gamma_{t_2,l}\ U_2^{-1}(\xi(t_2))\,U_1(\xi(t_2))\,\mathcal{I}^\gamma_{0,t_2} - \mathcal{I}^\gamma_{t_1,l}\,U_2^{-1}(\xi(t_1))\,U_1(\xi(t_1))\,\mathcal{I}^\gamma_{0,t_1}\Big)\,\eta_0, \zeta\Big\rangle_{g(y)} \\ = \int_{t_1}^{t_2} \frac{\partial}{\partial s}\Big\langle U_2(\xi(l))\,\mathcal{I}^\gamma_{s,l}\,U_2^{-1}(\xi(s))\,U_1(\xi(s))\,\mathcal{I}^\gamma_{0,s}\,\eta_0,\ \zeta\Big\rangle_{g(y)}\,\mathrm{d}s.\end{aligned} \tag{1.17}$$

Now we will evaluate the derivative on the right side of this equation. First we transpose the first two operators to get

$$\frac{\partial}{\partial s}\left\langle U_2(\xi(l))\,\mathcal{I}^{\gamma}_{s,l}\,U_2^{-1}(\xi(s))\;\; U_1(\xi(s))\,\mathcal{I}^{\gamma}_{0,s}\eta_0, \zeta\right\rangle_{g(y)}$$
$$= \frac{\partial}{\partial s}\left\langle U_2^{-1}(\xi(s))\,U_1(\xi(s))\,\mathcal{I}^{\gamma}_{0,s}\,\eta_0,\, \mathcal{I}^{\gamma}_{l,s}\,U_2^{*}(\xi(l))\,\zeta\right\rangle_{g(\gamma(s))}.$$

We simplify this by replacing ζ with $(U_2^*)^{-1}(\xi(l))\,\zeta$, and then use the compatibility of the metric g with the covariant derivative along with (1.9) to obtain

$$\frac{\partial}{\partial s}\left\langle U_2^{-1}\,U_1\,\mathcal{I}^{\gamma}_{0,s}\,\eta_0,\, \mathcal{I}^{\gamma}_{l,s}\,\zeta\right\rangle_{g(\gamma(s))}$$
$$= \left\langle \left(U_2^{-1}\,[P_{\xi(s)}f_1](\gamma(s))\,U_1 - U_2^{-1}\,[P_{\xi(s)}f_2](\gamma(s))\,U_1\right)\,\mathcal{I}^{\gamma}_{0,s}\,\eta_0,\, \mathcal{I}^{\gamma}_{l,s}\,\zeta\right\rangle_{g(\gamma(s))}.$$

Simplifying this last expression and plugging it into (1.17) we obtain

$$\left\langle \left(\mathcal{I}^{\gamma}_{t_2,l}\;U_2^{-1}(\xi(t_2))\,U_1(\xi(t_2))\,\mathcal{I}^{\gamma}_{0,t_2} - \mathcal{I}^{\gamma}_{t_1,l}\,U_2^{-1}(\xi(t_1))\,U_1(\xi(t_1))\,\mathcal{I}^{\gamma}_{0,t_1}\right)\eta_0, \zeta\right\rangle_{g(y)}$$
$$= \int_{t_1}^{t_2}\left\langle U_2^{-1}\left[P_{\xi(s)}\,(f_1 - f_2)\right](\gamma(s))\,U_1\,\mathcal{I}^{\gamma}_{0,s}\,\eta_0,\, \mathcal{I}^{\gamma}_{l,s}\,\zeta\right\rangle_{g(\gamma(s))}\mathrm{d}s. \tag{1.18}$$

Note that the left hand side has changed since we replaced ζ by $(U_2^*)^{-1}(\xi(l))\,\zeta$. Formula (1.18) is the main identity for which we were searching. It's utility becomes more apparent if we consider the special case when $t_1 = 0$ and $t_2 = l$. In this case (1.18) is

$$\left\langle \left(U_2^{-1}(\xi(l))\;\,U_1(\xi(l)) - E\right)\,\mathcal{I}^{\gamma}_{0,l}\,\eta_0, \zeta\right\rangle_{g(y)}$$
$$= \int_0^l\left\langle U_2^{-1}\left[P_{\xi(s)}\,(f_1 - f_2)\right](\gamma(s))\,U_1\,\mathcal{I}^{\gamma}_{0,s}\,\eta_0,\, \mathcal{I}^{\gamma}_{l,s}\,\zeta\right\rangle_{g(\gamma(s))}\mathrm{d}s \tag{1.19}$$

where E is the identity map. Since $\xi(l) \in \partial_+\Omega M$, when f_1 and f_2 have the same polarization data the left hand side of (1.19) vanishes. Thus, when the polarization data are the same we have

$$0 = \int_0^l\left\langle U_2^{-1}(\xi(s))\left[P_{\xi(s)}\,(f_1 - f_2)\right](\gamma(s))\,U_1(\xi(s))\,\mathcal{I}^{\gamma}_{0,s}\,\eta_0,\, \mathcal{I}^{\gamma}_{l,s}\,\zeta\right\rangle_{g(\gamma(s))}\mathrm{d}s. \tag{1.20}$$

We would like to make the integral in (1.20) into a bilinear form on T_xM, and so we replace ζ by $\mathcal{I}^{\gamma}_{0,l}\,\zeta$ where now $\zeta \in T_xM$. This gives

$$0 = \int_0^l\left\langle U_2^{-1}(\xi(s))\left[P_{\xi(s)}\,(f_1 - f_2)\right](\gamma(s))\,U_1(\xi(s))\,\mathcal{I}^{\gamma}_{0,s}\,\eta_0,\, \mathcal{I}^{\gamma}_{0,s}\,\zeta\right\rangle_{g(\gamma(s))}\mathrm{d}s. \tag{1.21}$$

This last formula shows that when the polarization data of f_1 and f_2 are equal, then $f = f_1 - f_2$ is in the kernel of the X-ray transform defined by the right hand side of (1.21) when U_1 and U_2 are fixed as weights. Since it will be useful later we also replace ζ by $\mathcal{I}^{\gamma}_{0,l}\,\zeta$ in (1.18). Doing this we obtain

$$\begin{aligned}&\left\langle \left(\mathcal{I}^{\gamma}_{t_2,0}\, U_2^{-1}(\xi(t_2))\, U_1(\xi(t_2))\, \mathcal{I}^{\gamma}_{0,t_2} - \mathcal{I}^{\gamma}_{t_1,0}\, U_2^{-1}(\xi(t_1))\, U_1(\xi(t_1))\, \mathcal{I}^{\gamma}_{0,t_1}\right) \eta_0, \zeta \right\rangle_{g(x)} \\ &\qquad = \int_{t_1}^{t_2} \left\langle U_2^{-1} \left[P_{\xi(s)}\,(f_1 - f_2)\right](\gamma(s))\, U_1\, \mathcal{I}^{\gamma}_{0,s}\, \eta_0,\, \mathcal{I}^{\gamma}_{0,s}\, \zeta \right\rangle_{g(\gamma(s))} \mathrm{d}s\end{aligned} \tag{1.22}$$

for any η_0 and $\zeta \in T_x M$.

1.5 Outline of the method

Our first step towards the proof of the Theorem 1 will be to fix U_1 and U_2 as some specific pair of semi-basic tensor fields, and then consider the linear transform given by the right hand side of (1.21) acting on $f = f_1 - f_2$. This transform will be written as

$$I_{U_1,U_2}[f].$$

See (2.1). In coordinates $I_{U_1,U_2}[f]$ is a sum of weighted X-ray transforms acting on the components of f, and so in chapter 2 we study in detail such transforms acting only on functions.

In particular, if I_{w_1} and I_{w_2} are two such transforms with weights given respectively by w_1 and w_2, then we consider the normal operator $N_{w_1,w_2} = I_{w_1} \circ I_{w_2}$. We show that on a simple manifold it is a pseudodifferential operator (ΨDO) of order -1, and also calculate the full symbol of $\mathcal{N}_{w_1,w_2}$ for arbitrary weights (see Theorem 6 and corollary 1).

In chapter 3 we return to analyzing I_{U_1,U_2} by using the results of chapter 2. As in the case of the scalar X-ray transform, we may introduce a normal operator $\mathcal{N}_{U_1,U_2} = I^*_{U_1,U_2} \circ I_{U_1,U_2}$ and show that on a simple manifold $\mathcal{N}_{U_1,U_2}$ is a ΨDO of order -1. In fact, on a simple manifold of dimension greater than 3 this operator is an elliptic ΨDO (see Theorem 9). In that case we apply a left parametrix to $\mathcal{N}_{U_1,U_2}$ to show that it has finite dimensional kernel, and prove a stability estimate, from L^2 to H^1, under the additional assumption that it is injective (see Theorem 12). In dimension three we must add an additional ΨDO corresponding to the requirement that $\mathrm{d}_\beta(f) = 0$ in order to obtain an elliptic system

(Theorem 10), and then apply a parametrix for this system to obtain the same types of results as in the higher dimensional case (Theorem 13). Theorems 12 and 13 also include a stability result for perturbations of U_1, U_2, and g about a given set of tensor fields for which the corresponding normal operator is known to be injective.

In Theorem 15 we finally establish injectivity of I_{U_1,U_2} when U_1, U_2, and g are real-analytic. The proof of this theorem uses analytic microlocal analysis, and essentially functions by showing that for f in the kernel of I_{U_1,U_2}, the analytic wavefront set of f must be empty. Of course in the three dimensional case we must also assume that $\mathrm{d}_\beta(f) = 0$. Combining this injectivity result with Theorems 12 and 13 shows that I_{U_1,U_2} is injective with a stability estimate for any U_1, U_2, and g sufficiently close to any given real-analytic tensor fields. This is what we mean by "generic uniqueness."

Finally, in chapter 4 we examine again the nonlinear problem. To do this, we initially show that when f_1 and f_2 are sufficiently close to a given real-analytic tensor field f, and g' is a metric sufficiently close to a real-analytic metric g, then the corresponding semi-basic tensor fields U_1' and U_2' given by f_1 and f_2 with respect to g' are close to the field U given by f with respect to g. Then U_1', U_2', and g' lie in the generic set where the linear transform $I_{U_1',U_1'}$ is injective. Combined with the identity (1.22), which relates the polarization data of f_1 and f_2 to $I_{U_1',U_2'}[f_1 - f_2]$, this allows us to use the results for the linear problem to analyze the nonlinear problem, and prove Theorem 1.

Chapter 2
WEIGHTED X-RAY TRANSFORMS

In Chapter 3 we will be considering the linearization of the polarization tomography problem obtained by fixing the weights in (1.21) as some specific pair of semi-basic tensor fields U_1 and $U_2 \in \beta_1^1(\Omega^{\mathbb{R}}M)$ which are both invertible maps on $T_{\pi(\xi)}M$ for every $\xi \in \Omega^{\mathbb{R}}M$. Replacing $f_1 - f_2$ by f in (1.21), we see that for every $\xi \in \partial_-\Omega^{\mathbb{R}}M$, f is in the kernel of the linear map $(I_{U_1,U_2})_\xi : \tau_1^1(M) \to (B_2)_\xi(\partial_-\Omega^{\mathbb{R}}M)$ defined by

$$(I_{U_1,U_2})_\xi\,[f](\eta,\zeta) = \int_0^l \left\langle U_2^{-1}(\dot\gamma_\xi(s)) \left[P_{\dot\gamma_\xi(s)}\,(f)\right](\gamma_\xi(s))\,U_1(\dot\gamma_\xi(s))\,\mathcal{I}_{0,s}^{\gamma_\xi}\,\eta,\ \mathcal{I}_{0,s}^{\gamma_\xi}\,\zeta\right\rangle_{g(\gamma_\xi(s))} \mathrm{d}s. \tag{2.1}$$

In Chapter 3 we will analyze this map directly in coordinates by considering how it acts on each of the components of f separately. The action on each of these components is the weighted X-ray transform of a function, and so we will now study in detail the weighted X-ray transform for functions. The development given here largely follows methods used in [23], [5], and [6], although I have tried to reformulate some of the arguments to give them a more geometric flavor. This is also done in considerably more generality than is needed for the problem of polarization tomography that we are considering. Indeed, except for section 2.3, in this chapter we will continue to assume only that M is a CNT manifold.

We shall refer to the space of maximally extended, directed, unit speed geodesics on M as Γ. We emphasize that the curves are directed, and so two parametrizations of the same point set in opposite directions are considered to be different elements of Γ. In the course of our analysis we will have occasion to require a smooth manifold structure and measure on Γ, which we will obtain by identifying Γ with the space of inward pointing unit vectors, $\partial_-\Omega^{\mathbb{R}}M$, as follows.

Since M is a CNT manifold, every $\gamma \in \Gamma$ must begin on the boundary ∂M, and, since the boundary is strictly convex, the initial tangent vector must be inward pointing. Thus we can define a bijective map $F : \partial_-\Omega^{\mathbb{R}}M \to \Gamma$. For $(x,v) \in \partial_-\Omega^{\mathbb{R}}M$ we will write $F(x,v) = \gamma_{x,v}$, and more generally for any $\xi \in T^{\mathbb{R}}M$ we will write γ_ξ for the maximally extended geodesic with initial conditions $\dot{\gamma}_\xi(0) = \xi$. We will also make use of the four different "exponential" maps defined as follows:

$$\begin{array}{ll} \exp : \mathcal{F} \subset T^{\mathbb{R}}M \to M & \exp(\xi) \equiv \gamma_\xi(1) \\ \exp_x : \mathcal{F}_x = \mathcal{F} \cap T^{\mathbb{R}}_x M \to M & \exp_x \equiv \exp\big|_{\mathcal{F}_x} \\ \mathrm{Exp} : \mathcal{F} \setminus \{0\} \to T^{\mathbb{R}}M & \mathrm{Exp}(\xi) \equiv \frac{\dot{\gamma}_\xi(1)}{|\dot{\gamma}_\xi(1)|_g} \\ \mathrm{Exp}_x : \mathcal{F}_x \setminus \{0\} \to T^{\mathbb{R}}M & \mathrm{Exp}_x = \mathrm{Exp}\big|_{\mathcal{F}_x}. \end{array}$$

We will identify Γ with $\partial_-\Omega^{\mathbb{R}}M$ through the map F, and in fact will refer to Γ and $\partial_-\Omega^{\mathbb{R}}M$ interchangeably. In particular, we may define a smooth structure on Γ by declaring that F is a diffeomorphism, and we may pull back any form or measure on $\partial_-\Omega^{\mathbb{R}}M$ by F^{-1} to obtain a form or measure on Γ. With such a measure we may then define $L^2(\Gamma)$, and make use of the inner product $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$.

In what follows, whenever N is a manifold with boundary we will use the notation $C_c^\infty(N)$ to refer to those smooth functions with support compactly contained in $N^{int} = N \setminus \partial N$. The same notation will also be used in other function spaces (e.g. we will refer to $L_c^2(N)$). Let $w \in C^\infty(T^{\mathbb{R}}M)$. Then for any $f \in C_c^\infty(M)$ we define the weigthed x-ray transform of f, which is a function on Γ, as

$$I_w[f](\gamma) = \int_0^l w(\dot{\gamma}(s))\, f(\gamma(s))\, \mathrm{d}s. \tag{2.2}$$

Here $\gamma \in \Gamma$ has length l. We have the following theorem.

Theorem 3 *The map $I_w : C_c^\infty(M) \to C_c^\infty(\Gamma)$ is continuous.*

Remark 2: The topologies on $C_c^\infty(M)$ and $C_c^\infty(\Gamma)$ are defined by fixing a set of local coordinates charts and a partition of unity. When I say below that the derivatives of a function can be bounded, I mean that there is a single uniform bound for each derivative that applies within the support of each element of the partition of unity in the corresponding coordinate chart.

Proof: For any $(x, v) \in \partial_-\Omega^{\mathbb{R}}M$, let $l(x, v)$ be the positive endpoint of the domain of the maximally extended geodesic $\gamma_{x,v}$ (when v is a unit vector this is the length). The first step in the proof is to show that l is a smooth function on $\partial_-\Omega^{\mathbb{R}}M$, which is identified with Γ.

To accomplish this, let $\rho \in C^\infty(M)$ be a defining function for ∂M (ie. a function such that $\partial M = \{\rho = 0\}$, and $\mathrm{d}\rho|_{\partial M} \neq 0$). Then consider the equation

$$\rho(\exp_x(lv)) = 0. \tag{2.3}$$

By convexity of the boundary, $\dot\gamma_{x,v}(l(x,v))$ is transverse to ∂M for every $(x,v) \in \partial_-\Omega^{\mathbb{R}}M$, and so

$$\left.\frac{\mathrm{d}}{\mathrm{d}l}\right|_{l(x,v)} \rho(\exp_x(lv)) = \mathrm{d}\rho(\dot\gamma_{x,v}(l(x,v))) \neq 0$$

since $\mathrm{d}\rho$ is the conormal to ∂M. Therefore, by the implicit function theorem (2.3) shows that l is a smooth function of $(x,v) \in \partial_-\Omega^{\mathbb{R}}M$.

Next, by the basic theory of ODEs (see for example [28] or [4]), $\gamma_{x,v}(s)$ and $\dot\gamma_{x,v}(s)$ are smooth functions of (x,v) and s. Thus there is no problem differentiating the integral in (2.2) which defines $I_w[f]$, and we see that it is a smooth function of $(x,v) \in \partial_-\Omega^{\mathbb{R}}M$. Further, in any particular system of coordinates we can perform this differentiation and bound derivatives of $I_w[f]$ in terms of derivatives of f and w, and derivatives of l at points (x,v) such that $\gamma_{x,v}$ intersects the support of f. Thus in order to complete the proof we must show that the support of $I_w[f]$ is compact, and that on the set of (x,v) where $\gamma_{x,v}$ intersects any given compact set $K \Subset M^{int}$ the derivatives of $l(x,v)$ can be bounded.

From (2.3) it is possible to bound the derivatives of $l(x,v)$ on any subset of $\partial_-\Omega^{\mathbb{R}}M$ where the quantity $\mathrm{d}\rho(\dot\gamma_{x,v}(l(x,v)))$ is bounded away from zero. On the other hand, looking in local coordinates and using the convexity of the boundary it is clear that a subset of $\partial_-\Omega^{\mathbb{R}}M$ is compact if and only if $\mathrm{d}\rho(\dot\gamma_{x,v}(l(x,v)))$ is uniformly bounded away from zero on that set. Finally, if we fix a compact subset $K \Subset M^{int}$ where the support of f lies, then K must be at a positive distance from ∂M and from this we can show, using local coordinates once again, that on the set of $(x,v) \in \partial_-\Omega^{\mathbb{R}}M$ such that $\gamma_{x,v}$ passes through K, $\mathrm{d}\rho(\dot\gamma_{x,v}(l(x,v)))$ must be bounded away from zero. Therefore the support of $I_w[f]$ is compact, and the map is continuous as claimed.

□

Next we would like to define an adjoint of I_w, but in order to do this we need an L^2 structure on Γ. There is a way to define such a structure by taking a "natural" measure on

$\partial_-\Omega^{\mathbb{R}}M$ which we will now describe.

2.1 Volume form on $\partial_-\Omega^{\mathbb{R}}M$

Locally $\partial_-\Omega^{\mathbb{R}}M$ is a product of an open neighborhood in ∂M, and the open $n-1$ dimensional unit ball. We will use this fact to define an orientation form on $\partial_-\Omega^{\mathbb{R}}M$, and then show that this form behaves well in relation to the volume form on TM induced by the metric g. This orientation form will be our volume form on $\partial_-\Omega^{\mathbb{R}}M$, and thus provide us with our "natural" measure on $\partial_-\Omega^{\mathbb{R}}M$.

Let $(x^1, \dots, x^n)$ be a set of boundary normal coordinates for ∂M defined on a set $U \subset M$ so that on the domain of the coordinates $\partial M = \{x^n = 0\}$, and the inward pointing unit normal to ∂M is given by $\partial/\partial x^n$. Let $(x^1, \dots, x^n, v^1, \dots, v^n)$ denote the corresponding natural coordinates on $T^{\mathbb{R}}U$. Then we may parametrize $\partial_-\Omega^{\mathbb{R}}M \cap T^{\mathbb{R}}U$ by the set

$$\{(x, v) \in T^{\mathbb{R}}\partial M \,:\, v^i v^j g_{ij}(x) < 1\}$$

via the map

$$\phi^{-1}(x^1, \dots, x^{n-1}, v^1, \dots, v^{n-1}) = \left(\sqrt{1 - \sum_{i,j=1}^{n-1} v^i v^j g_{ij}}\, \frac{\partial}{\partial x^n} + \sum_{i=1}^{n-1} v^i \frac{\partial}{\partial x^i}\right|_{(x^1, \dots, x^{n-1}, 0)}.$$

This expresses $\partial_-\Omega^{\mathbb{R}}M$ locally as a product of an open neighborhood in ∂M, and an open $n-1$ dimensional ball. The inverse of this map, ϕ, provides a local coordinate system on $\partial_-\Omega^{\mathbb{R}}M \cap T^{\mathbb{R}}U$. In these coordinates we define locally a $2n-2$ form by

$$\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M} = \det(g)\, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{n-1} \wedge \mathrm{d}v^1 \wedge \dots \wedge \mathrm{d}v^{n-1}$$

Since $(x^1, \dots, x^{n-1}, x^n)$ are boundary normal coordinates, $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$ actually does not depend on the choice of coordinates and is thus invariantly defined on all of $\partial_-\Omega^{\mathbb{R}}M$. Therefore $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$ is an orientation form for $\partial_-\Omega^{\mathbb{R}}M$. It is this form that we will use to define our measure and L^2 structure on Γ (which we identify with $\partial_-\Omega^{\mathbb{R}}M$). Indeed, for f and $h \in C_c^\infty(\partial_-\Omega^{\mathbb{R}}M)$ we set

$$\langle f,\, h\rangle_{L^2(\partial_-\Omega^{\mathbb{R}}M)} = \int_{\partial_-\Omega^{\mathbb{R}}M} f(x, v)\, \overline{h(x, v)}\, \left|\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x, v)\right|.$$

Here $\left|\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)\right|$ is the density defined from $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$. The completion of $C_c^\infty(\partial_-\Omega^{\mathbb{R}}M)$ with respect to the norm coming from this inner product is $L^2(\partial_-\Omega^{\mathbb{R}}M)$.

With the assistance of $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$, we are now able to define the adjoint I_w^* of the transform I_w. Indeed for $h \in C_c^\infty(M)$ and $f \in C_c^\infty(\partial_-\Omega^{\mathbb{R}}M)$ we set

$$\langle I_w^*[f],\, h\rangle_{L^2(M)} = \langle f,\, I_w[h]\rangle_{L^2(\partial_-\Omega^{\mathbb{R}}M)}.$$

By Theorem 3 this defines a continuous map $I_w^* : \mathcal{D}'(\partial_-\Omega^{\mathbb{R}}M) \to \mathcal{D}'(M)$. In the next few sections we will be concerned with so called *normal operator*s. If w_1 and $w_2 \in C^\infty(TM)$ are two, potentially different, weight functions, then the normal operator with respect to w_1 and w_2, $\mathcal{N}_{w_1,w_2}$, is defined by

$$\mathcal{N}_{w_1,w_2} = I_{w_1}^* \circ I_{w_2}. \tag{2.4}$$

At present we only know that $\mathcal{N}_{w_1,w_2} : C_c^\infty(M) \to \mathcal{D}'(M)$, but below in section 2.3 we will derive an integral formula for $\mathcal{N}_{w_1,w_2}$ which shows that the operator has better regularity properties. We will also do the same for I_w^* in section 2.2. Furthermore, we will show that with the additional assumption that M is a simple manifold, $\mathcal{N}_{w_1,w_2}$ is a Pseudo-Differential Operator (ΨDO).

As a first step towards deriving the integral formulae for $\mathcal{N}_{w_1,w_2}$ and I_w^* mentioned above, we will now show that $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$ behaves well with respect to the $2n$-form on $T^{\mathbb{R}}M$ obtained by pulling back the natural $2n$-form on $(T^*)^{\mathbb{R}}M$ by the musical isomorphism $\flat : T^{\mathbb{R}}M \to (T^*)^{\mathbb{R}}M$ corresponding to the metric g. The natural $2n$-form on $(T^*)^{\mathbb{R}}M$ is defined in natural coordinates $(x^1,\, \dots, x^n,\, \xi_1,\, \dots, \xi_n)$ by

$$\mathrm{d}x^1 \wedge \,\dots\, \wedge \mathrm{d}x^n \wedge \mathrm{d}\xi_1 \wedge \,\dots\, \wedge \mathrm{d}\xi_n = (-1)^{\mathrm{floor}(n/2)}\frac{\omega^{\wedge n}}{n!}. \tag{2.5}$$

Here ω is the canonical symplectic form on $(T^*)^{\mathbb{R}}(M)$. If we work in a set of local coordinates $(x^1,\, \dots, x^n)$ on M, then in the corresponding natural coordinates $(x^1,\, \dots, x^n, v^1,\, \dots, v^n)$ for $T^{\mathbb{R}}M$ a calculation shows that the so obtained $2n$-form is given by

$$\mathrm{d}V = \det(g)\, \mathrm{d}x^1 \wedge \,\dots\, \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \,\dots\, \mathrm{d}v^n. \tag{2.6}$$

We will show that $T^{\mathbb{R}}M \setminus (\{0\} \cup T\partial M)$ is given, loosely speaking, as a product of $\partial_-\Omega^{\mathbb{R}}M$ with another manifold, and that $\mathrm{d}V$ is given by the product of $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$ with a form on the other manifold. To do this we must first introduce some notations and definitions.

Let $m : \mathbb{R} \times \partial_- \Omega^{\mathbb{R}} M \to T^{\mathbb{R}} M$ be defined by $m(t,(x,v)) = (x, tv)$, and let $\mathcal{F}_R = m^{-1}(\mathcal{F})$. Recall that $\mathcal{F}$ is the domain of the exponential map. Define $p_\Omega : \mathcal{F}_R \to \Omega^{\mathbb{R}} M$ by

$$p_\Omega(t,(x,v)) = \dot{\gamma}_{x,v}(t).$$

Observe that p_Ω is the flow of the geodesic vector field on TM restricted to $\Omega^{\mathbb{R}} M$ beginning at points in $\partial_- \Omega^{\mathbb{R}} M$, and so is smooth up to the boundary of $\mathcal{F}_R$. Further, since $\partial_- \Omega^{\mathbb{R}} M$ is transverse to the geodesic vector field, p_Ω is a local diffeomorphism. From this and the non-trapping assumption we see that p_Ω is a diffeomorphism onto its image which is all of $\Omega^{\mathbb{R}} M \setminus T\partial M$.

Next we introduce two related maps $p_\pm : \mathbb{R}^+ \times \mathcal{F}_R \to T^{\mathbb{R}} M \setminus (\{0\} \cup T\partial M)$ which provide the "product structure" for $T^{\mathbb{R}} M \setminus (\{0\} \cup T\partial M)$ mentioned above. These maps are defined respectively by

$$p_\pm(s,(t,(x,v))) = \pm s\, \dot{\gamma}_{x,v}(t).$$

Since both $p_\pm$ are injective and $(p_\pm)_* \frac{\partial}{\partial s}$ is transverse to ΩM, $p_\pm$ are both diffeomorphisms onto their images. Therefore, if $(x^1, \dots, x^{n-1}, v^1, \dots, v^{n-1})$ are one of the sets of local coordinates for $\partial_- \Omega^{\mathbb{R}} M$ used to define $\mathrm{d}V_{\partial_- \Omega^{\mathbb{R}} M}$ above, then the maps

$$(x^1, \dots, x^{n-1}, t, v^1, \dots, v^{n-1}, s) \mapsto p_\pm(s,(t,(x^1, \dots, x^{n-1}, v^1, \dots, v^{n-1})$$

are the inverses of two different coordinate maps on some subsets of $T^{\mathbb{R}} M \setminus (\{0\} \cup T\partial M)$. We will now calculate $\mathrm{d}V$ in these coordinates. Indeed, we will calculate $\mathrm{d}V$ first in these coordinates at points where $t = 0$, which correspond to the points in $\partial_\pm T^{\mathbb{R}} M$ respectively. Starting from (2.6) we have

$$\mathrm{d}V = (\pm 1)^n \det(g)\, \mathrm{d}(\gamma^1_{x,v}(t)) \wedge \dots \wedge \mathrm{d}(\gamma^n_{x,v}(t)) \wedge \mathrm{d}(s\, \dot{\gamma}^1_{x,v}(t)) \wedge \dots \mathrm{d}(s\, \dot{\gamma}^n_{x,v}(t)) \tag{2.7}$$

where

$$(x,v) = \left(x^1, \dots, x^{n-1}, 0, v^1, \dots, v^{n-1}, \sqrt{1 - \sum_{i,j=1}^{n-1} v^i v^j g_{ij}} \right).$$

When we evaluate at $t = 0$, after some simplification (2.7) becomes

$$\begin{aligned}
\mathrm{d}V &= (\pm 1)^n \det(g)\, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{n-1} \wedge \left(\sqrt{1-\sum_{i,j=1}^{n-1} v^i v^j g_{ij}}\, \mathrm{d}t\right) \\
&\qquad \wedge (v^1 \mathrm{d}s + s\, \mathrm{d}v^1) \wedge \dots \wedge \left(\left(\sqrt{1-\sum_{i,j=1}^{n-1} v^i v^j g_{ij}}\right) \mathrm{d}s - s \sum_{i,j=1}^{n-1} \frac{v^i g_{ij}\, \mathrm{d}v^j}{\sqrt{1-\sum_{i,j=1}^{n-1} v^i v^j g_{ij}}}\right) \\
&= (\pm 1)^n s^{n-1} \det(g)\, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{n-1} \wedge \mathrm{d}t \wedge \mathrm{d}v^1 \dots \wedge \mathrm{d}v^{n-1} \wedge \mathrm{d}s.
\end{aligned}$$

Now, from (2.5) we see that $\mathrm{d}V$ can be written in terms of the symplectic form $\flat^*\omega$ on $T^{\mathbb{R}}M$. Since the geodesic flow is the Hamiltonian flow for $H(x,v) = \frac{1}{2} g_{ij} v^i v^j$ relative to this symplectic form, by Liouville's theorem $\mathrm{d}V$ is invariant under the geodesic flow (see [28]). In each of the coordinate systems $(x^1, \dots, x^{n-1}, t, v^1, \dots, v^{n-1}, s)$, flowing by time $\tilde{t}$ is simply given by

$$(x^1, \dots, x^{n-1}, t, v^1, \dots, v^{n-1}, s) \mapsto (x^1, \dots, x^{n-1}, t \pm s\tilde{t}, v^1, \dots, v^{n-1}, s),$$

and so we see that

$$\mathrm{d}V = (\pm 1)^n s^{n-1} \left(\det(g)|_{(x^1, \dots, x^{n-1}, 0)}\right) \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{n-1} \wedge \mathrm{d}t \wedge \mathrm{d}v^1 \dots \wedge \mathrm{d}v^{n-1} \wedge \mathrm{d}s \tag{2.8}$$

everywhere that the coordinates are defined (not just when $t = 0$). At this point we note that the two coordinate systems, corresponding to p_+ and p_-, have the same orientation exactly when n is even, and that the one corresponding to p_+ is always positively oriented (this explains the $(\pm 1)^n$ in (2.8)).

By identifying $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$ with its pullback by the projection map

$$(s, (t, (x, v))) \mapsto (x, v)$$

from $\mathbb{R}^+ \times \mathcal{F}_R$ to $\partial_-\Omega^{\mathbb{R}}M$, we see from (2.8) that

$$p_\pm^*(\mathrm{d}V) = (\mp 1)^n s^{n-1}\, \mathrm{d}s \wedge \mathrm{d}t \wedge \mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M} \tag{2.9}$$

at least on the domain of the coordinates used in (2.8). Since $T^{\mathbb{R}}M \setminus (\{0\} \cup T\partial M)$ can be covered by such coordinate patches this formula actually holds on all of $\mathbb{R}^+ \times \mathcal{F}_R$.

There is also a "natural" orientation form on $\Omega^{\mathbb{R}}M$ which is obtained by considering $\Omega^{\mathbb{R}}M$ as the boundary of the closed unit ball bundle over M. Indeed we define an orientation form on $\Omega^{\mathbb{R}}M$ by

$$\mathrm{d}V_{\Omega^{\mathbb{R}}M} = \left(i_{(p_+)_* \frac{\partial}{\partial s}} (\mathrm{d}V) \right|_{\Omega^{\mathbb{R}}M}$$

where $i_{(p_+)_* \frac{\partial}{\partial s}}$ denotes interior multiplication by $\frac{\partial}{\partial s}$. Using (2.9) and the relationship between p_Ω and p_+ this becomes

$$p_\Omega^*(\mathrm{d}V_{\Omega^{\mathbb{R}}M}) = (-1)^n \,\mathrm{d}t \wedge \mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M} \tag{2.10}$$

where now we identify $\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}$ with the form on $\mathcal{F}_R$ obtained from pull-back by the obvious projection. Rewriting (2.9) and (2.10) in terms of the corresponding densities we have

$$p_\pm^*(|\mathrm{d}V|) = s^{n-1} \left|\mathrm{d}s \wedge \mathrm{d}t \wedge \mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}\right| \tag{2.11}$$

and

$$p_\omega^*(|\mathrm{d}V_{\Omega^{\mathbb{R}}}|) = \left|\mathrm{d}t \wedge \mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}\right|. \tag{2.12}$$

2.2 Integral formula for I_w^*

In section 4.3 we will need one result concerning the continuity of I_w^*, and in order to establish this we will derive an integral formula for this adjoint operator. The method here is almost exactly the same as the method used in the next section to derive a formula for $\mathcal{N}_{w_1,w_2}$, and so this also serves as a warm-up to that slightly more difficult problem.

Let $f \in C_c^\infty(\partial_-\Omega^{\mathbb{R}}M)$ and $h \in C_c^\infty(M)$. Then from the definition of I_w^* we have

$$\begin{aligned}
\langle I_w^*[f],\, h\rangle_{L^2(M)} &= \langle f, I_w[h]\rangle_{L^2(\partial_-\Omega^{\mathbb{R}}M)} \\
&= \int_{\partial_-\Omega^{\mathbb{R}}M} \int_0^{l(x,v)} f(x,v)\, \overline{w(\dot\gamma_{x,v}(t))\, h(\gamma_{x,v}(t))}\, \mathrm{d}t\, \left|\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)\right| \\
&= \int_{\Omega^{\mathbb{R}}M} f(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(\xi)))\, \overline{w(\xi)\, h(\pi(\xi))}\, |\mathrm{d}V_{\Omega^{\mathbb{R}}M}(\xi)|
\end{aligned}$$

where $\pi_{\partial_-\Omega^{\mathbb{R}}M} : \mathcal{F}_R \to \partial_-\Omega^{\mathbb{R}}M$ is the projection mapping. The last equality in the previous calculation follows from (2.12). The equivalent equality for general functions defined on $\Omega^{\mathbb{R}}M$ is called Santaló's formula (see [19]). Suppose now that we have local coordinates $x = (x^1, \ldots, x^n)$ defined on a set $U \subset M$ where an orthonormal frame $\{e_1, \ldots, e_n\}$ is also defined. Let $(x, \xi) = (x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ be the corresponding coordinates on TU, and note that $(p_+)_* \frac{\partial}{\partial s} = \xi^i \frac{\partial}{\partial \xi^i}$ is the radial vector field at points on $\Omega^{\mathbb{R}} M$. Thus the portion of the previous integral in U is given in coordinates by

$$\int_U \overline{h(x)} \left(\int_{\mathbb{S}^{n-1}} \overline{w(x,\xi)}\, f(\pi_{\partial_- \Omega^{\mathbb{R}} M}(p_\Omega^{-1}(x,\xi))\, \mathrm{d}\mathbb{H}_{\mathbb{S}^{n-1}}(\xi) \right) \sqrt{\det(g)}\, \mathrm{d}x.$$

Therefore

$$I_w^*[f](x) = \int_{\Omega_x^{\mathbb{R}} M} \overline{w(\xi)}\, f(\pi_{\partial_- \Omega^{\mathbb{R}} M}(p_\Omega^{-1}(\xi)) \left| \mathrm{d}V_{\Omega_x^{\mathbb{R}} M}(\xi) \right|. \tag{2.13}$$

We can use this formula to prove the following theorem.

Theorem 4 *I_w^* maps $C_c^\infty(\partial_- \Omega^{\mathbb{R}} M)$ continuously to $C^\infty(M)$.*

Proof: Let $f \in C_c^\infty(\partial_- \Omega^{\mathbb{R}} M)$ have support contained within a given compact set $K' \Subset \partial_- \Omega^{\mathbb{R}} M$. From (2.13) it is immediately clear that $\|I_w^*[f]\|_{C^0(M)} \leq C \|f\|_{C^0(\partial_- \Omega^{\mathbb{R}} M)}$. Derivatives of $\|I_w^*[f]\|$ may be similarly estimated from (2.13) as we show in the next paragraph.

Let $U \subset M$ be the domain of some coordinates $(x^1, \dots, x^n)$. Also assume that we have an orthonormal frame $\{e_1, \dots, e_n\}$ for TU, and that $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ are the corresponding coordinates for TU. In order to take derivatives of f we also introduce a partition of unity on K' consisting of functions $\{\psi_j\}_{j=1}^l$ which all lie in $C_c^\infty(\partial_- \Omega^{\mathbb{R}} M)$, and such that the support of each ψ_j is contained within the domain of a single coordinate chart on $\partial_- \Omega^{\mathbb{R}} M$. Using the coordinates $(x,\xi) = (x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ on TU, (2.13) becomes

$$I_w^*[f](x) = \sum_{j=1}^l \int_{\mathbb{S}^{n-1}} \overline{w(x,\xi)}\, \psi_j(\pi_{\partial_- \Omega^{\mathbb{R}} M}(p_\Omega^{-1}(x,\xi)))\, f(\pi_{\partial_- \Omega^{\mathbb{R}} M}(p_\Omega^{-1}(x,\xi)))\, \mathrm{d}\mathbb{H}_{\mathbb{S}^{n-1}}(\xi). \tag{2.14}$$

From this formula we may estimate derivatives of $I_w^*[f]$ in these coordinates in terms of derivatives of f in various coordinate charts, derivatives of the functions ψ_j, derivatives of $\overline{w}$, and derivatives of p_Ω^{-1}. Since all of the ψ_j are zero except on a compact set we only need bounds on the derivatives of p_Ω^{-1} on a compact subset of $\partial_- \Omega^{\mathbb{R}} M$, which certainly exist since p_Ω^{-1} is a diffeomorphism. This completes the proof.

□

Of course it is an immediate corollary of this result that I_w extends to a continuous operator from $\mathcal{E}'(M)$ to $\mathcal{D}'(\partial_- \Omega^{\mathbb{R}} M)$. Later in Theorem 7 we will also use (2.13) to show that $I_w^* : H_c^1(\partial_- \Omega^{\mathbb{R}} M) \to H_{loc}^1(M)$ continuously.

2.3 Integral form of the normal operator

We will now find an integral formula for the normal operator $\mathcal{N}_{w_1,w_2} = I^*_{w_1} \circ I_{w_2}$ introduced above. Let f and $h \in C^\infty_c(M)$. From the definition of $I^*_{w_1}$ we have

$$\begin{aligned}\langle \mathcal{N}_{w_1,w_2}[f],\, h\rangle_{L^2(M)} =& \langle I_{w_2}[f], I_{w_1}[h]\rangle_{L^2(\partial_-\Omega^{\mathbb{R}}M)} \\ =& \int_{\partial_-\Omega^{\mathbb{R}}M} \left(\int_0^{l(x,v)} w_2(\dot\gamma_{x,v}(s))\, f(\gamma_{x,v}(s))\, \mathrm{d}s \right) \times \\ & \overline{\left(\int_0^{l(x,v)} w_1(\dot\gamma_{x,v}(t))\, h(\gamma_{x,v}(t))\, \mathrm{d}t \right)} \left|\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)\right| \\ =& \int_{\partial_-\Omega^{\mathbb{R}}M} \int_0^{l(x,v)} \int_0^{l(x,v)} w_2(\dot\gamma_{x,v}(s))\, \overline{w_1(\dot\gamma_{x,v}(t))}\, f(\gamma_{x,v}(s)) \times \\ & \overline{h(\gamma_{x,v}(t))}\, \mathrm{d}s\, \mathrm{d}t \left|\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)\right| .\end{aligned}$$

We next split this integral into two parts and make the change of variables $s \mapsto -s$ in one of the parts to obtain

$$\begin{aligned}\langle \mathcal{N}_{w_1,w_2}[f],\, h\rangle_{L^2(M)} =& \int_{\partial_-\Omega^{\mathbb{R}}M} \int_0^{l(x,v)} \int_0^{t} w_2(\dot\gamma_{\dot\gamma_{x,v}(t)}(-s))\, \overline{w_1(\dot\gamma_{x,v}(t))}\, f(\gamma_{\dot\gamma_{x,v}(t)}(-s)) \times \\ & \overline{h(\gamma_{x,v}(t))}\, \mathrm{d}s\, \mathrm{d}t \left|\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)\right| \\ & + \int_{\partial_-\Omega^{\mathbb{R}}M} \int_0^{l(x,v)} \int_0^{l(x,v)-t} w_2(\dot\gamma_{\dot\gamma_{x,v}(t)}(s))\, \overline{w_1(\dot\gamma_{x,v}(t))}\, f(\gamma_{\dot\gamma_{x,v}(t)}(s)) \times \\ & \overline{h(\gamma_{x,v}(t))}\, \mathrm{d}s\, \mathrm{d}t \left|\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)\right| .\end{aligned}$$

Recall that $\mathcal{F} \subset T^{\mathbb{R}}M$ is the domain of the exponential map, and for each $x \in M$, $\mathcal{F}_x = \mathcal{F} \cap T^{\mathbb{R}}_x M$. Using (2.11) we see that each of these two parts is an integral over $\mathcal{F} \setminus \{0\}$ in the coordinates corresponding to the maps p_- and p_+ respectively. Thus we have

$$\begin{aligned}\langle \mathcal{N}_{w_1,w_2}[f],\, h\rangle_{L^2(M)} \quad = \quad & \int_{\mathcal{F}\setminus\{0\}} w_2(\mathrm{Exp}(\xi))\, \overline{w_1(\xi/|\xi|_g)}\, f(\exp(\xi))\, \overline{h(\pi(\xi))}\, \frac{|\mathrm{d}V(\xi)|}{|\xi|_g^{n-1}} \\ & + \int_{\mathcal{F}\setminus\{0\}} w_2(-\mathrm{Exp}(\xi))\, \overline{w_1(-\xi/|\xi|_g)}\, f(\exp(\xi))\, \overline{h(\pi(\xi))}\, \frac{|\mathrm{d}V(\xi)|}{|\xi|_g^{n-1}} .\end{aligned} \tag{2.15}$$

In any set of natural coordinates $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ for $T^{\mathbb{R}}M$, we have that $\mathrm{d}V$ is given by (2.6). Thus, if U is the domain of this coordinate chart, then the portion of the integrals from (2.15) contained in $\pi^{-1}(U)$ is given by

$$\int_{\pi(U)} \overline{h(x)} \left(\int_{\mathcal{F}_x \setminus \{0\}} w_2(\mathrm{Exp}_x(v))\, \overline{w_1(v/|v|_g)} f(\exp_{\mathrm{x}}(v)) \sqrt{\det(g)}\, \frac{\mathrm{d}v}{|v|_g^{n-1}} \right) \sqrt{\det(g)}\, \mathrm{d}x$$
$$+ \int_{\pi(U)} \overline{h(x)} \left(\int_{\mathcal{F}_x \setminus \{0\}} w_2(-\mathrm{Exp}_x(v))\, \overline{w_1(-v/|v|_g)} f(\exp_{\mathrm{x}}(v)) \sqrt{\det(g)}\, \frac{\mathrm{d}v}{|v|_g^{n-1}} \right) \sqrt{\det(g)}\, \mathrm{d}x. \tag{2.16}$$

From this we see that

$$\begin{aligned} \mathcal{N}_{w_1,w_2}[f](x) &= \int_{\mathcal{F}_x \setminus \{0\}} w_2(\mathrm{Exp}_x(v))\, \overline{w_1(v/|v|_g)} f(\exp_{\mathrm{x}}(v)) \sqrt{\det(g)}\, \frac{\mathrm{d}v}{|v|_g^{n-1}} \\ &\quad + \int_{\mathcal{F}_x \setminus \{0\}} w_2(-\mathrm{Exp}_x(v))\, \overline{w_1(-v/|v|_g)} f(\exp_{\mathrm{x}}(v)) \sqrt{\det(g)}\, \frac{\mathrm{d}v}{|v|_g^{n-1}}. \end{aligned} \tag{2.17}$$

For ease of notation we combine these integrals into one by setting

$$A_x(v) = w_2(\mathrm{Exp}_x(v))\, \overline{w_1(v/|v|_g)} + w_2(-\mathrm{Exp}_x(v))\, \overline{w_1(-v/|v|_g)}$$

so that (2.17) becomes

$$\mathcal{N}_{w_1,w_2}[f](x) = \int_{\mathcal{F}_x \setminus \{0\}} A_x(v)\, f(\exp_x(v)) \sqrt{\det(g)}\, \frac{\mathrm{d}v}{|v|_g^{n-1}}. \tag{2.18}$$

Now let $(t, \omega) \in \mathbb{R} \times \Omega^{\mathbb{R}} M$, and note that $A_x(t\omega)$ can be written as

$$A_x(t\omega) = w_2(\dot{\gamma}_{x,\omega}(t))\overline{w_1(\omega)} + w_2(-\dot{\gamma}_{x,\omega}(t))\overline{w_1(-\omega)}$$

which is a smooth function of $(x,\, t\,, \omega) \in M \times \mathbb{R} \times \Omega^{\mathbb{R}} M$ even if we extend the domain to include $\{t = 0\}$. If we take $t > 0$ and ω as polar coordinates on $\mathcal{F}_x \setminus \{0\}$, then (2.18) becomes

$$\mathcal{N}_{w_1,w_2}[f](x) = \int_{\Omega_x^{\mathbb{R}} M} \int_0^{l(x,\omega)} A_x(t\omega)\, f(\exp_x(t\omega))\, \mathrm{d}t\, \mathrm{d}\omega \tag{2.19}$$

where $\mathrm{d}\omega$ is the natural measure on $\Omega_x M$. If we assume that $f \in C_c^\infty(M)$, then we can extend the integration in (2.19) to

$$\mathcal{N}_{w_1,w_2}[f](x) = \int_{\Omega_x^{\mathbb{R}} M} \int_0^{\infty} A_x(t\omega)\, f(\exp_x(t\omega))\, \mathrm{d}t\, \mathrm{d}\omega. \tag{2.20}$$

Since the integrand in (2.20) is compactly supported and is a smooth function of all variables we can use this formula to establish the following theorem by differentiating under the integral.

Theorem 5 *The map $\mathcal{N}_{w_1,w_2} : C_c^\infty(M) \to C^\infty(M)$ is continuous.*

Of course this result also follows from Theorems 3 and 4.

2.4 $\mathcal{N}$ on simple manifolds

The purpose of this section is to prove that when the manifold M is simple, the operator $\mathcal{N}_{w_1,w_2}$ is a ΨDO, and to record a few simple corollaries of this result. To make the notation simpler and streamline the presentation we will simply write $\mathcal{N}$ for some operator of the form (2.20). As part of the proof we will also calculate the symbol of such operators $\mathcal{N}$, which will in turn give us the symbol of $\mathcal{N}_{w_1,w_2}$ for any particular choice of the weights w_1 and w_2.

We first recall the definition of a simple manifold.

Definition 3 *A Riemannian manifold (M, g) with boundary is called simple if the following two conditions are met.*

1. *The boundary of M is convex with respect to g.*

2. *For every $x \in M$, $\exp_x$ is a diffeomorphism.*

If there is only one metric g defined on M and (M, g) is simple, we will say only that M is simple. Also, when we vary the metric g on a given manifold M, we will say that g is a simple metric when (M, g) is simple.

By taking the exponential map based at a point in the interior of M, and then transferring the domain of this map to a closed ball we can see that any simple manifold is diffeomorphic to a closed ball, and thus there is in particular a global coordinate system. For the remainder of this section we will assume that we have such a set of global coordinates $(x^1, \ldots, x^n)$, and that their range $B \subset \mathbb{R}^n$ is a closed ball. In the case when M is simple we may take advantage of the inverse exponential map $\exp_x^{-1}$ to express (2.19) as an integral operator on $C_c^\infty(M)$ in these global coordinates. Indeed, for each $x \in M$ let $(\mathcal{F}_r)_x \subset \mathbb{R} \times \Omega_x^{\mathbb{R}} M$ denote the maximal domain of the map $(t, \omega) \mapsto \gamma_x(t\omega)$, and consider the map $F_x : (\mathcal{F}_r)_x \to \mathbb{R} \times \mathbb{S}^{n-1}$ defined by

$$F_x(t,\omega) = \left(\text{sign}(t)|\exp_x(t\omega) - x|,\, \text{sign}(t)\frac{\exp_x(t\omega) - x}{|\exp_x(t\omega) - x|}\right) = (\rho,\theta) \tag{2.21}$$

where the algebraic operations on the right are being performed in the global coordinates. This formula only applies when $t \neq 0$, however as we will see below it extends to a smooth map on all of $(\mathcal{F}_r)_x$ by continuity. Roughly speaking, F_x takes polar coordinates $(t,\omega) \in \mathbb{R} \times \Omega^{\mathbb{R}}_x M$ with respect to the exponential map at x to polar coordinates $(\rho,\theta) \in \mathbb{R} \times \mathbb{S}^{n-1}$ in the global coordinate system centered at x. Note that

$$\exp_x(t\omega) - x = t\int_0^1 \dot\gamma_{x,\omega}(rt)\mathrm{d}r.$$

By the simplicity assumption this quantity is only zero when $t = 0$, and the integral on the right is not zero when $t = 0$. From this we can see that

$$\theta(t,\omega) = \frac{\displaystyle\int_0^1 \dot\gamma_{x,\omega}(rt)\,\mathrm{d}r}{\left|\displaystyle\int_0^1 \dot\gamma_{x,\omega}(rt)\,\mathrm{d}r\right|} \quad \text{and} \quad \rho(t,\omega) = t\left|\int_0^1 \dot\gamma_{x,\omega}(rt)\,\mathrm{d}r\right| \tag{2.22}$$

are both smooth functions of x, t, and ω. Further, we have that

$$(t,\omega) = \left(\text{sign}(\rho)\left|\exp_x^{-1}(\rho\,\theta + x)\right|_g,\, \text{sign}(\rho)\frac{\exp_x^{-1}(\rho\,\theta + x)}{\left|\exp_x^{-1}(\rho\,\theta + x)\right|_g}\right) = F_x^{-1}(\rho,\theta)$$

wherever the functions on the right are defined. From this we can see that F_x is a diffeomorphism away from $\{t = 0\}$. If we note that $F_x(0,\omega) = (0, \dot\gamma_{x,\omega}(0))$, $\frac{\partial\rho}{\partial t}(0,\omega) = |\dot\gamma_{x,\omega}(0)|$, and that $\frac{\partial\rho}{\partial\omega}(0,\omega) = 0$ (with respect to any choice of coordinates for ω on $\Omega^{\mathbb{R}}_x$), then we can see that by the inverse function theorem F_x is a local diffeomorphism near $\{t = 0\}$, and so F_x is a diffeomorphism onto its range. Also, from the arguments above and the facts that $\exp_x$ and $\exp_x^{-1}$ depend smoothly on x, we can see that the maps F_x and F_x^{-1} depend smoothly on the parameter x. Now set

$$A'(x,\rho,\theta) = A_x(t(\rho\,\theta)\,\omega(\rho\,\theta))\left|\frac{\partial F_x^{-1}}{\partial(\rho,\theta)}\right|, \tag{2.23}$$

which by the above arguments is a smooth function on its domain, which includes $\{t = 0\}$, and is even in (ρ,θ) (ie. $A'(x,-\rho,-\theta) = A'(x,\rho,\theta)$). Since A' is smooth up to the boundary of its domain, it can be extended to a function in $C_c^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1})$ that is still even in (ρ, θ). We can now use F_x to change coordinates in (2.20) to obtain

$$\mathcal{N}[f](x) = \int_{\mathbb{S}^{n-1}} \int_0^\infty A'(x, \rho, \theta) f(x + \rho\,\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{2.24}$$

Since we have extended the integral kernel A' to a function on all of $\mathbb{R}^n$ in these coordinates, we can use equation (2.24) to define an operator $\mathcal{N} : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. Following [5], we will now show that any operator of the form (2.24) is a ΨDO of order -1. Indeed, we have the following theorem.

Theorem 6 *Any operator defined on $C_c^\infty(\mathbb{R}^n)$ of the form (2.24) with $A' \in C_c^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1})$ is a ΨDO of order -1. Further, if A' is an even function of (ρ, θ) and for $\xi \in \mathbb{R}^n$ we define $f_\xi(\omega') = \langle \omega', \xi \rangle$ on $\mathbb{S}^{n-1}$, then the symbol $\sigma_\mathcal{N}$ of the operator $\mathcal{N}$ has the asymptotic expansion*

$$\sigma_\mathcal{N}(x, \xi) \sim \sum_{k=0}^\infty |\xi|^{-1-k} \frac{\pi}{i^k k!} \left\langle f^*_{\frac{\xi}{|\xi|}} \delta^k, \frac{\partial^k A'}{(\partial \rho)^k}(x, 0, \cdot) \right\rangle. \tag{2.25}$$

Remark 3: The functions making up the asymptotic expansion in (2.25) are not smooth near $\{|\xi| = 0\}$, but they can each be modified on a compact set to put the kth term in $S^{-1-k}(\mathbb{R}^n \times \mathbb{R}^n)$.

Remark 4: We can use this result to calculate the symbol of $\mathcal{N}$ even in the case that A' is not even since we may replace A' by the even part of A' in (2.24) without changing $\mathcal{N}[f]$.

Proof: Let $\mathcal{N}$ be an operator of the form (2.24) with $A' \in C_c^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1})$. Then in (2.24) we can change from the polar coordinates centered at x back to rectangular coordinates $y = x + \rho\,\theta$ to get

$$\mathcal{N}[f](x) = \int_{\mathbb{R}^n} A'(x, |y - x|, (y - x)/|y - x|)\, f(y)\, \frac{\mathrm{d}y}{|y - x|^{n-1}}.$$

Thus $\mathcal{N}$ has a Schwarz kernel that is compactly supported and smooth except along the diagonal where it has an integrable singularity. Therefore, $\mathcal{N}[f]$ can be rewritten as

$$\mathcal{N}[f](x) = \frac{1}{(2\pi)^n} \int e^{i\langle x, \xi \rangle} \sigma_\mathcal{N}(x, \xi)\, \hat{f}(\xi)\,\mathrm{d}\xi$$

where

$$\sigma_\mathcal{N}(x, \xi) = e^{-i\langle x, \xi \rangle} \mathcal{N}[e^{i\langle \cdot, \xi \rangle}](x) = \int_{\mathbb{S}^{n-1}} \int_0^\infty e^{i\langle \rho\,\theta, \xi \rangle}\, A'(x, \rho, \theta)\,\mathrm{d}\rho\,\mathrm{d}\theta \tag{2.26}$$

is the symbol of $\mathcal{N}$. Since A' has compact support there is no problem differentiating under the integral in (2.26), and so we see that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta}\sigma_{\mathcal{N}}(x,\xi)\right| = \left|\int_{\mathbb{S}^{n-1}}\int_0^\infty (i\rho)^{|\beta|}(\theta)^{\beta}e^{i\langle \rho\,\theta,\xi\rangle}\left(\frac{\partial}{\partial x}\right)^{\alpha}A'(x,\rho,\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta\right|.$$

If $\xi \neq 0$, we can make the change of variables $r = \rho|\xi|$ to get

$$\begin{aligned}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta}\sigma_{\mathcal{N}}(x,\xi)\right| &= |\xi|^{-1-|\beta|}\left|\int_{\mathbb{S}^{n-1}}\int_0^\infty (ir)^{|\beta|}(\theta)^{\beta}e^{i\langle r\,\theta,\frac{\xi}{|\xi|}\rangle}\left(\frac{\partial}{\partial x}\right)^{\alpha}A'(x,\frac{r}{|\xi|},\theta)\,\mathrm{d}r\,\mathrm{d}\theta\right| \\ &\leq C_{\alpha,\beta}\,|\xi|^{-1-\beta}.\end{aligned}$$

Therefore $\sigma_{\mathcal{N}} \in S^{-1}(\mathbb{R}^n \times \mathbb{R}^n)$, and so $\mathcal{N}$ is a ΨDO of order -1 as claimed.

We now assume that A' is even in (ρ,θ) and complete the proof by showing that the asymptotic expansion (2.25) holds. To accomplish this we first expand A' in a Taylor series around $\rho = 0$ as follows

$$A'(x,\rho,\theta) = \sum_{k=0}^{m}\frac{\rho^k}{k!}\frac{\partial^k A'}{(\partial\rho)^k}(x,0,\theta) + \rho^{m+1}R_{m+1}(x,\rho,\theta).$$

By the assumption that A' is even in (ρ,θ), for each k the function $\partial^k A'/(\partial\rho)^k(x,0,\theta)$ has parity in θ corresponding to the parity of k. To find the asymptotic expansion for $\sigma_{\mathcal{N}}$ we split up the integral in (2.26) into a sum of terms, which must now be interpreted as tempered distributions, each corresponding to one of the terms in the Taylor series. Note that each of these terms is $(2\pi)^n e^{-i\langle x,\xi\rangle}$ times the inverse Fourier transform of the term in the Taylor series frozen at x. We will analyze these integrals individually. The general term is

$$\int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{r'^k}{k!}\frac{\partial^k A'}{(\partial r')^k}(x,0,\omega')\,e^{i\langle r'\omega',\xi\rangle}\,\mathrm{d}r'\,\mathrm{d}\omega'$$

which, for each x, can be written as the following limit in the sense of distributions

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{r'^k}{k!}\frac{\partial^k A'}{(\partial r')^k}(x,0,\omega')\,e^{i\langle r'\omega',\xi\rangle-\epsilon r'^2}\,\mathrm{d}r'\,\mathrm{d}\omega'.$$

Using the symmetry of A' in ω' described above this can be manipulated by changing variables $(r',\omega') \mapsto (-r',-\omega')$ in half of the integral to obtain

$$\lim_{\epsilon\to 0^+}\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^\infty \frac{r'^k}{k!}\frac{\partial^k A'}{(\partial r')^k}(x,0,\omega')\,e^{i\langle r'\omega',\xi\rangle-\epsilon r'^2}\,\mathrm{d}r'\,\mathrm{d}\omega'.$$

Performing the integral in r' now gives

$$\lim_{\epsilon\to 0^+} \frac{\sqrt{\pi}}{2i^k k!} \int_{\mathbb{S}^{n-1}} \frac{\partial^k A'}{(\partial r')^k}(x,0,\omega') \left(\frac{\partial^k}{\partial s^k} \frac{e^{\frac{-s^2}{4\epsilon}}}{\sqrt{\epsilon}} \right|_{s=\langle \omega',\xi\rangle} \mathrm{d}\omega'.$$

In the sense of distributions

$$\lim_{\epsilon\to 0^+} \frac{e^{\frac{-s^2}{4\epsilon}}}{\sqrt{\epsilon}} = 2\sqrt{\pi}\,\delta$$

and so by the continuity of the derivative and pull-back operations on distributions the limit of the last integral becomes

$$\frac{\pi}{i^k k!} \left\langle f_\xi^* \delta^k, \frac{\partial^k A'}{(\partial r')^k}(x,0,\cdot) \right\rangle = |\xi|^{-1-k} \frac{\pi}{i^k k!} \left\langle f_{\frac{\xi}{|\xi|}}^* \delta^k, \frac{\partial^k A'}{(\partial r')^k}(x,0,\cdot) \right\rangle$$

where we have used properties of the pull-back and the delta function to show the last equality.

To complete the proof we must show that the contribution to $\sigma_{\mathcal{N}}$ coming from the remainder $r^{m+1}R(x,r,\omega)$ in the Taylor series expansion satisfies the proper estimates. Indeed, by the above argument and the linearity of the inverse Fourier transform, we can see that the following tempered distribution

$$\tilde{R}_{m+1}(x,\xi) = \mathcal{F}_y^{-1}\left\{ |y|^{m-n+2} R_{m+1}(x,|y|,\frac{y}{|y|}) \right\}(\xi) \tag{2.27}$$

is a smooth function of ξ away from $\{\xi = 0\}$. By [11, Proposition 18.1.4] to complete the proof it is sufficient to show that

$$|\tilde{R}_{m+1}(x,\xi)| \le C|\xi|^{-1-m} \tag{2.28}$$

for some constant C, and $|\xi|$ sufficiently large. To accomplish this we will split the argument of the Fourier transform in (2.27) into two pieces. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a bump function that is equal to 1 in the neighborhood $B_l(0)$ of the origin. Then $\phi(y)|y|^{m-n+2} R_{m+1}(x,|y|,\frac{y}{|y|})$ is compactly supported and integrable in y. Thus we may apply precisely the same argument that we used on A' in the first paragraph of this proof to show that the estimate (2.28) holds for $\mathcal{F}_y^{-1}\{\phi(y)|y|^{m-n+2} R_{m+1}(x,|y|,\frac{y}{|y|})\}$. Now let us consider $(1 - \phi(y))|y|^{m-n+2} R_{m+1}(x,|y|,\frac{y}{|y|})$ which is smooth although not integrable in y. By the formula for the remainder in the Taylor series, and the fact that A' is compactly supported we can see that this function is homogeneous of order $m-n+1$ for $|y|$ sufficiently large. We consider the inverse Fourier transform as an oscillatory integral

$$\int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} \left((1-\phi(y))\, |y|^{m-n+2} R_{m+1}(x,|y|,\frac{y}{|y|}) \right) \mathrm{d}y.$$

Here we have ignored the factor of $(2\pi)^n$ to simplify the notation. Since the function in parentheses is homogeneous of degree $m-n+1$ when $|y|$ is suffiently large, if k is large enough this distribution is represented by the following function away from $\xi = 0$

$$R(x,\xi) = \int_{\mathbb{R}^n} \frac{e^{i\langle y,\xi\rangle}}{|\xi|^{2k}} (-\Delta_y)^k \left[(1-\phi(y))\, |y|^{m-n+2} R_{m+1}(x,|y|,\frac{y}{|y|}) \right] \mathrm{d}y.$$

If we assume that $|\xi| = 1$ and take any $\lambda > 1$, then by changing variables to $y' = \lambda y$ the previous formula gives

$$|R(x,\lambda\xi)| = \lambda^{-2k-n} \left| \int_{\mathbb{R}^n} e^{i\langle y',\xi\rangle} (-\Delta_y)^k \left((1-\phi(y)\, |y|^{m-n+2} R_{m+1}(x,|y|,\frac{y}{|y|}) \right|_{y=y'/\lambda} \mathrm{d}y' \right|.$$

The differentiated function in this integral is homogeneous of degree $m-n+1-2k$ for $|y|$ sufficiently large, and is thus bounded by $C|y|^{m-n+1-2k}$. Therefore

$$|R(x,\lambda\xi)| \leq \lambda^{-2k-n} \int_{\mathbb{R}^n \setminus B_l(0)} C|y'|^{m-n+1-2k} \lambda^{-m+n-1+2k} \mathrm{d}y' \leq C\lambda^{-1-m}$$

where the constants C change at each step. This completes the proof.

□

If $\mathcal{N}$ is an operator as in Theorem 6, then its principal symbol is given by the first term in the asymptotic series (2.25), which is

$$\sigma_{\mathcal{N},p}(x,\xi) = |\xi|^{-1}\, \pi \int_{\omega' \in \mathbb{S}^{n-1} \cap \xi^\perp} A'(x,0,\omega')\, \mathrm{d}\mathbb{H}_{\omega'}.$$

Here $\xi^\perp \subset T^{\mathbb{R}}_{\pi(\xi)} \mathbb{R}^n$ is the set of vectors annihilated by the covector ξ and $\mathrm{d}\mathbb{H}_{\omega'}$ is the natural (Hausdorff) measure on the set $\mathbb{S}^{n-1} \cap \xi^\perp$. Now we return to the case of the normal operators $\mathcal{N}_{w_1,w_2}$ defined by (2.4). In this case A' is given by (2.23), and so

$$A'(x,0,\omega') = w_2(\omega(0,\omega'))\, \overline{w_1(\omega(0,\omega'))} + w_2(-\omega(0,\omega'))\, \overline{w_1(-\omega(0,\omega'))}$$

Therefore

$$\sigma_{\mathcal{N}_{w_1,w_2},p}(x,\xi) = |\xi|_g^{-1}\, 2\pi \int_{\Omega_x^{\mathbb{R}}M\cap\xi^{\perp}} w_2(\omega)\, \overline{w_1(\omega)}\, \mathrm{d}\omega. \tag{2.29}$$

Applying Theorem 6 to our case and using (2.29) we have the following corollary.

Corollary 1 *If M is a simple manifold, w_1 and $w_2 \in C^\infty(TM)$ are weight functions, and $\mathcal{N}_{w_1,w_2}$ is the normal operator defined by (2.4), then $\mathcal{N}_{w_1,w_2}$ is a ΨDO of order -1 with principal symbol $\sigma_{\mathcal{N}_{w_1,w_2},p} \in C^\infty((T^*)^{\mathbb{R}}M \setminus \{0\})$ given by (2.29).*

From the expression (2.29) for $\sigma_{\mathcal{N}_{w_1,w_2},p}$ it is possible to find conditions on w_1 and w_2 that are sufficient to imply that $\mathcal{N}_{w_1,w_2}$ is elliptic. For example, if w_1 and w_2 are both real and ≥ 0, then for $\mathcal{N}_{w_1,w_2}$ to be elliptic it is sufficient to require that for every $\xi \in (T^*)^{\mathbb{R}}M \setminus \{0\}$ there exists an $\omega \in \Omega_x^{\mathbb{R}}M \cap \xi^{\perp}$ such that $w_1(\omega)$, $w_2(\omega) \neq 0$. If $w_1 = w_2$, then $\mathcal{N}_{w_1,w_2}$ is elliptic if for every $\xi \in (T^*)^{\mathbb{R}}M \setminus \{0\}$ there exists an $\omega \in \Omega_x^{\mathbb{R}}M \cap \xi^{\perp}$ such that $w_1(\omega) \neq 0$.

It is also useful to note that Theorem 6 implies that $\mathcal{N}_{w_1,w_2}$ extends to a continuous operator from $\mathcal{E}'(M)$ to $\mathcal{D}'(M)$, and that for any $s \in \mathbb{R}$, $\mathcal{N}_{w_1,w_2} : H_c^s(M) \to H_{loc}^{s+1}(M)$ is continuous (see [29]). Here $H_c^s(M)$ refers to the subspace of $H^s(M)$ consisting of distributions with compact support. A sequence converges to zero in $H_c^s(M)$ if all its elements have support within a given compact set, and they converge to zero in the $H^s(M)$ norm. The space $H_{loc}^s(M)$ is the set of distributions on M^{int} that are in $H^s(K)$ for every compact $K \Subset M^{int}$. The topology of $H_{loc}^s(M)$ is that of H^s convergence on compact sets. By taking $w = w_1 = w_2$ in Corollary 1, we have the following result concerning the continuity of I_w.

Corollary 2 *If M is a simple manifold, and $w \in C^\infty(TM)$, then I_w is a continuous operator from $L_c^2(M)$ to $L_c^2(\partial_-\Omega^{\mathbb{R}}M)$.*

Proof: If $f \in C_c^\infty(M)$ has support contained in a given compact set $K \Subset M^{int}$, then $I_w[f]$ has support within a corresponding compact subset $K' \Subset \partial_-\Omega^{\mathbb{R}}M$. Thus, applying Corollary 1 with $w = w_1 = w_2$ and using the continuity properties of ΨDOs we have

$$\|I_w[f]\|^2_{L^2(\partial_-\Omega^{\mathbb{R}}M)} = \|I_w[f]\|^2_{L^2(K')} = \langle \mathcal{N}[f], f\rangle_{L^2(K')} \leq C\|f\|^2_{L^2(M)}.$$

Thus I_w is a continuous operator on $L_c^2(M)$ by approximation.

□

We now apply corollary 2 to prove a continuity result for I_w^*.

Theorem 7 *I_w^* maps $H_c^1(\partial_-\Omega^{\mathbb{R}}M)$ to $H_{loc}^1(M)$ continuously.*

Proof: Suppose that $f \in C_c^\infty(\partial_-\Omega^{\mathbb{R}}M)$ has support contained within a given compact set $K' \Subset \partial_-\Omega^{\mathbb{R}}M$, and suppose that $h \in L_c^2(M)$ with $\|h\|_{L^2(M)} = 1$ has support contained within a compact set $K \Subset M^{int}$. Then by corollary 2

$$\langle h, I_w^*[f]\rangle_{L^2(M)} = \langle I_w[h], f\rangle_{L^2(\partial_-\Omega^{\mathbb{R}}M)} \leq C\|h\|_{L^2(M)}\|f\|_{L^2(\partial_-\Omega^{\mathbb{R}}M)} = C\|f\|_{L^2(\partial_-\Omega^{\mathbb{R}}M)}$$

for some constant $C > 0$ possibly depending on K. This shows that $I_w^* : L^2(\partial_-\Omega^{\mathbb{R}}M) \to L_{loc}^2(M)$ is continuous.

The remainder of the proof is very similar to the proof of Theorem 4, but now we use the fact that we have already established the L^2 continuity in order to bootstrap up to H^1 continuity. In parallel to the proof of Theorem 4, let $U \Subset M$ be the domain of some coordinates $(x^1, \dots, x^n)$ and assume that we have an orthonormal frame $\{E_1, \dots, E_n\}$ for TU. Suppose that $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ are the corresponding coordinates for TU, and also introduce a cut-off function $\phi \in C_c^\infty(U)$. To prove the continuity property claimed it is now sufficient to show that for any f and h as defined in the previous paragraph

$$\langle h, \phi \frac{\partial}{\partial x^i} I_w^*[f]\rangle_{L^2(M)} \leq C\|f\|_{H^1(\partial_-\Omega^{\mathbb{R}}M)}, \tag{2.30}$$

since we may apply a partition of unity on K by functions like ϕ with support contained in the domains of coordinate charts to estimate the full $H^1(K)$ norm of $I_w^*[f]$ by using (2.30) in each chart. Still following the proof of Theorem 4 we introduce a partition of unity on K' consisting of functions $\{\psi_j\}_{j=1}^l$ which all lie in $C_c^\infty(\partial_-\Omega^{\mathbb{R}}M)$, and such that the support of each ψ_j is contained within the domain of a single coordinate chart. For each j, suppose these coordinates are given by $(y_j^1, \dots, y_j^{2n-2})$. Using the coordinates $(x, \xi) = (x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ on TU, (2.13) becomes

$$I_w^*[f](x) = \sum_{j=1}^{l} \int_{\mathbb{S}^{n-1}} \overline{w(x,\xi)}\, \psi_j(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi)))\, f(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi))\, \mathrm{d}\mathbb{H}_{\mathbb{S}^{n-1}}(\xi). \tag{2.31}$$

Now we may calculate $\frac{\partial}{\partial x^i} I_w^*[f](x)$ by differentiating under the integral in (2.31). Doing this we obtain

$$\phi \frac{\partial}{\partial x^i} I_w^*[f](x) = \sum_{j=1}^{l} \phi \Bigg(\int_{\mathbb{S}^{n-1}} \overline{\frac{\partial w}{\partial x^i}(x,\xi)}\, \psi_j(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi)))\, f(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi)))$$

$$+\overline{w(x,\xi)}\, \frac{\partial}{\partial x^i}(p_\Omega^{-1}(x,\xi))^k \frac{\partial \psi_j}{\partial y_j^k}(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi)))\, f(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi)))$$

$$+\, \overline{w(x,\xi)}\, \frac{\partial}{\partial x^i}(p_\Omega^{-1}(x,\xi))^k\, \psi_j(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi)))$$

$$\cdot \frac{\partial f}{\partial y_j^k}(\pi_{\partial_-\Omega^{\mathbb{R}}M}(p_\Omega^{-1}(x,\xi)))\, \mathrm{d}\mathbb{H}_{\mathbb{S}^{n-1}}(\xi) \Bigg).$$

From this equation we see that $\phi \frac{\partial}{\partial x^i} I_w^*[f](x)$ is given as a sum of terms $I_{w_{ij}}^*[f](x)$ and $I_{w_{ij}^k}^*[\frac{\partial f}{\partial y_j^k}](x)$ where the w_{ij} and w_{ij}^k are an indexed set of weights, and $\frac{\partial f}{\partial y_j^k}$ are derivatives of f in local coordinates that are defined on the required respective domains. These weights are all in $C_c^\infty(\Omega M)$ and depend on ϕ as well as the choice of partition $\{\psi\}_{j=1}^l$, but not on either f or h. Therefore (2.30) holds by applying the same argument that established the L^2 continuity of I_w^* to each of these terms. Now the constant C may depend on both K and K'. This completes the proof.

□

Remark 5: A similar argument to that found in the proof of Theorem 7 could also be used to show that $I_w^* : H_c^k(\partial_-\Omega^{\mathbb{R}}M) \to H_{loc}^k(M)$ is continuous for any non-negative integer k. However, we will not need the result for $k > 1$, and so have not included that case in the theorem.

Chapter 3

THE LINEARIZED PROBLEM

We now return to the linearization of the polarization problem described at the beginning of the previous chapter. This linearized problem is the inversion of the transform I_{U_1,U_2} defined by (2.1) which takes $f \in \tau_1^1(M)$ to a (possibly rough) section of the vector bundle $B_2(\partial_-\Omega^{\mathbb{R}}M)$ defined in Chapter 1. For each $\xi \in \partial_-\Omega^{\mathbb{R}}M$ and $\eta,\ \zeta \in T_{\pi(\xi)}M$, the value of $I_{U_1,U_2}[f](\eta,\zeta)$ is given by (2.1). We will assume that M is a simple manifold (see definition 3), and also assume that we have a set of global coordinates $(x^1,\ \dots,x^n)$ for M, which provide natural global coordinates on TM as well. If we simply write out (2.1) with respect to these coordinates, taking $R(\dot\gamma_{x,v}(s))_{ab} = g(\gamma_{x,v}(s))_{dj}\,(\mathcal{I}_{0,s}^{\gamma_{x,v}})_b^j\,(U_2^{-1})(\dot\gamma_{x,v}(s))_a^d$ and $Q(\dot\gamma_{x,v}(s))_k^m = (U_1)(\dot\gamma_{x,v}(s))_p^m\,(\mathcal{I}_{0,s}^{\gamma_{x,v}})_k^p$, we have

$$((I_{U_1,U_2})[f])(x,v)_{kb} = \int_0^l R(\dot\gamma_{x,v}(s))_{ab}\,[P_{\dot\gamma_{x,v}(s)}\,f](\gamma_{x,v}(s))_m^a\,Q(\dot\gamma_{x,v}(s))_k^m\,\mathrm{d}s. \tag{3.1}$$

To fully analyze this operator in coordinates we also need the expansion

$$[P_v\,f]_m^a = \left(\delta_r^a - \frac{v^a\,v_r}{|v|_g^2}\right)\,f_u^r\,\left(\delta_m^u - \frac{v^u\,v_m}{|v|_g^2}\right). \tag{3.2}$$

Using (3.2) we can expand the integrand in (3.1) in terms of the components f_u^r. From this, we can see that $((I_{U_1,U_2})[f])(\xi)_{kb}$ is a sum of weighted X-ray transforms of each of these components. Therefore, if f has support compactly contained in the interior of M, then we may apply Theorem 3 to conclude that each of the components of $I_{U_1,U_2}[f]$ are in $C_c^\infty(\partial_-\Omega M)$. This reasoning gives the following theorem.

Theorem 8 *The map $I_{U_1,U_2} : (\tau_1^1)_c(M) \to (\beta_2)_c(\partial_-\Omega M)$ is continuous.*

Much of the remainder of our analysis of this transform, I_{U_1,U_2}, will proceed in a very similar manner. We will look at the individual components of I_{U_1,U_2} in coordinates as individual X-ray transforms and apply the results of Chapter 2.

In section 3.1 we define and analyze the normal operator $\mathcal{N}_{U_1,U_2}$ in a manner analogous to the treatment of $\mathcal{N}_{w_1,w_2}$ in Chapter 2. Having done this we apply the results to the linear problem in section 3.2. Finally, to complete our study of the linear problem in section 3.3 we apply analytic microlocal methods to prove generic injectivity.

3.1 The operators $I^*_{U_1,U_2}$ and $\mathcal{N}_{U_1,U_2}$

In order to define $\mathcal{N}_{U_1,U_2}$ we first introduce some new norms and spaces. Both T^1_1M and $B_2(\partial_-\Omega^{\mathbb{R}}M)$ are vector bundles with inner products on each fiber induced by the metric g. These are defined with respect to any given frame as

$$\langle f,h\rangle_{(T^1_1)_xM} = f^r_u\, g_{r\alpha}(x)\, g^{u\epsilon}(x)\, \overline{h}^{\alpha}_{\epsilon} \quad \text{and} \quad \langle F,H\rangle_{(B_2)_{(x,v)}(\partial_-\Omega^{\mathbb{R}}M)} = F_{kb}\, g^{k\kappa}(x)\, g^{b\nu}(x)\, \overline{H}_{\kappa\nu}.$$

Using these we may introduce the corresponding L^2 inner products and spaces for the sections of each vector bundle (see for example [16]). These inner products are given by

$$\langle f,h\rangle_{L^2\tau^1_1(M)} = \int_M \langle f,h\rangle_{(T^1_1)_xM}\,\mathrm{d}v_g \tag{3.3}$$

and

$$\langle F,H\rangle_{L^2\beta_2(\partial_-\Omega^{\mathbb{R}}M)} = \int_{\partial_-\Omega^{\mathbb{R}}M} \langle F,H\rangle_{(B_2)_\xi(\partial_-\Omega^{\mathbb{R}}M)}\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}. \tag{3.4}$$

We now define the transpose $I^*_{U_1,U_2}$ in an analogous manner to the definition of I^*_w in Chapter 2. Indeed, for $F\in\beta_2(\partial_-\Omega^{\mathbb{R}}M)$ and $h\in\tau^1_1(M)$ we define $I^*_{U_1,U_2}[F]$ by

$$\langle I^*_{U_1,U_2}[F],h\rangle_{L^2\tau^1_1(M)} = \langle F, I_{U_1,U_2}[h]\rangle_{L^2\beta_2(\partial_-\Omega^{\mathbb{R}}M)}.$$

Similarly we define the normal operator $\mathcal{N}_{U_1,U_2}$ by

$$\mathcal{N}_{U_1,U_2} = I^*_{U_1,U_2}\circ I_{U_1,U_2}. \tag{3.5}$$

The important properties of this operator are contained in the next two theorems. As we might expect from section 1.3, there is a difference between the 3 dimensional case, and the case of more than 3 dimensions. We will first deal with the case of greater than 3 dimensions since it is simpler, and most of the analysis there actually applies to the 3 dimensional case as well.

Theorem 9 *If M is a simple manifold, then the operator $\mathcal{N}_{U_1,U_2}$ defined by (3.5) is a ΨDO of order -1 on the sections of the vector bundle $T^1_1 M$. Furthermore, if the dimension of M is greater than 3, then $\mathcal{N}_{U_1,U_2}$ is elliptic.*

Proof: The first step is derive an integral formula for $\mathcal{N}_{U_1,U_2}$ similar to the one found in section 2.3 for $\mathcal{N}_{w_1,w_2}$. Indeed, as in that section we have according to (3.4)

$$\begin{aligned}
\langle \mathcal{N}_{U_1,U_2}[f],\, h\rangle_{L^2\tau^1_1(M)} &= \langle I_{U_1,U_2}[f], I_{U_1,U_2}[h]\rangle_{L^2\beta_2(\partial_-\Omega^{\mathbb{R}}M)} \\
&= \Bigg(\int_{\partial_-\Omega^{\mathbb{R}}M} |\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)|\, g(x)^{bb'}\, g(x)^{kk'} \times \\
&\quad \left(\int_0^{l(x,v)} R(\dot\gamma_{x,v}(s))_{ab}\, [P_{\dot\gamma_{x,v}(s)}\, f](\gamma_{x,v}(s))^a_m\, Q(\dot\gamma_{x,v}(s))^m_k\, \mathrm{d}s \right) \times \\
&\quad \overline{\left(\int_0^{l(x,v)} R(\dot\gamma_{x,v}(t))_{a'b'}\, [P_{\dot\gamma_{x,v}(t)}\, h](\gamma_{x,v}(t))^{a'}_{m'}\, Q(\dot\gamma_{x,v}(t))^{m'}_{k'}\, \mathrm{d}t \right)} \Bigg) \\
&= \Bigg(\int_{\partial_-\Omega^{\mathbb{R}}M} |\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)| \times \\
&\quad \Bigg(\int_0^{l(x,v)} (\mathcal{I}^{\gamma_{\dot\gamma_{x,v}(s)}}_{0,-s})^{b'}_b\, (U_2^{-1})(\dot\gamma_{x,v}(s))^b_a\, [P_{\dot\gamma_{x,v}(s)}\, f](\gamma_{x,v}(s))^a_m \times \\
&\quad (U_1)(\gamma_{x,v}(s))^m_p\, (\mathcal{I}^{\gamma_{\dot\gamma_{x,v}(s)}}_{0,-s})^{k'}_k\, g(\gamma_{x,v}(s))^{kp}\, \mathrm{d}s \Bigg) \times \\
&\quad \overline{\left(\int_0^{l(x,v)} R(\dot\gamma_{x,v}(t))_{a'b'}\, [P_{\dot\gamma_{x,v}(t)}\, h](\gamma_{x,v}(t))^{a'}_{m'}\, Q(\dot\gamma_{x,v}(t))^{m'}_{k'}\, \mathrm{d}t \right)} \Bigg).
\end{aligned}$$

In this last inequality we have used the following property of parallel translation in relation to the metric

$$g(x)^{bb'}\,(\mathcal{I}^{\gamma_{x,v}}_{0,s})^j_b = (\mathcal{I}^{\gamma_{x,v}}_{s,0})^{b'}_o\,(\mathcal{I}^{\gamma_{x,v}}_{0,s})^o_\rho\, g(x)^{\rho b}\,(\mathcal{I}^{\gamma_{x,v}}_{0,s})^j_b = (\mathcal{I}^{\gamma_{x,v}}_{s,0})^{b'}_b\, g(\gamma_{x,v}(s))^{bj}.$$

We could also use (3.2) to fully expand the above integrands in coordinates, but will not because the notation already seems excessive. With such a full expansion we would have a sum of the form

$$\int_{\partial_-\Omega^{\mathbb{R}}M} \Big(\int_0^{l(x,v)} w_2(\dot\gamma_{x,v}(s))^{uk'b'}_r\, f(\gamma_{x,v}(s))^r_u\, \mathrm{d}s \Big) \times \overline{\left(\int_0^{l(x,v)} w_1(\dot\gamma_{x,v}(t))^{\epsilon}_{\alpha k'b'}\, h(\gamma_{x,v}(t))^{\alpha}_{\epsilon}\, \mathrm{d}t \right)} |\mathrm{d}V_{\partial_-\Omega^{\mathbb{R}}M}(x,v)|.$$

For any $(x, v) \in \Omega M$ the weights $w_1(x,v)^{\epsilon}_{\alpha k' b'}$ and $w_2(x,v)^{uk'b'}_{r}$ are given by

$$w_1(x,v)^{\epsilon}_{\alpha k'b'} = g(x)_{d'j'}\,(\mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0})^{j'}_{b'}\,(U_2^{-1})(x,v)^{d'}_{a'}\,[P_{x,v}]^{a'\epsilon}_{m'\alpha}\,(U_1)(x,v)^{m'}_{p'}\,(\mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0})^{p'}_{k'} \tag{3.6}$$

and

$$w_2(x,v)^{uk'b'}_{r} = (\mathcal{I}^{\gamma_{x,v}}_{0,-l(x,-v)})^{b'}_{b}\,(U_2^{-1})(x,v)^{b}_{a}\,[P_{x,v}]^{au}_{mr}\,(U_1)(x,v)^{m}_{p}\,(\mathcal{I}^{\gamma_{x,v}}_{0,-l(x,-v)})^{k'}_{k}\,g(x)^{kp}. \tag{3.7}$$

The components of $[P_{x,v}]^{a'\epsilon}_{m'\alpha}$ can be calculated from (3.2), although as we will see below this is unnecessary for our purposes. If we apply the steps from section 2.3 to each of these terms separately then the equivalent of (2.16) in this case is (note that since we have global coordinates $\pi(U) = M$)

$$\begin{aligned}\int_M \overline{h(x)^{\alpha}_{\epsilon}}\Bigg(\int_{\mathcal{F}_x\setminus\{0\}} & w_2(\mathrm{Exp}_x(v))^{uk'b'}_{r}\,\overline{w_1(v/|v|_g)^{\epsilon}_{\alpha k'b'}}\\ &\times f(\mathrm{exp_x}(v))^{r}_{u}\,\sqrt{\det(g)}\,\frac{\mathrm{d}v}{|v|_g^{n-1}}\Bigg)\sqrt{\det(g)}\,\mathrm{d}x\\ &+\int_M \overline{h(x)^{\alpha}_{\epsilon}}\Bigg(\int_{\mathcal{F}_x\setminus\{0\}} w_2(-\mathrm{Exp}_x(v))^{uk'b'}_{r}\,\overline{w_1(-v/|v|_g)^{\epsilon}_{\alpha k'b'}}\\ &\qquad\times f(\mathrm{exp_x}(v))^{r}_{u}\,\sqrt{\det(g)}\,\frac{\mathrm{d}v}{|v|_g^{n-1}}\Bigg)\sqrt{\det(g)}\,\mathrm{d}x.\end{aligned}$$

Considering now (3.3), we can see that

$$\begin{aligned}\mathcal{N}_{U_1,U_2}[f](x)^{\alpha}_{\epsilon} = g(x)^{\alpha\alpha'}\,g(x)_{\epsilon\epsilon'}\Bigg(&\int_{\mathcal{F}_x\setminus\{0\}} w_2(\mathrm{Exp}_x(v))^{uk'b'}_{r}\,\overline{w_1(v/|v|_g)^{\epsilon'}_{\alpha'k'b'}}\\ &\times f(\mathrm{exp_x}(v))^{r}_{u}\,\sqrt{\det(g)}\,\frac{\mathrm{d}v}{|v|_g^{n-1}}\\ &+\int_{\mathcal{F}_x\setminus\{0\}} w_2(-\mathrm{Exp}_x(v))^{uk'b'}_{r}\,\overline{w_1(-v/|v|_g)^{\epsilon'}_{\alpha'k'b'}}\\ &\qquad\times f(\mathrm{exp_x}(v))^{r}_{u}\,\sqrt{\det(g)}\,\frac{\mathrm{d}v}{|v|_g^{n-1}}\Bigg).\end{aligned} \tag{3.8}$$

By continuing to analyze each of the terms in this sum separately, the remaining results in Chapter 2 show that this is a system of ΨDOs of order -1, and therefore a ΨDO on the sections of the vector bundle $T^1_1(M)$. Furthermore, the principal symbol for this system is given, according to (2.29), by

$$\sigma_{\mathcal{N}_{U_1,U_2},p}(x,\xi)^{u\alpha}_{r\epsilon} = |\xi|_g^{-1} g(x)^{\alpha\alpha'}\, g(x)_{\epsilon\epsilon'}\, 2\pi \int_{\Omega_x^{\mathbb{R}} M\cap\xi^\perp} w_2(\omega)_r^{uk'b'}\, \overline{w_1(\omega)^{\epsilon'}_{\alpha'k'b'}}\, \mathrm{d}\omega. \tag{3.9}$$

Recall that $\xi^\perp$ is the set of vectors annihilated by the covector ξ. The principal symbol $\sigma_{\mathcal{N}_{U_1,U_2},p}(x,\xi)$ is a linear map from $(T_1^1)_x(M)$ to $(T_1^1)_x(M)$ for each $(x,\xi)\in(T^*)^{\mathbb{R}}M\setminus\{0\}$.

We will now prove that $\mathcal{N}_{U_1,U_2}$ is elliptic by showing that $\sigma_{\mathcal{N}_{U_1,U_2},p}(x,\xi)$ is injective at every point $(x,\xi)\in(T^*)^{\mathbb{R}}M\setminus\{0\}$. Indeed, suppose that $f\in(T_1^1)_xM$ is not zero. We must show that this implies that $\sigma_{\mathcal{N}_{U_1,U_1},p}(x,\xi)[f]$ is not zero. By (3.9) we have

$$\langle\sigma_{\mathcal{N}_{U_1,U_1},p}(x,\xi)[f],f\rangle_{(T_1^1)_xM} = \frac{2\pi}{|\xi|_g}\int_{\Omega_x^{\mathbb{R}} M\cap\xi^\perp} w_2(\omega)_r^{uk'b'}\,\overline{w_1(\omega)^{\epsilon}_{\alpha k'b'}}\, f_u^r\,\overline{f_\epsilon^\alpha}\,\mathrm{d}\omega.$$

Applying (3.6), and (3.7) we can rewrite the integrand in this formula invariantly as

$$\|U_2^{-1}(\omega)\circ[P_\omega f]\circ U_1(\omega)\|^2_{(T_1^1)_xM}. \tag{3.10}$$

Since this quantity is always non-negative, to show that $\sigma_{\mathcal{N}_{U_1,U_1},p}(x,\xi)[f]$ is not zero it is sufficient to prove that $U_2^{-1}(\omega)\circ[P_\omega f]\circ U_1(\omega)$ is not zero at some point $\omega\in\Omega_x^{\mathbb{R}}M\cap\xi^\perp$. We note here that all of the steps in the proof until now apply equally well in dimension 3. It is the next step that requires dimension greater than 3. Since f is not zero, there exists some $v\in T_xM$ such that $f(v)\neq 0$. When the dimension is greater than 3, it is always possible to find a vector $\omega_v\in\Omega_x^{\mathbb{R}}M\cap\xi^\perp$ that is simultaneously perpendicular to v and $f(v)$. Given such an ω_v we have

$$\langle U_2^{-1}(\omega_v)\circ[P_{\omega_v} f]\circ U_1(\omega_v)(U_1^{-1}(\omega_v)v),\, U_2^*(\omega_v)f(v)\rangle_{g(x)} = \|f(v)\|^2_{g(x)} > 0.$$

Therefore $U_2^{-1}(\omega_v)\circ[P_{\omega_v} f]\circ U_1(\omega_v)\neq 0$, and so $\mathcal{N}_{U_1,U_2}$ is elliptic.

□

We now turn to the case of dimension 3. For the statement of the theorem in this case we use the operator d_β defined by (1.15), and take Λ^2 to be a ΨDO on M which is a parametrix for the positive Laplace-Beltrami operator $-\Delta_g$.

Theorem 10 *If M is simple and the dimension of M is 3, then the system of ΨDOs $(\mathcal{N}_{U_1,U_2},\Lambda^2\,\mathrm{d}_\beta)^T$ from sections of $T_1^1(M)$ to sections of $T_1^1(M)\bigoplus\Lambda^0(M)$ is elliptic. Furthermore, this remains true if $\Lambda^2\,\mathrm{d}_\beta$ is replaced with the same operator with respect to any other metric g' that is sufficiently close to g.*

Proof: We will first calculate the operator $\mathrm{d}_\beta : \tau^1_1(M) \to \Lambda^0(M)$ in coordinates and then determine its principal symbol. Taking any $f \in \tau^1_1(M)$ we have

$$\begin{aligned}
\mathrm{d}_\beta(f^j_i \,\mathrm{d}x^i \otimes \tfrac{\partial}{\partial x^j}) &= *\mathrm{d}\left(\sum_{i<j}(g_{ik}\, f^k_j - g_{jk}\, f^k_i)\,\mathrm{d}x^i \wedge \mathrm{d}x^j\right) \\
&= \tfrac{1}{\sqrt{\det(g)}}\Big(\tfrac{\partial(g_{1k}\, f^k_2)}{\partial x^3} - \tfrac{\partial(g_{2k}\, f^k_1)}{\partial x^3} - \tfrac{\partial(g_{1k}\, f^k_3)}{\partial x^2} + \tfrac{\partial(g_{3k}\, f^k_1)}{\partial x^2} \\
&\quad + \tfrac{\partial(g_{2k}\, f^k_3)}{\partial x^1} - \tfrac{\partial(g_{3k}\, f^k_2)}{\partial x^1}\Big)
\end{aligned}$$

From this we can see that the principal symbol $\sigma_{\mathrm{d}_\beta,p}(x,\xi)$ is given by

$$\sigma_{\mathrm{d}_\beta,p}(x,\xi)[f] = \frac{i}{\sqrt{\det(g)}}(\xi_1(g_{2k}\, f^k_3 - g_{3k}\, f^k_2) - \xi_2(g_{1k}\, f^k_3 - g_{3k}\, f^k_1) + \xi_3(g_{1k}\, f^k_2 - g_{2k}\, f^k_1)).$$

Since composition of ΨDOs corresponds to multiplication at the level of principal symbols and the principal symbol of Λ^2 is $|\xi|^{-2}_g$, this implies that the principal symbol $\sigma_{\Lambda^2\,\mathrm{d}_\beta,p}$ of $\Lambda^2\,\mathrm{d}_\beta$ is given by

$$\sigma_{\Lambda^2\,\mathrm{d}_\beta,p}(x,\xi)[f] = \frac{i}{|\xi|^{-2}_g\sqrt{\det(g)}}(\xi_1(g_{2k}\, f^k_3 - g_{3k}\, f^k_2) - \xi_2(g_{1k}\, f^k_3 - g_{3k}\, f^k_1) + \xi_3(g_{1k}\, f^k_2 - g_{2k}\, f^k_1)). \tag{3.11}$$

To complete the proof we must show that for any $(x,\xi) \in (T^*)^{\mathbb{R}}M \setminus \{0\}$, the map

$$(\sigma_{\mathcal{N}_{U_1,U_2},p}(x,\xi), \sigma_{\Lambda^2\,\mathrm{d}_\beta,p}(x,\xi))^T : (T^1_1)_x M \to (T^1_1)_x M \bigoplus \Lambda^0(M) \tag{3.12}$$

is injective.

Indeed, let us suppose that $f \in (T^1_1)_x M$ is such that $(\sigma_{\mathcal{N}_{U_1,U_2},p}, \sigma_{\Lambda^2\,\mathrm{d}_\beta,p})^T[f] = 0$. We must show that this implies $f = 0$. Most of the proof of Theorem 9 still applies to this case. In fact, following through that proof we see that it is sufficient to show that if $f \neq 0$ and $\sigma_{\Lambda^2\,\mathrm{d}_\beta}(x,\xi)[f] = 0$, then for any $(x,\xi) \in (T^*)^{\mathbb{R}}M$ there exists $\omega \in \Omega^{\mathbb{R}}_x M \cap \xi^\perp$ such that $U_2^{-1}(\omega) \circ [P_\omega f] \circ U_1(\omega)$ is not zero. Indeed, suppose that f satisfies these hypotheses and split f into symmetric and anti-symmetric parts by writing

$$f = f^s + f^a$$

where $f^s = (f + f^t)/2$ and $f^a = (f - f^t)/2$. Recall that f^t is the transpose of f given by (1.14). Since $f \neq 0$, one of f^s or f^a must be non-zero, and so we break the proof into two cases.

Case 1: f^s is not zero. In this case there must be a real vector $v \in T_xM$ such that $\langle f^s(v), v\rangle_{g(x)} \neq 0$. Take $\omega_v \in \Omega_x^{\mathbb{R}} M \cap \xi^\perp$ such that ω_v is perpendicular to v. Then since v is real we have

$$\langle U_2^{-1}(\omega_v) \circ [P_{\omega_v} f] \circ U_1(\omega_v)(U_1^{-1}(\omega_v)v), U_2^*(\omega_v)v\rangle_{g(x)} = \langle f(v), v\rangle_{g(x)} = \langle f^s(v), v\rangle_{g(x)} \neq 0.$$

Therefore $U_2^{-1}(\omega_v) \circ [P_{\omega_v} f] \circ U_1(\omega_v) \neq 0$, and the proof is complete in this case.

Case 2: f^s is zero, but f^a is not zero. Since this entire calculation takes place in a single fiber over M, we assume that we are working with respect to an orthonormal frame and so $g_{ij} = \delta_{ij}$. By assumption f is represented with respect to this frame by an anti-symmetric matrix

$$f = \begin{pmatrix} 0 & f_2^1 & f_3^1 \\ -f_2^1 & 0 & f_3^2 \\ -f_3^1 & -f_3^2 & 0 \end{pmatrix}.$$

If we also write $v = (v^1, v^2, v^3)^T$ in this frame, then it is easy to check that

$$f(v) = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \times \begin{pmatrix} f_3^2 \\ -f_3^1 \\ f_2^1 \end{pmatrix} \tag{3.13}$$

where $\times$ denotes the Euclidean cross product. Also, in this orthonormal coordinate frame (3.11) becomes

$$\sigma_{\Lambda^2 \mathrm{d}_\beta, p}(x, \xi)[f] = \frac{2i}{|\xi|^{-2}}(\xi_1 f_3^2 - \xi_2 f_3^1 + \xi_3 f_2^1).$$

Therefore, if we take $\omega = (f_3^2, -f_3^1, f_2^1)^T / |(f_3^2, -f_3^1, f_2^1)|$, then $\sigma_{\Lambda^2 \mathrm{d}_\beta}(x, \xi)[f] = 0$ implies that $\omega \cdot \xi = 0$ and so $\omega \in \Omega_x^{\mathbb{R}} \cap \xi^\perp$. Finally, if we take v to be nonzero and perpendicular to both ω and $\xi^\sharp$ (this is ξ with the index raised), then by (3.13) $f(v)$ must be nonzero, parallel to $\xi^\sharp$, and perpendicular to ω (see figure (3.1)). Therefore

$$\langle U_2^{-1}(\omega) \circ [P_\omega f] \circ U_1(\omega)(U_1^{-1}(\omega)v), U_2^*(\omega)\, \xi^\sharp\rangle_{g(x)} \neq 0.$$

This shows that the map (3.12) is injective which completes the proof except for the final statement. To prove the last statement of the theorem we simply note that if we replace $\Lambda^2 \mathrm{d}_\beta$ by the same operator with respect to a different metric g', then the principal symbol

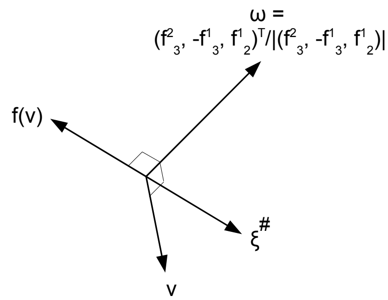

Figure 3.1: By the condition that $\sigma_{\Lambda^2\,\mathrm{d}_\beta,p}[f] = 0$, when ω is chosen as indicated ω and $\xi^\sharp$ are perpendicular. Then if v is perpendicular to both ω and $\xi^\sharp$, by (3.13) $f(v)$ will be perpendicular to both ω and v, and therefore necessarily parallel to $\xi^\sharp$.

of this new operator is given by (3.11) with g replaced by g'. Therefore, if $\|g - g'\|_{S_2M}$ is sufficiently small, then (3.12) is still injective when the replacement is made. This completes the proof of Theorem 10.

$\square$

Remark 6: In the proofs of both Theorem 9 and Theorem 10 the final step was to prove the algebraic fact that for f non-zero, and in the case of Theorem 10 satisfying the additional identity $\sigma_{\Lambda^2\,d_\beta,p}(x,\xi)[f] = 0$, the semi-basic tensor field $U_2^{-1}(\omega)\,[P_\omega f]\,U_1(\omega)$ cannot vanish for all $\omega \in \Omega_x^{\mathbb{R}} M \cap \xi^\perp$. This fact will also be useful later in the proof of lemma 6

The remainder of this section is dedicated to the proof of two results that will be required later. First we have the following lemma.

Lemma 2 *In dimension* 3*, the* ΨDO $\mathcal{N}_{U_1,U_1} \circ (\sharp \circ *\mathrm{d})$ *from* $\Lambda^0(M)$ *to sections of* $T^1_1(M)$ *is of order* -1*. Here* $\sharp$ *raises the first index.*

Remark 7: Note that $\sharp \circ *\mathrm{d}$ is a ΨDO of order 1 while $\mathcal{N}_{U_1,U_2}$ is of order -1, and so we would naively expect $\mathcal{N}_{U_1,U_2} \circ (\sharp \circ *\mathrm{d})$ to be of order 0.

Proof: By the calculus of ΨDOs, it is sufficient to show that for every $(x,\xi) \in (T^*)^{\mathbb{R}}M \setminus \{0\}$, $\sigma_{\mathcal{N}_{U_1,U_2},p}(x,\xi) \circ \sigma_{\sharp\circ *\mathrm{d},p}(x,\xi) = 0$. Based on (3.10), it is sufficient to show that for any nonzero complex number $\tilde{\beta}$ and any $\omega \in \Omega^{\mathbb{R}}_x M \cap \xi^{\perp}$

$$P_\omega[\sigma_{\sharp\circ*\mathrm{d},p}(x,\xi)\,\tilde{\beta}] = 0.$$

Let us now proceed to calculate $\sigma_{\sharp\circ*\mathrm{d},p}$. Assume that we have an oriented coordinate system (x^1, x^2, x^3) whose coordinate vectors are orthonormal at the point x, $(x,\xi) = \mathrm{d}x^1$, and $\omega = \frac{\partial}{\partial x^2}$. Then if $\beta \in \Lambda^0(M)$, in these coordinates we have

$$\sharp \circ *\mathrm{d}(\beta) = \frac{\partial \beta}{\partial x^1}\,\partial_{x^2} \wedge \mathrm{d}x^3 - \frac{\partial \beta}{\partial x^2}\,\partial_{x^1} \wedge \mathrm{d}x^3 + \frac{\partial \beta}{\partial x^3}\,\partial_{x^1} \wedge \mathrm{d}x^2.$$

Therefore, if $\tilde{\beta} = \beta(x)$ then

$$\sigma_{\sharp\circ*\mathrm{d},p}(x,\xi)\,\tilde{\beta} = i\,\tilde{\beta}\,\partial_{x^2} \wedge \mathrm{d}x^3.$$

Similar to (3.13), if $v \in T_x M$ is represented as the vector $(v^1, v^2, v^3)^T$ in these coordinates, then

$$\sigma_{\sharp\circ*\mathrm{d},p}(x,\xi)\,\tilde{\beta}(v) = i\,\tilde{\beta}\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v^3 \\ -v^2 \end{pmatrix}$$

where $\times$ refers to the Euclidean cross product. This implies (recall that $\omega = \frac{\partial}{\partial x^2}$)

$$P_\omega[\sigma_{\sharp\circ*\mathrm{d},p}(x,\xi)\,\tilde{\beta}](v) = i\,\tilde{\beta}\,\pi_\omega \begin{pmatrix} 0 \\ v_3 \\ 0 \end{pmatrix} = 0.$$

This completes the proof of the lemma.

□

We also require in section 4.3 a continuity result for $I^*_{U_1,U_2}$ which we will now prove. This theorem is not necessary to prove local uniqueness for the nonlinear problem, but is required for local stability. Note that for this result we do not require M to be simple.

Theorem 11 *The operator $I^*_{U_1,U_2}$ is continuous from $H^1_c\beta_2(\partial_-\Omega^{\mathbb{R}}M)$ to $H^1_{loc}\tau^1_1 M$.*

Proof: The proof involves working in coordinates and showing that each coordinate of $I^*_{U_1,U_2}[F]$ is given by a sum of x-ray transforms I^*_w for some set of weights w applied to the coordinates of F. To do this, let $F \in (\beta_2)_c\, \partial_-\Omega^{\mathbb{R}} M$ with support contained in a given compact set $K' \Subset \partial_-\Omega^{\mathbb{R}} M$ and take any $h \in (\tau_1^1)_c\, M^{int}$ with support contained within a set $K \Subset M$. Let $\{\phi_{j'}\} \subset C_c^\infty(\partial_-\Omega^{\mathbb{R}} M)$ and $\{\psi_j\} \subset C_c^\infty(M^{int})$ be finite partitions of unity on K' and K respectively such that each of the cut-off functions has support contained within a coordinate chart. Then

$$I^*_{U_1,U_2}[F]\big|_K = \sum_{j,j'} \psi_j\, I^*_{U_1,U_2}[\phi_{j'}\, F],$$

and so it is sufficient to show that for each pair of j and j', $(I^*_{U_1,U_2})_{jj'} = \psi_j^m \circ I^*_{U_1,U_2} \circ \phi_{j'}^m : H_c^1 \beta_2 \partial_-\Omega^{\mathbb{R}} M \to H_c^1 \tau_1^1 M$ is continuous. Here ψ_j^m denotes multiplication by ψ_j, and similarly for $\phi_{j'}^m$.

Next we derive a formula for the components of $(I^*_{U_1,U_2})_{jj'}[F]$ in the local coordinates. Proceeding in a manner similar to the proof of Theorem 9 we have

$$\begin{aligned}
\langle (I^*_{U_1,U_2})_{jj'}[F],\, h\rangle_{L^2\tau_1^1(M)} &= \langle \phi_{j'}\, F, I_{U_1,U_2}[\psi_j h]\rangle_{L^2\beta_2(\partial_-\Omega^{\mathbb{R}} M)} \\
&= \int_{\partial_-\Omega^{\mathbb{R}} M} \int_0^{l(x,v)} \phi_{j'}\, \psi_j\, F(x,v)_{bk}\, (\mathcal{I}_{0,-t}^{\gamma_{\gamma_{x,v}(t)}})^b_{b'}\, \overline{(U_2^{-1})(\dot\gamma_{x,v}(t))^{b'}_{a'}} \times \\
&\quad [P_{\dot\gamma_{x,v}(t)}\, \overline{h}](\gamma_{x,v}(t))^{a'}_{m'}\, \overline{(U_1)(\dot\gamma_{x,v}(t))^{m'}_{p'}}\, (\mathcal{I}_{0,-t}^{\gamma_{\gamma_{x,v}(t)}})^k_{k'} \times \\
&\quad g(\gamma_{x,v}(t))^{k'p'}\, \mathrm{d}t\, \mathrm{d}V_{\partial_-\Omega^{\mathbb{R}} M}(x,v) \\
&= \sum_{b,k,\alpha,\epsilon} \langle \phi_{j'}\, F_{bk}, I_{w_\alpha^{bk\epsilon}}[\psi_j\, h^\alpha_\epsilon]\rangle_{L^2(\partial_-\Omega M)} \\
&= \sum_{b,k,\alpha,\epsilon} \langle I^*_{w_\alpha^{bk\epsilon}}[\phi_{j'}\, F_{bk}], \psi_j\, h^\alpha_\epsilon\rangle_{L^2(M)}
\end{aligned}$$

where

$$w_\alpha^{bk\epsilon}(\xi) = (\mathcal{I}_{0,-l(-\xi)}^{\gamma_\xi})^b_{b'}\, (U_2^{-1})(\xi)^{b'}_{a'}\, (P_\xi)^{a'\epsilon}_{m'\alpha}\, (U_1)(\xi)^{m'}_{p'}\, (\mathcal{I}_{0,-l(-\xi)}^{\gamma_\xi})^k_{k'}\, g(\pi(\xi))^{k'p'}.$$

From this calculation we see that in local coordinates

$$(I^*_{U_1,U_2})_{jj'}[F]^\alpha_\epsilon = \sum_{b,k,\alpha',\epsilon'} g(x)^{\alpha\alpha'}\, g(x)_{\epsilon\epsilon'}\, \psi_j^m(x)\, I^*_{w_{\alpha'}^{bk\epsilon'}}[\phi_{j'}^m\, F_{bk}].$$

Therefore by Theorem 7 $\|(I^*_{U_1,U_2})_{jj'}[F]^\alpha_\epsilon\|_{H^1(M)} \le C \|F\|_{H^1\beta_2(\partial_-\Omega^{\mathbb{R}} M)}$ for a constant $C > 0$, which in turn implies the result.

$\square$

In the next section we will apply Theorems 9 and 10 to the problem of inverting the transform I_{U_1,U_2}.

3.2 Stability estimate for the linear problem

We now consider explicitly the linear problem of inverting the transform I_{U_1,U_2} defined by (2.1). In the case of dimension greater than 3 the results of this section together with the next will show that I_{U_1,U_2} is injective for a generic set of metrics g, and weights U_1 and U_2. The precise meaning of "generic" in this context will be explained below, but in particular we may say that the result holds for a set of metrics and weights which is open and dense in the respective C^4 topologies.

The case of dimension 3 is, as we may expect based on Theorem 2, more complicated. In this case we can show that the same results hold as in the higher dimensional case if we restrict to the space of tensor fields $f \in \tau_1^1(M)$ such that $\mathrm{d}_\beta(f) = 0$. Recall that d_β is defined in (1.15). For convenience we introduce the notation $L^2_\beta \tau_1^1(M)$ for the space of tensor fields such that $\mathrm{d}_\beta(f) = 0$, which is a closed subspace of $L^2_\beta \tau_1^1(M)$. It may be tempting to think that the elements of $L^2 \tau_1^1(M)$ which can be written as $(*\mathrm{d}(\beta))^\#$ are in the kernel of I_{U_1,U_2}, however this is not true. In fact they are only in this kernel according to Theorem 2 if U_1 and U_2 have a specific form. Lemma 2 shows that these elements are in some sense in the kernel of $\mathcal{N}_{U_1,U_2}$ to leading order.

In order to formulate our results, we must introduce a few new objects. We will first of all take a new manifold with boundary M_1 such that $M \Subset M_1^{int}$. This is always possible by taking a collar neighborhood of ∂M that is diffeomorphic to $\partial M \times [0, \epsilon)$, and then taking M_1 to be $M \cup (\partial M \times [-\epsilon, 0))$ where the charts for M_1 across ∂M are defined in the obvious way. Furthermore, since g is smooth up to ∂M, g can be extended smoothly to a metric on M_1 that agrees with g on M. If we assume that M is a simple manifold, then we may assume that with the extended metric M_1 is also a simple manifold. In what follows we will assume that we have such an M_1. Note that this extension is necessitated by the fact that we wish to consider I_{U_1,U_2} and N_{U_1,U_2} acting on functions that are nonzero up to ∂M, but general ΨDOs over M only act on elements of $\mathcal{E}'(M)$.

Now if U_1 and $U_2 \in \beta_1^1(M_1)$, then we may apply all of the previous results on either M

or M_1. Thus we may have normal operators $\mathcal{N}^M_{U_1,U_2}$ and $\mathcal{N}^{M_1}_{U_1,U_2}$. However, note that by the integral formula for $\mathcal{N}_{U_1,U_2}$, if $\text{supp}(f) \subset M$ then $\mathcal{N}^M_{U_1,U_2}[f] = \mathcal{N}^{M_1}_{U_1,U_2}[f]\Big|_M$. Thus, if we only consider $\mathcal{N}_{U_1,U_2}$ acting on $L^2\tau^1_1(M)$, and we identify $L^2\tau^1_1(M)$ with the subspace of $L^2_c\tau^1_1(M_1)$ of tensor fields having support contained in M, then there is no need to distinguish between $\mathcal{N}^M_{U_1,U_2}$ and $\mathcal{N}^{M_1}_{U_1,U_2}$. Considering these comments together with the continuity properties of ΨDOs, we see that $\mathcal{N}_{U_1,U_2} : L^2\tau^1_1(M) \to H^1_{loc}\tau^1_1(M_1)$, and composing with the restriction map we obtain that $\mathcal{N}_{U_1,U_2} : L^2\tau^1_1(M) \to H^1\tau^1_1(M)$ is continuous. From this we have the following corollary whose proof is formally the same as Corollary 2.

Corollary 3 *If M is a simple manifold, then I_{U_1,U_2} can be extended to a continuous operator from $L^2\tau^1_1(M)$ to $L^2\beta_2(\partial_-\Omega^{\mathbb{R}}M)$.*

For the remainder of this chapter we will be studying the inversion of I_{U_1,U_2} on the space $L^2\tau^1_1(M)$. The main result in the higher dimensional case is the next theorem. In the statement and proof we use the annulus $(\Omega^b_a)^{\mathbb{R}}M_1 = \{(x,v) \in T^{\mathbb{R}}M_1 \mid a < \|v\|_g < b\}$ where $0 < a < b$. For this proof and the proof of Theorem 13 we use methods developed in [6] and [24].

Theorem 12 *If U_1 and $U_2 \in \beta^1_1(T^{\mathbb{R}}M_1 \setminus \{0\})$ are everywhere invertible and M_1 is a simple manifold of dimension greater than 3, then the kernel of I_{U_1,U_2} acting on $L^2\tau^1_1(M)$ is at most finite dimensional and contains only elements of $\tau^1_1(M)$ that are zero to infinite order on ∂M (ie. the elements of the kernel are all smooth and vanish to infinite order on ∂M). Furthermore, if I_{U_1,U_2} is injective, then there is a stability estimate*

$$\|f\|_{L^2\tau^1_1(M)} \leq C\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau^1_1(M_1)}. \tag{3.14}$$

The constant C in (3.14) can be chosen so that there exists $\epsilon > 0$ such that the estimate (3.14) remains valid if U_1, U_2, and g are replaced by U_1', $U_2' \in C^3\beta^1_1(T^{\mathbb{R}}M_1 \setminus \{0\})$, and $g' \in C^4S_2M_1$ with $\|U_1 - U_1'\|_{C^3\beta^1_1((\Omega^b_a)^{\mathbb{R}}M_1)} < \epsilon$, $\|U_2 - U_2'\|_{C^3\beta^1_1((\Omega^b_a)^{\mathbb{R}}M_1)} < \epsilon$, and $\|g - g'\|_{C^4S_2M_1} < \epsilon$ assuming that the unit sphere bundles with respect to both g and g' are contained in $(\Omega^b_a)^{\mathbb{R}}M_1$.

Remark 8: We require bounds on $U_1 - U_1'$ and $U_2 - U_2'$ in $\beta^1_1((\Omega^b_a)^{\mathbb{R}}M_1)$ rather than just $\beta^1_1(\Omega^{\mathbb{R}}M_1)$ since when we vary the metric g to g', the set $\Omega^{\mathbb{R}}M_1$ changes, and we want to compare the operators corresponding to these different metrics. Indeed, note that the operator $\mathcal{N}_{U_1,U_2}$ depends on the metric g, even though that dependence is not explicitly indicated by the notation as is the dependence on U_1 and U_2. The conclusions of Theorem 12 show that the set of (U_1, U_2, g) for which $\mathcal{N}_{U_1,U_2}$ is injective is open with respect to the C^4 norm on the product space $\beta^1_1((\Omega^b_a)^{\mathbb{R}}M_1) \times \beta^1_1((\Omega^b_a)^{\mathbb{R}}M_1) \times S_2M_1$.

Proof: For this proof we will require an intermediate manifold $M_{1/2}$ such that $M \Subset M^{int}_{1/2} \Subset M^{int}_1$. For example, if M_1 is constructed as described above, then $M_{1/2}$ can be taken to be $M \cup (\partial M \times [-\epsilon/2, 0)$. Now, take a cut-off function $\phi \in C^\infty_c(M_1)$ such that $\phi = 1$ on $M_{1/2}$. By Theorem 9 the ΨDO $\mathcal{N}_{U_1,U_2}$ is elliptic of order -1, and therefore there exists a parametrix $\mathcal{A}$ for $\mathcal{N}_{U_1,U_2}$ which is a ΨDO of order 1. This means that, if we denote multiplication by ϕ as ϕ^m, for any $f \in L^2\tau^1_1(M)$ we have

$$\mathcal{A} \circ (\phi^m \circ \mathcal{N}_{U_1,U_2} \circ \phi^m)[f] = f + \mathcal{K}[f] \quad \text{on } M^{int}_{1/2} \tag{3.15}$$

where $\mathcal{K} : \mathcal{E}'\tau^1_1(M_1) \to \tau^1_1(M_1)$ is a properly supported smoothing operator. The addition of the cut-off functions in this formula is required for the composition of the two ΨDOs to be well-defined. Rearranging this last formula slightly, using the continuity properties of $\mathcal{A}$ and $\mathcal{K}$, and using the fact that $\phi f = f$, we obtain the estimate

$$\|f\|_{L^2\tau^1_1(M)} = \|f\|_{L^2\tau^1_1(M_1)} \leq C(\|\mathcal{N}_{U_1,U_1}[f]\|_{H^1\tau^1_1(M_1)} + \|f\|_{H^{-s}\tau^1_1(M_1)}) \tag{3.16}$$

for any $s > 0$. For $f \in L^2\tau^1_1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$ this estimate becomes

$$\|f\|_{L^2\tau^1_1(M_1)} \leq C\|f\|_{H^{-s}\tau^1_1(M_1)}.$$

Since the inclusion map $L^2\tau^1_1(M_1) \hookrightarrow H^{-s}\tau^1_1(M_1)$ is compact when $s > 0$ this shows that the identity map restricted to $f \in L^2\tau^1_1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$ must be compact and so $L^2\tau^1_1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$ must be finite dimensional. From the definition of $\mathcal{N}_{U_1,U_2}$ it is clear that $L^2\tau^1_1(M) \cap \ker(\mathcal{N}_{U_1,U_2}) = L^2\tau^1_1(M) \cap \ker(I_{U_1,U_2})$, and so as claimed the kernel of I_{U_1,U_2} acting on $L^2\tau^1_1(M)$ is finite dimensional. Further, if $f \in L^2\tau^1_1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$, then by (3.15) $f = -\mathcal{K}[f]$ on $M^{int}_{1/2}$, and in particular $f \in \tau^1_1(M_{1/2})$. Since $f = 0$ on $M_{1/2} \setminus M$, this proves result that if $I_{U_1,U_2}[f] = 0$, then f vanishes to infinite order on ∂M.

The stability estimate (3.14) follows from (3.16) and the following lemma which is similar to [27, Prop. V.3.1]. See also [23, Lemma 2]. A proof is included for completeness.

Lemma 3 *If X, Y, and Z are all Banach spaces, $A : X \to Y$ is a continuous and injective linear operator, $K : X \to Z$ is a compact linear operator, and we have the estimate*

$$\|x\|_X \leq C(\|Ax\|_Y + \|Kx\|_Z) \quad \forall x \in X, \tag{3.17}$$

then in fact we have

$$\|x\|_X \leq \tilde{C}\|Ax\|_Y \quad \forall x \in X.$$

Proof of Lemma 3: Suppose that the conclusion were not true and so we could find a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\|_X = 1$ for all n, and such that $\|Ax_n\|_Y \to 0$ as $n \to \infty$. By compactness of K, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ so that $\{Kx_{n_k}\}_{k=1}^{\infty}$ converges as $k \to \infty$ and is therefore Cauchy. Thus using (3.17) and the fact $\|Ax_{n_k}\|_Y \to 0$ we see that $\{x_{n_k}\}_{k=1}^{\infty}$ is Cauchy. Thus $\{x_{n_k}\}$ converges to some $x \in X$. The continuity and injectivity of A show that $x = 0$, but this is a contradiction of the original assumption that $\|x_n\|_X = 1$ for all n.

□

Now take $X = L^2\tau_1^1(M)$, $Y = H^1\tau_1^1(M_1)$, $Z = H^{-s}\tau_1^1(M_1)$, $A = \mathcal{N}_{U_1,U_2}$, and K the inclusion map from $L^2\tau_1^1(M)$ to $H^{-s}\tau_1^1(M_1)$. Then (3.16) gives (3.17), and so if I_{U_1,U_2}, and therefore $\mathcal{N}_{U_1,U_2}$, is injective (3.14) is proved.

The proof of the final statement of the theorem relies on Theorem 14 which is stated at the end of this section. This theorem is used in the proof of both Theorem 12 and Theorem 13, and so we delay its statement. In the present case, if U_1', U_2', and g' satisfy the hypotheses of this theorem for any $\epsilon > 0$ sufficiently small, then Theorem 14 gives

$$\|\mathcal{N}_{U_1,U_2} - \mathcal{N}_{U_1',U_2'}\|_{L^2\tau_1^1(M) \to H^1\tau_1^1(M_1)} \leq C'\epsilon$$

for some new constant C'. If ϵ is taken to be less than $1/(2CC')$ where C is the constant from (3.14) then we have

$$\begin{aligned}
\|f\|_{L^2\tau_1^1(M)} &\leq C\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} \\
&\leq C\|\mathcal{N}_{U_1',U_2'}[f]\|_{H^1\tau_1^1(M_1)} + C\|(\mathcal{N}_{U_1,U_2} - \mathcal{N}_{U_1',U_2'})[f]\|_{H^1\tau_1^1(M_1)} \\
&\leq C\|\mathcal{N}_{U_1',U_2'}[f]\|_{H^1\tau_1^1(M_1)} + \tfrac{1}{2}\|f\|_{L^2\tau_1^1(M)}.
\end{aligned}$$

Therefore

$$\|f\|_{L^2\tau_1^1(M)} \leq 2C\|\mathcal{N}_{U_1',U_2'}[f]\|_{H^1\tau_1^1(M_1)}$$

and so the proof is complete.

□

The next theorem is the analog of Theorem 12 in the three dimensional case. Before its statement we introduce one more definition

Definition 4 *For $f \in H^1\tau_1^1(M)$, we say that f satisfies the tangential boundary condition if the tangential part of the antisymmetric part of f vanishes on ∂M.*

Recall that the antisymmetric part of f is $f^a = (f - f^t)/2$. In the notation of section 1.3, f satisfies the tangential boundary condition if and only if $\mathbf{t} f^a = 0$.

Theorem 13 *If U_1 and $U_2 \in \beta_1^1(T^{\mathbb{R}}M_1 \setminus \{0\})$ are everywhere invertible and M_1 is a simple manifold of dimension 3, then the kernel of I_{U_1,U_2} acting on $L^2_\beta\tau_1^1(M)$ is at most finite dimensional and consists entirely of functions that are smooth on M^{int}. If we additionally assume that $f \in C^3_\beta\tau_1^1(M)$ is in the kernel of I_{U_1,U_2} and satisfies the tangential boundary condition, then in fact f is smooth up to the boundary of M and vanishes to infinite order there. Furthermore, if I_{U_1,U_2} is injective on any closed subspace $\mathcal{L} \subset L^2_\beta\tau_1^1(M)$ then there is a stability estimate*

$$\|f\|_{L^2\tau_1^1(M)} \leq C\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} \tag{3.18}$$

which holds for any $f \in \mathcal{L}$. Furthermore, the constant C in (3.18) can be chosen so that there exists $\epsilon > 0$ such that the estimate (3.18) remains valid if U_1, U_2, and g are replaced by U_1', $U_2' \in C^3\beta_1^1(T^{\mathbb{R}}M_1 \setminus \{0\})$, and $g' \in C^4S_2M_1$ with $\|U_1 - U_1'\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < \epsilon$, $\|U_2 - U_2'\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < \epsilon$, and $\|g - g'\|_{C^4S_2M_1} < \epsilon$ assuming that the unit sphere bundles of both g and g' are contained in $(\Omega_a^b)^{\mathbb{R}}M$.

Proof: Take $M_{1/2}$ and ϕ to be defined as in the proof of Theorem 12. By Theorem 10 the system of ΨDOs $(\mathcal{N}_{U_1,U_2}, \Lambda^2\,\mathrm{d}_\beta)^T$ is elliptic, and so there exists a parametrix for this system which we will denote by (A, B) where A is a ΨDO from sections of $T_1^1(M_1)$ to $T_1^1(M_1)$, and

B is a ΨDO from sections of $\Lambda^0(M_1)$ to sections of $T_1^1(M_1)$. For any $f \in L^2\tau_1^1(M_1)$, the analog of (3.15) in this case is

$$(A \circ \phi^m \circ \mathcal{N}_{U_1,U_2} \circ \phi^m)[f] + (B \circ \phi^m \circ \Lambda^2 \, \mathrm{d}_\beta \circ \phi^m)[f] = f + \mathcal{K}[f] \quad \text{on } M_{1/2}^{int} \tag{3.19}$$

where as before $\mathcal{K} : \mathcal{E}'\tau_1^1(M_1) \to \tau_1^1(M_1)$ is a properly supported smoothing operator.

Now let us suppose that $f \in L^2_\beta\tau_1^1(M)$. Unfortunately this does not mean that $f \in L^2_\beta\tau_1^1(M_1)$ since derivatives of f extended as zero to M_1 will in general be singular on ∂M. To overcome this problem we use the decomposition (1.13) on M_1. Indeed, take $\beta \in H_0^1\Lambda^0(M_1)$ to be the solution of the Dirichlet problem

$$\Delta_g \beta = \mathrm{d}_\beta f \qquad \beta|_{\partial M_1} = 0. \tag{3.20}$$

We will denote the operator which takes $\mathrm{d}_\beta f$ to the function $\beta \in H_0^1(M_1)$ by Δ_g^{-1}, and so with this notation $\beta = \Delta_g^{-1}\,\mathrm{d}_\beta\, f$. Now define $f^\beta = f - (*\mathrm{d}\beta)^\#$, and so $\mathrm{d}_\beta f^\beta = 0$ on M_1^{int}. Note also that β is harmonic on M^{int} and $M_1 \setminus M$ since $\mathrm{d}_\beta f = 0$ on both of those sets, but that β may be singular on ∂M.

Next we apply (3.19) to f^β. This yields

$$(A\circ\phi^m\circ\mathcal{N}_{U_1,U_2}\circ\phi^m)[f^\beta]+(B\circ\phi^m\circ\Lambda^2\circ *)[(\mathrm{d}\,\phi)\wedge\mathrm{Alt}(f^\beta)^\flat] = f^\beta+\mathcal{K}[f^\beta] \quad \text{on } M_{1/2}^{int}. \tag{3.21}$$

Note that the second term on the right hand side of (3.21) is a properly supported ΨDO of order -1 applied to f^β, and so we may rewrite (3.21) as

$$(A \circ \phi^m \circ \mathcal{N}_{U_1,U_2} \circ \phi^m)[f^\beta] = f^\beta + \mathcal{K}_1[f^\beta] \quad \text{on } M_{1/2}^{int}. \tag{3.22}$$

where $\mathcal{K}_1 = \mathcal{K} - (\phi^m \circ B \circ \phi^m \circ \Lambda^2 \circ *(\mathrm{d}\phi)^\wedge \circ \mathrm{Alt} \circ \flat)$ is a properly supported ΨDO of order -1.

Now recall that Λ^2 is a parametrix for $-\Delta_g$ on M_1, and therefore $(\Delta_g^{-1} + \Lambda^2) \circ \mathrm{d}_\beta = \tilde{\mathcal{K}} : \mathcal{E}'\tau_1^1(M_1) \to \Lambda^0(M_1)$ is a smoothing operator. Using this fact we may write

$$f^\beta = f - (*\mathrm{d}\,\tilde{\mathcal{K}}\,[f])^\# + (*\mathrm{d}(\Lambda^2 \circ \mathrm{d}_\beta)[f])^\#.$$

Plugging this into (3.22), using lemma 2, and making use of the fact that $[\phi^m, *\mathrm{d}]$ is a ΨDO of order 0, we have

$$(A \circ \phi^m \circ \mathcal{N}_{U_1,U_2} \circ \phi^m)[f] = f^\beta + \mathcal{K}_2[f] \quad \text{on } M_{1/2}^{int} \tag{3.23}$$

where $\mathcal{K}_2$ is a new operator with the same properties as $\mathcal{K}_1$.

From the last equation we have the following estimate which is similar to (3.16)

$$\|f^\beta\|_{L^2\tau_1^1(M_1)} \leq C(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-1}\tau_1^1(M_1)}). \tag{3.24}$$

If we were able to replace f^β by f on the left hand side of this estimate, then the remainder of the proof would follow as in the proof of Theorem 12. We will now proceed to show that it is possible to make this replacement. To start note that since $f = 0$ on $M_{1/2}^{int} \setminus M$, from (3.23) we have

$$(*\mathrm{d}\beta)^\# = -(A \circ \phi^m \circ \mathcal{N}_{U_1,U_2} \circ \phi^m)[f] + \mathcal{K}_2[f] \quad \text{on } M_{1/2}^{int} \setminus M. \tag{3.25}$$

We will use this fact along with the definition of β, (3.20), to estimate $\|\beta\|_{H^1(M)}$. Directly from (3.25), we have the estimate

$$\|\mathrm{d}\beta\|_{L^2\tau_1(M_{1/2}\setminus M^{int})} = \|(*\mathrm{d}\beta)^\#\|_{L^2\tau_1^1(M_{1/2}\setminus M^{int})} \leq C\left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-1}\tau_1^1(M_1)}\right). \tag{3.26}$$

Next we derive a Poincaré type inequality for β in order to estimate its norm in $H^1(M_{1/2} \setminus M^{int})$. From the way we have defined $M_{1/2}$, $M_{1/2} \setminus M^{int}$ is diffeomorphic to $\partial M \times [0, \epsilon/2]$ and $(x,t) \in \partial M \times [0, \epsilon/2]$ provide boundary normal coordinates on $M_{1/2} \setminus M^{int}$. Using the covector field $t\,\mathrm{d}t$ defined in these coordinates on $M_{1/2} \setminus M^{int}$ together with Stokes' theorem we have

$$\begin{aligned}
\|\beta\|^2_{L^2(M_{1/2}\setminus M^{int})} &= \int_{M_{1/2}\setminus M^{int}} |\beta|^2 \,\mathrm{d}v_g \\
&= -\int_{M_{1/2}\setminus M^{int}} \mathrm{d}(|\beta|^2) \wedge *(t\,\mathrm{d}t) + \int_{\partial(M_{1/2}\setminus M^{int})} |\beta|^2 \, *(t\,\mathrm{d}t) \\
&= -\int_{M_{1/2}\setminus M^{int}} 2\,\mathrm{Re}(\overline{\beta}\,\mathrm{d}\beta) \wedge *(t\,\mathrm{d}t) + \frac{\epsilon}{2}\int_{\partial M_{1/2}} |\beta|^2 \,\mathrm{d}v_{\tilde{g}} \\
&\leq C\left(\|\beta\|_{L^2(M_{1/2}\setminus M)}\,\|\mathrm{d}\beta\|_{L^2\tau_1(M_{1/2}\setminus M)} + \|\beta\|^2_{L^2(\partial M_{1/2})}\right) \\
&\leq C\left(\tfrac{1}{2C}\|\beta\|^2_{L^2(M_{1/2}\setminus M)} + \tfrac{C}{2}\|\mathrm{d}\beta\|^2_{L^2\tau_1(M_{1/2}\setminus M)} + \|\beta\|^2_{L^2(\partial M_{1/2})}\right).
\end{aligned}$$

Now, using the operators defined above we have that $\beta = \tilde{\mathcal{K}}[f] - (\Lambda \circ \mathrm{d}_\beta)[f]$. If we take a function $\psi \in C_c^\infty(M_{1/2})$ that is equal to 1 on M, and $\tilde{\psi} \in C_c^\infty(M_1 \setminus \mathrm{supp}(\psi))$ that is equal to 1 on a neighborhood of $\partial M_{1/2}$, then on some neighborhood of $\partial M_{1/2}$ we will have

$$\beta = (\tilde{\psi}^m \circ \tilde{\mathcal{K}} \circ \psi^m - \tilde{\psi}^m \circ \Lambda^2 \circ \mathrm{d}_\beta \circ \psi^m)[f].$$

The operator involved in the previous equation now has a properly supported C^∞ kernel, and so maps $H^{-1}\tau_1^1(M_1) \to H^1(M_1)$ continuously. Combining this with the trace theorem we have

$$\|\beta\|_{L^2(\partial M_{1/2})} \leq C \, \|f\|_{H^{-1}(M_1)}.$$

Together with the calculation from above this shows that

$$\|\beta\|_{H^1(M_{1/2}^{int} \setminus M)} \leq C \left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-1}\tau_1^1(M_1)} \right).$$

Once more by the trace theorem

$$\|\beta\|_{H^{1/2}(\partial M)} \leq C \left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-1}\tau_1^1(M_1)} \right).$$

Since β satisfies $\Delta_g \beta = \mathrm{d}_\beta f = 0$ on M^{int}, standard estimates prove that

$$\|\beta\|_{H^1(M)} \leq C \, \|\beta\|_{H^{1/2}(\partial M)} \leq C \left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-1}\tau_1^1(M_1)} \right). \tag{3.27}$$

Finally, since $f = f^\beta + (*\mathrm{d}\,\beta)^\#$ from (3.24) and (3.27) we obtain

$$\|f\|_{L^2\tau_1^1(M)} \leq C \left(\|f^\beta\|_{L^2\tau_1^1(M)} + \|\beta\|_{H^1(M)} \right) \leq C \left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-1}\tau_1^1(M_1)} \right).$$

The fact that the kernel of I_{U_1,U_2} acting on $L^2_\beta \tau_1^1(M)$ is finite dimensional now follows as in the proof of Theorem 12. Also, by the pseudolocal property of ΨDOs, (3.21) implies that $f = (*\mathrm{d}\,\beta)^\#$ modulo a smooth function on the interior of M. Since β is harmonic on M^{int} it is smooth there, and so this implies that f is smooth on M^{int}.

Now assume that $f \in C^3_\beta \tau_1^1(M)$ is in the kernel of I_{U_1,U_2} and satisfies the tangential boundary condition with respect to g'. Then assuming g' is sufficiently close to g so that lemma 4 applies we get that $f \in C^1\tau_1^1(M_1)$. Since $\mathrm{d}_\beta(f) = 0$ on M and $M_1 \setminus M$, this implies that $\mathrm{d}_\beta(f) = 0$ on all of M_1. From this we see that $f^\beta = f$, and so from (3.21) we conclude that $f \in \tau_1^1(M_1)$, which finally implies that f must vanish to infinite order on ∂M.

If $\mathcal{L}$ is a closed subspace of $L^2_\beta \tau_1^1(M)$, then the stability estimate (3.18) follows just as in the proof of (3.14) by applying lemma 3 with $X = \mathcal{L}$, and then using Theorem 14. This completes the proof of Theorem 13.

□

Note that both the tangential boundary condition and the operator d_β actually depend on the metric g. Thus when we perturb the metric to g', the estimate (3.18) still holds only for $f \in \mathcal{L}$, which is not defined by g', but rather by the reference metric g. We would like to establish an estimate like (3.18) that holds for f in a subspace defined from the perturbed metric.

The following technical lemma was required in the proof of Theorem 13. The proof is similar to the proof in [9] that the full jet of the symmetric and normal parts of $f \in \tau_1^1(M)$ may be recovered at the boundary from the polarization data.

Lemma 4 *Suppose that (M, g), U_1, and U_2 are as in Theorem 13. If $f \in C^3\tau_1^1(M)$ is in the kernel of I_{U_1,U_2}, $\mathrm{d}_\beta(f) = 0$, and f satisfies the tangential boundary condition, then $f \in C^1\tau_1^1(M_1)$. This result still holds if g is only in $C^4S_2M_1$. Furthermore, there is an $\epsilon > 0$ such that if g' is another metric with $\|g - g'\|_{S_2M} < \epsilon$, then the result is still true if $\mathrm{d}_\beta(f) = 0$ and f satisfies the tangential boundary condition with respect to g'.*

Proof: Since f is identically zero on $M_1 \setminus M$, it is sufficient to show that all components of f and all derivatives of those components are zero on ∂M. Let us pick a point $x_0 \in \partial M$ and attempt to show that f vanishes to first order at this point. As in [9] we will use normal coordinates with respect to g, $(x', x^3) = (x^1, x^2, x^3)$, centered at x_0 such that the inward pointing normal to ∂M with respect to g, $-\nu_g$, is given by $\frac{\partial}{\partial x^3}$. By the convexity of ∂M, in these coordinates the boundary of M is given by the graph of a function $\phi(x') = x^3$ where $\phi(0) = 0$, $D\phi(0) = 0$, and $D^2\phi(0) > 0$.

Now choose any vectors η and $\zeta \in T^{\mathbb{R}}_{x_0}M$ such that either η and ζ are parallel, or one of η or ζ is parallel to $\frac{\partial}{\partial x^3}$. In either of these cases it is possible to choose a third vector $\xi \in T^{\mathbb{R}}_{x_0}(\partial M)$ such that ξ is perpendicular to both η and ζ. We also define $\eta' = U_1^{-1}(\xi)\,\eta$, and $\zeta' = (U_2^*)(\xi)\,\zeta$. For $\tau \in (0, \delta)$ for some small $\delta > 0$, let us consider the straight line γ_τ connecting $x_0 = 0$ to the point $c(\tau) = (\tau\,\xi, \phi(\tau\xi))$ (see figure 3.2). Here we are identifying the vector $\xi = \xi^1\frac{\partial}{\partial x^1} + \xi^2\frac{\partial}{\partial x^2}$ with the 2-tuple (ξ^1, ξ^2). Since we work in normal coordinates centered at x_0, this line is a geodesic. Thus $I_{U_1,U_2}[f](\gamma_\tau)(\eta', \zeta') = 0$ for all $\tau \in (0, \delta)$, and so

$$\frac{\partial}{\partial \tau}\left(I_{U_1,U_2}[f](\gamma_\tau)(\eta', \zeta')\right) = 0.$$

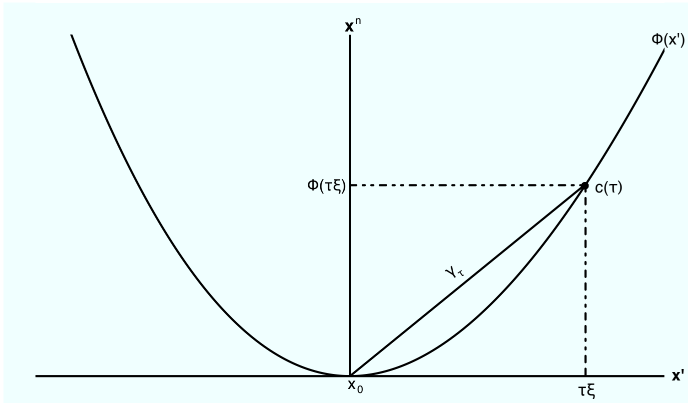

Figure 3.2: Boundary diagram

Writing out this formula in detail we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_0^{t(\tau)}\left\langle U_2^{-1}(\dot\gamma_\tau(s))\left[P_{\dot\gamma_\tau(s)}\left(f\right)\right](\gamma_\tau(s))\,U_1(\dot\gamma_\tau(s))\,\mathcal{I}_{0,s}^{\gamma_\tau}\,\eta',\;\mathcal{I}_{0,s}^{\gamma_\tau}\,\zeta'\right\rangle_{g(\gamma_\tau(s))}\mathrm{d}s\right)=0$$

where $t(\tau)=\tau\sqrt{|\xi|^2+\left(\frac{\phi(\tau\xi)}{\tau}\right)^2}$ is the length of γ_τ. By the fundamental theorem of calculus

$$\begin{aligned} t'(\tau)&\left\langle U_2^{-1}(\dot\gamma_\tau(t(\tau)))\left[P_{\dot\gamma_\tau(t(\tau))}\left(f\right)\right](c(\tau))\,U_1(\dot\gamma_\tau(t(\tau)))\,\mathcal{I}_{0,t(\tau)}^{\gamma_\tau}\,\eta',\;\mathcal{I}_{0,t(\tau)}^{\gamma_\tau}\,\zeta'\right\rangle_{g(c(\tau))}\\ &+\int_0^{t(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\langle U_2^{-1}(\dot\gamma_\tau(s))\left[P_{\dot\gamma_\tau(s)}\left(f\right)\right](\gamma_\tau(s))\,U_1(\dot\gamma_\tau(s))\,\mathcal{I}_{0,s}^{\gamma_\tau}\,\eta',\;\mathcal{I}_{0,s}^{\gamma_\tau}\,\zeta'\right\rangle_{g(\gamma_\tau(s))}\mathrm{d}s=0. \end{aligned} \tag{3.28}$$

Since the derivatives of the integrand are uniformly bounded on M, taking the limit as $\tau\to 0^+$ this last formula becomes

$$\begin{aligned} 0&=\left\langle U_2^{-1}(\xi)\left[P_\xi\left(f\right)\right](x_0)\,U_1(\xi)\,U_1^{-1}(\xi)\,\eta,\;U_2^*(\xi)\,\zeta\right\rangle_{g(x_0)}\\ &=\left\langle\left[P_\xi\left(f\right)\right](x_0)\,\eta,\;\zeta\right\rangle_{g(x_0)}=\left\langle f(x_0)\,\eta,\;\zeta\right\rangle_{g(x_0)}. \end{aligned} \tag{3.29}$$

This identity holds whenever η and $\zeta\in T_{x_0}^{\mathbb{R}}M$ are parallel, or when ζ is parallel to the outer unit normal vector $\nu_g=-\frac{\partial}{\partial x^3}$.

Now suppose that f satisfies the tangential boundary condition with respect to another metric g'. We can now rewrite the previous identity in terms of g' as follows.

$$0=\left\langle f(x_0)\,\eta,\;\sharp_{g'}\circ\flat_g(\zeta)\right\rangle_{g'(x_0)}=\left\langle f(x_0)\,\eta,\;\sharp_{g'}\circ\flat_g(\zeta)-\zeta\right\rangle_{g'(x_0)}+\left\langle f(x_0)\,\eta,\;\zeta\right\rangle_{g'(x_0)}. \tag{3.30}$$

Therefore

$$\left\langle f(x_0)\,\eta,\ \zeta - \sharp_{g'} \circ \flat_g(\zeta)\right\rangle_{g'(x_0)} = \langle f(x_0)\,\eta,\ \zeta\rangle_{g'(x_0)}\,. \tag{3.31}$$

Since the only imaginary part of (3.31) (and (3.30)) comes from $f(x_0)$, we can replace the sesquilinear inner products with the real inner product, and then (3.31) splits into two identities for the real and imaginary parts of $f(x_0)$. Thus it is sufficient to assume that $f(x_0)$ is real, and then the same argument applies to the real and imaginary parts of f.

We will first prove that the symmetric part f^s of $f(x_0)$ with respect to g' is zero. From (3.31) we have

$$\|f^s\| = \sup_{\eta \in (\Omega')^{\mathbb{R}}_{x_0} M} \Big(\langle f(x_0)\,\eta,\ \eta\rangle_{g'(x_0)}\Big) \leq \|f^s\|\,\|\mathrm{Id} - \sharp_{g'} \circ \flat_g\|. \tag{3.32}$$

The norms in the above equation are all the operator norm from $T^{\mathbb{R}}_{x_0}M$ with the metric g' to itself, and the space $(\Omega')^{\mathbb{R}}_{x_0}M$ is the real unit sphere over x_0 with respect to g'. In coordinates we have

$$\left(\mathrm{Id} - \sharp_{g'} \circ \flat_g\right)^i_j = \delta^i_j - (g')^{ik} g_{kj} = (g')^{ik}\left((g')_{kj} - g_{kj}\right).$$

Therefore $\|\mathrm{Id} - \sharp_{g'} \circ \flat_g\|$ in (3.32) is bounded by $C\|g - g'\|_{S_2M}$ for some constant C which can be chosen uniformly for any $x_0 \in \partial M$. Therefore, if $\|g - g'\|_{S_2M}$ is sufficiently small, $\|\mathrm{Id} - \sharp_{g'} \circ \flat_g\| < 1$, and so (3.32) implies that $f^s = 0$.

Now we prove that $f(x_0)$ is zero. To do this, we first note that when ν_g and $\nu_{g'}$ are respectively the outer unit normal vectors with respect to g and g', then $\sharp_{g'} \circ \flat_g(\nu_g)$ is a multiple of $\nu_{g'}$. Thus from (3.30) we have

$$0 = \langle f(x_0)\eta,\ \nu_{g'}\rangle_{g'(x_0)} \tag{3.33}$$

for any $\eta \in T^{\mathbb{R}}_{x_0}M$. Since we have already shown that $f(x_0)$ is antisymmetric with respect to g', (3.33) also implies that

$$0 = \langle f(x_0)\nu_{g'},\ \eta\rangle_{g'(x_0)} \tag{3.34}$$

also for any $\eta \in T^{\mathbb{R}}_{x_0}M$. Now for arbitrary $\zeta \in T^{\mathbb{R}}_{x_0}M$ we may decompose ζ and η as

$$\zeta = a\nu_{g'} + \zeta_T \quad \text{and} \quad \eta = b\nu_{g'} + \eta_T$$

where ζ_T and $\eta_T \in T^{\mathbb{R}}_{x_0}\partial M$. Thus, using both (3.33) and (3.34) we obtain

$$\langle f(x_0)\eta, \zeta\rangle_{g'(x_0)} = \langle f(x_0)(b\nu_{g'} + \eta_T),\ a\nu_{g'} + \zeta_T\rangle_{g'(x_0)} = \langle f(x_0)\eta_T,\ \zeta_T\rangle_{g'(x_0)}.$$

Finally, since f satisfies the tangential boundary condition with respect to g', this last equality shows that $f(x_0) = 0$. Repeating the same argument for every $x_0 \in \partial M$ proves that f vanishes on ∂M. Therefore $f \in C^0\tau^1_1(M_1)$.

What remains is to show that the normal derivatives of f pointing towards the interior of M also vanish on ∂M. To do this we return to (3.28). Note that since we have proved that f vanishes on ∂M, the first term in (3.28) vanishes. Now we continue to take two additional derivatives with respect to τ of (3.28). Omitting the details of the computation, which can be found in [9], we obtain using the facts that $f \in C^3\tau^1_1(M)$ and $g \in C^4S_2M_1$

$$\begin{aligned} 0 &= \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^2 \int_0^{t(\tau)} \frac{\mathrm{d}}{\mathrm{d}\tau} \left\langle U_2^{-1}(\dot\gamma_\tau(s)) \left[P_{\dot\gamma_\tau(s)}\,(f)\right](\gamma_\tau(s))\, U_1(\dot\gamma_\tau(s))\, \mathcal{I}^{\gamma_\tau}_{0,s}\,\eta',\ \mathcal{I}^{\gamma_\tau}_{0,s}\,\zeta'\right\rangle_{g(\gamma_\tau(s))} \mathrm{d}s \\ &= t'(\tau)\,\rho'(\tau) \left\langle U_2^{-1}(\dot\gamma_\tau(t(\tau))) \left[P_{\dot\gamma_\tau(t(\tau))}\left(\frac{\partial f}{\partial x^3}\right)\right](c(\tau))\, U_1(\dot\gamma_\tau(t(\tau)))\,\eta',\ \zeta'\right\rangle_{g(c(\tau))} + \mathcal{O}(\tau) \end{aligned}$$

where $\mathcal{O}(\tau)$ is a function that goes to zero when $\tau \to 0^+$ and $\rho(\tau) = \phi(\tau\xi)/\tau$. Taking the limit as $\tau \to 0^+$ in the above formula then gives

$$\begin{aligned} 0 &= (\xi^t \cdot D^2\phi(0)\cdot\xi) \left\langle U_2^{-1}(\xi) \left[P_\xi\left(\frac{\partial f}{\partial x^3}\right)\right](0)\, U_1(\xi)\,\eta',\ \zeta'\right\rangle_{g(x_0)} \\ &= (\xi^t \cdot D^2\phi(0)\cdot\xi) \left\langle \frac{\partial f}{\partial x^3}\eta,\ \zeta\right\rangle_{g(x_0)}. \end{aligned}$$

Just as above this holds exactly when either η and ζ are parallel, or when one of η or ζ is parallel to ν_g. Since $D^2\phi(0) > 0$, the first of these two cases implies that the symmetric part of $\frac{\partial f}{\partial x^3}(x_0)$ with respect to g must vanish. The second of these cases implies that the normal part of the antisymmetric part of $\frac{\partial f}{\partial x^3}(x_0)$ with respect to g must vanish. Therefore the only components of $\frac{\partial f}{\partial x^3}(x_0)$ which may not be zero are $\frac{\partial f^1_2}{\partial x^3}(x_0)$ and $\frac{\partial f^2_1}{\partial x^3}(x_0) = -\frac{\partial f^1_2}{\partial x^3}(x_0)$. On the other hand, using this and the fact already established that $f(x_0) = 0$, the condition $\mathrm{d}_\beta(f) = 0$ with respect to g' becomes

$$(g'_{11} + g'_{22})\frac{\partial f^1_2}{\partial x^3}(x_0) = 0.$$

Therefore $\frac{\partial f}{\partial x^3}(x_0) = 0$. This completes the proof of the lemma.

□

To end the section we will prove a stability result used in the proofs of Theorem 12 and Theorem 13.

Theorem 14 *Let M, M_1, U_1, U_2, and g be all as in either Theorem 12 or Theorem 13. If U_1', $U_2' \in C^3\beta_1^1(T^{\mathbb{R}}M_1 \setminus \{0\})$, and $g' \in C^4 S_2 M$ with $\|U_1 - U_1'\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < \epsilon$, $\|U_2 - U_2'\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < \epsilon$, $\|g - g'\|_{C^4S_2M} < \epsilon$ for ϵ sufficiently small, and the unit spheres with respect to both g and g' are both contained in $(\Omega_a^b)^{\mathbb{R}}M_1$, then (M, g') is still a simple manifold (in the sense that the exponential map is a C^3 diffeomorphism at every point), and*

$$\|\mathcal{N}_{U_1,U_2} - \mathcal{N}_{U_1',U_2'}\|_{L^2\tau_1^1(M)\to H^1\tau_1^1(M_1)} \leq C'\epsilon$$

for some constant C' which depends only on U_1, U_2, and g.

Proof: To prove this theorem we will carefully compare the kernels of the two operators $\mathcal{N}_{U_1,U_2}$ and $\mathcal{N}_{U_1',U_2'}$ in the global coordinates on M_1. The same method is also applied in [6] and [10] to similar problems. Let us assume that U_1, U_2, and g are as in the statement, and that $\|U_1 - U_1'\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < \epsilon$, $\|U_2 - U_2'\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < \epsilon$, and $\|g - g'\|_{C^4S_2M} < \epsilon$ for some $\epsilon > 0$.

We begin by considering the two maps F_x and F_x' defined by (2.21) corresponding to g and g' respectively. Here and through out this proof unprimed functions, operators, and sets correspond to g, while primed functions, operators and sets correspond to g'. By possibly extending g and g' continuously in the C^4 norm beyond M_1, we may assume that F_x and F_x' are both defined on the same domain. As a first step we would like to estimate $\|F_x(r,\omega) - F_x'(r,\omega)\|_{C^3_{x,r,\omega}}$. To accomplish this we use the following result whose proof may be found in [4].

Lemma 5 *Let x and $\tilde{x}$ solve the ODE systems*

$$x' = G(t,x), \quad \tilde{x}' = \tilde{G}(t,\tilde{x}),$$

where G, $\tilde{G}$ are continuous functions from $[0,T] \times U$ to a Banach space $\mathcal{B}$, where $U \subset \mathcal{B}$ is open. Let G be Lipschitz w.r.t. x with a Lipschitz constant $k > 0$. Assume that

$$\|G(t,x) - \tilde{G}(t,x)\| \leq \delta, \quad \forall t \in [0,T],\ \forall x \in U,$$

and that $x(t)$, $\tilde{x}(t)$ stay in U for $0 \le t \le T$. Then for $0 \le t \le T$

$$\|x(t) - \tilde{x}(t)\| \le e^{kt}\|x(0) - \tilde{x}(0)\| + \frac{\delta}{k}\left(e^{kt} - 1\right).$$

We first apply this lemma to the exponential map by recalling that the geodesics $\gamma_{x,\omega}(t)$ and $\gamma'_{x,\omega}(t)$ satisfy respectively the initial value problems

$$\begin{cases} \dot{\gamma}_{x,\omega}(t)^j = g(\gamma_{x,\omega}(t))^{jk}\,\xi_{x,\omega}(t)_k \\ \dot{\xi}_{x,\omega}(t)_k = -\frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}(\gamma_{x,\omega}(t))\,\xi_{x,\omega}(t)_i\,\xi_{x,\omega}(t)_j \\ \gamma_{x,\omega}(0)^j = x^j, \quad \xi_{x,\omega}(0)_k = g(x)_{kj}\,\omega^j, \end{cases} \text{and} \begin{cases} \dot{\gamma}'_{x,\omega}(t)^j = g'(\gamma'_{x,\omega}(t))^{jk}\,\xi'_{x,\omega}(t)_k \\ \dot{\xi}'_{x,\omega}(t)_k = -\frac{1}{2}\frac{\partial g'^{ij}}{\partial x^k}(\gamma'_{x,\omega}(t))\,\xi'_{x,\omega}(t)_i\,\xi'_{x,\omega}(t)_j \\ \gamma'_{x,\omega}(0)^j = x^j, \quad \xi'_{x,\omega}(0)_k = g'(x)_{kj}\,\omega^j. \end{cases}$$

By differentiating these systems with respect to the initial conditions we may obtain similar systems for the derivatives of $\gamma_{x,\omega}(t)$ and $\gamma'_{x,\omega}(t)$. Applying lemma 5 to these systems and using the fact that $\|g - g'\|_{C^4S_2(M)} < \epsilon$ we obtain

$$\|\gamma_{x,\omega}(t) - \gamma'_{x,\omega}(t)\|_{C^3_{x,t,\omega}} + \|\dot{\gamma}_{x,\omega}(t) - \dot{\gamma}'_{x,\omega}(t)\|_{C^3_{x,t,\omega}} < C\epsilon \tag{3.35}$$

for some constant $C > 0$ depending only on g and M_1. This estimate shows that if ϵ is taken small enough then (M_1, g') is still simple (ie. the exponential maps at every point are C^3 diffeomorphisms). Using the expressions from (2.22) for $F_x(t,\omega)$ and $F'_x(t,\omega)$ in terms of $\gamma_{x,\omega}(t)$ and $\gamma'_{x,\omega}(t)$ together with the estimate (3.35) gives

$$\|F_x(t,\omega) - F'_x(t,\omega)\|_{C^3_{x,t,\omega}} < C\epsilon \tag{3.36}$$

for a new constant $C > 0$ which still only depends on g and M_1.

Next we will use (3.36) to estimate $\|F_x^{-1}(\rho,\theta) - (F')_x^{-1}(\rho,\theta)\|_{C^3_{x,\theta,\rho}}$. As in (2.21), we denote the variables in the range of F_x and F'_x as (ρ,θ). Now, note that $(F_x^{-1} \circ F'_x - Id) = (F_x^{-1} \circ F'_x - F_x^{-1} \circ F_x)$, and so, working in some appropriate set of local coordinates for $\omega \in \mathbb{S}^{n-1}$, we have

$$(F_x^{-1} \circ F'_x - Id)(t,\omega) = \left(\int_0^1 DF_x^{-1}(sF'_x(t,\omega) + (1-s)F_x(t,\omega))\,\mathrm{d}s\right) \cdot (F'_x(t,\omega) - F_x(t,\omega)).$$

Taking derivatives of this last equation we see that $\|F_x^{-1} \circ F'_x - Id\|_{C^3_{x,t,\omega}}$ can be bounded in terms of (3.36), and $\|DF_x^{-1}\|_{C^3_{x,\rho,\theta}}$. By (3.36), $\|F'_x\|_{C^3_{x,t,\omega}}$ and $\|(DF'_x)^{-1}\|_{C^2_{x,\rho,\theta}}$ are uniformly bounded if ϵ is sufficiently small. Thus, if we precompose $F_x^{-1} \circ \tilde{F}_x - Id$ with $\tilde{F}_x^{-1}$ we see that

$$\|F_x^{-1}(\rho,\theta) - \tilde{F}_x^{-1}(\rho,\theta)\|_{C^3_{x,\rho,\theta}} < C\epsilon \tag{3.37}$$

for a new constant $C > 0$.

With estimate (3.37) in hand, we will now apply the same analysis used in Chapter 2 on the x-ray transform to derive a formula for the kernels of $\mathcal{N}_{U_1,U_2}$ and $\mathcal{N}'_{U_1,U_2}$. Beginning from (3.8), first define

$$\begin{aligned} A_x(v)^{u\alpha}_{r\nu} =& g^{\alpha\alpha'}(x)\, g_{\nu\nu'}(x)(w_2(\mathrm{Exp}_x(v))^{uk'b'}_r\, \overline{w_1(v/|v|_g)}^{\nu'}_{\alpha'k'b'} \\ &+ w_2(-\mathrm{Exp}_x(v))^{uk'b'}_r\, \overline{w_1(-v/|v|_g)}^{\nu'}_{\alpha'k'b'}), \end{aligned} \tag{3.38}$$

and let $A'_x(v)$ be defined in the same way with g replaced by g', w_1 replaced by w'_1, and w_2 replaced by w'_2. Let $f \in \tau^1_1(M)$ be extended as zero to M_1. Following (3.8), with this notation we have for $x \in M_1^{int}$

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\mathcal{F}_x\setminus\{0\}} A_x(v)^{u\alpha}_{r\nu}\, f(\exp_x(v))^r_u \sqrt{\det(g)}\, \frac{\mathrm{d}v}{|v|_g^{n-1}}.$$

Switching to polar coordinates (t,ω) on $\mathcal{F}_x M_1 \setminus \{0\}$ as in (2.19) gives

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\Omega_x^{\mathbb{R}}} \int_0^{l(x,\theta)} A_x(t\theta)^{u\alpha}_{r\nu}\, f(\exp_x(t\theta))^r_u\, \mathrm{d}t\, \mathrm{d}\omega. \tag{3.39}$$

A similar formula also holds for $\mathcal{N}_{U'_1,U'_2}[f]$. Next we introduce a cut-off function $\chi \in C_c^\infty(M_1^{int})$ that equals 1 on M. Since f vanishes on $M_1 \setminus M$, we may multiply the integrand in (3.39) by $\chi(\exp_x(t\omega))$ and this does not change $\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu$. Finally, we change variables by the map F_x^{-1} in (3.39) to get (in parallel with (2.24))

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\mathbb{S}^{n-1}} \int_0^\infty \chi(x+\rho\theta) A_x((F_x^{-1})_t(\rho,\theta)\,(F_x^{-1})_\omega(\rho,\theta))^{u\alpha}_{r\nu} f(x+\rho\theta)^r_u \left|\frac{\partial F_x^{-1}}{\partial(\rho,\theta)}\right| \mathrm{d}\rho\, \mathrm{d}\theta. \tag{3.40}$$

Motivated by (3.40), and still following the analysis of the x-ray transform from Chapter 2, let us now define

$$\tilde{A}(x,\rho,\theta)^{u\alpha}_{r\nu} = \chi(x+\rho\theta) A_x((F_x^{-1})_t(\rho,\theta)\,(F_x^{-1})_\omega(\rho,\theta))^{u\alpha}_{r\nu} \left|\frac{\partial F_x^{-1}}{\partial(\rho,\theta)}\right|. \tag{3.41}$$

As in Chapter 2 we observe that each of the functions $\tilde{A}(x,\rho,\theta)^{u\alpha}_{r\nu}$ may be extended a function in $C_c^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1})$ that is even with respect to (ρ,θ). With these extended functions we have

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\mathbb{S}^{n-1}} \int_0^\infty \tilde{A}(x,\rho,\theta)^{u\alpha}_{r\nu}\, f(x+\rho\theta)^r_u \,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{3.42}$$

Next we take the linear approximation to $\tilde{A}(x,\rho,\theta)^{u\alpha}_{r\nu}$ near $\rho = 0$

$$\tilde{A}(x,\rho,\theta)^{u\alpha}_{r\nu} = \tilde{A}(x,0,\omega)^{u\alpha}_{r\nu} + \rho\, R(x,\rho,\theta)^{u\alpha}_{r\nu},$$

and plug this into (3.42). At the same time we also change the integration from polar to Cartesian coordinates ($y = x+\rho\theta$) to get

$$\begin{aligned}\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = &\int_{\mathbb{R}^n} \tilde{A}\left(x,0,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu} f(y)^r_u \frac{\mathrm{d}y}{|y-x|^{n-1}} \\ &+ \int_{\mathbb{R}^n} R\left(x,|y-x|,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu} f(y)^r_u \frac{\mathrm{d}y}{|y-x|^{n-2}}.\end{aligned} \tag{3.43}$$

As before, we also have a similar formula for $\mathcal{N}_{U_1',U_2'}[f]$.

Using (3.43), we now compare $\mathcal{N}_{U_1,U_2}$ and $\mathcal{N}_{U_1',U_2'}$. Indeed, we have

$$\begin{aligned}&\left(\mathcal{N}_{U_1,U_2} - \mathcal{N}_{U_1',U_2'}\right)[f](x)^\alpha_\nu = \\ &\quad\int_{\mathbb{R}^n} \left(\tilde{A}\left(x,0,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu} - \tilde{A}'\left(x,0,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu}\right) f^r_u(y) \frac{\mathrm{d}y}{|y-x|^{n-1}} \\ &\quad+ \int_{\mathbb{R}^n} \left(R\left(x,|y-x|,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu} - R'\left(x,|y-x|,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu}\right) f(y)^r_u \frac{\mathrm{d}y}{|y-x|^{n-2}}.\end{aligned} \tag{3.44}$$

For the moment assume that

$$\left\|\tilde{A}\,(x,0,\omega)^{u\alpha}_{r\nu} - \tilde{A}'\,(x,0,\omega)^{u\alpha}_{r\nu}\right\|_{C^1(\mathbb{R}^n_x\times\mathbb{S}^{n-1}_\omega)} < C\epsilon \tag{3.45}$$

and

$$\left\|R\,(x,r,\omega)^{u\alpha}_{r\nu} - R'\,(x,r,\omega)^{u\alpha}_{r\nu}\right\|_{C^1(\mathbb{R}^n_x\times\mathbb{R}_\rho\times\mathbb{S}^{n-1}_\omega)} < C\epsilon \tag{3.46}$$

where $C > 0$ is some new constant. Since $|y-x|^{n-1}$ and $|y-x|^{n-2}$ are integrable singularities in each variable individually, we may apply [28, Proposition A.5.1] together with the above estimates and (3.44) to conclude that

$$\left\|\mathcal{N}_{U_1,U_2} - \mathcal{N}_{U_1',U_2'}\right\|_{L^2\tau^1_1(M)\to L^2\tau^1_1(M_1)} < C\epsilon.$$

It remains to estimate the L^2 norms of the derivatives of the components of $(\mathcal{N}_{U_1,U_2} - \mathcal{N}_{U_1',U_2'})[f]$. Indeed, we may simply differentiate with respect to x under the second integral in (3.44), and after doing this we have a new integral operator applied to f whose kernel is still integrable in each of the variables. Furthermore, these integrals can be uniformly bounded using (3.46), and so the desired estimates follow once again from [28, Proposition A.5.1]. Estimating the derivatives of the first integral in (3.44) poses a problem since when we differentiate that kernel with respect to x the result is no longer integrable. However, since $\tilde{A}(x,0,\omega)$ and $\tilde{A}'(x,0,\omega)$ are even with respect to ω, we may still apply the Calderón-Zygmund Theorem to estimate the singular integral which results from differentiating the kernel, and by [14, Theorem XI.11.1] this is the derivative of the integral. This argument combined with (3.45) shows that the derivatives of the components of the first integral are bounded by $C\epsilon \|f\|_{L^2\tau_1^1(M)}$ in $L^2(M_1)$, and so this completes the proof assuming (3.45) and (3.46).

All that remains now is to prove the estimates (3.45) and (3.46). Since

$$R(x,\rho,\theta)^{u\alpha}_{r\nu} = \int_0^1 \frac{\partial \tilde{A}}{\partial \rho}(x,\rho s,\omega)^{u\alpha}_{r\nu}\,\mathrm{d}s,$$

to prove (3.46) it is sufficient to show that

$$\left\| \tilde{A}(x,\rho,\theta)^{u\alpha}_{r\nu} - \tilde{A}'(x,\rho,\theta)^{u\alpha}_{r\nu} \right\|_{C^2(\mathbb{R}^n_x \times \mathbb{R}_\rho \times \mathbb{S}^{n-1}_\theta)} < C\epsilon. \tag{3.47}$$

Recall that $\tilde{A}$ and $\tilde{A}'$ are defined by (3.41), and note that

$$\begin{aligned} A_x\big((F_x^{-1})_t(\rho,\theta)\,(F_x^{-1})_\omega(\rho,\theta)\big)^{u\alpha}_{r\nu} - A'_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu} = \\ \Big(A_x\left((F_x^{-1})_t(\rho,\theta)\,(F_x^{-1})_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu} - A_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu}\Big) \\ + \Big(A_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu} - A'_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu}\Big). \end{aligned}$$

Therefore, using also (3.37) and the fact that all derivatives of $A_x(t\omega)$ are bounded (by (3.49) below), we see that in order to prove (3.47) it is sufficient to show that

$$\|A_x(t\omega)^{u\alpha}_{r\nu} - A'_x(t\omega)^{u\alpha}_{r\nu}\|_{C^2_{x,t,\omega}}. \tag{3.48}$$

Returning to the definition (3.38) of $A_x(t\omega)$ and $A'_x(t\omega)$ we see that

$$\begin{aligned} A_x(t\omega)^{u\alpha}_{r\nu} &= g(x)^{\alpha\alpha'}\,g(x)_{\nu\nu'} \\ &\Big(w_2(\dot{\gamma}_{x,\omega}(t))^{uk'b'}_r\,\overline{w_1((x,\omega))}^{\nu'}_{\alpha'k'b'} + w_2(-\dot{\gamma}_{x,\omega}(t))^{uk'b'}_r\,\overline{w_1((x,-\omega))}^{\nu'}_{\alpha'k'b'}\Big). \end{aligned} \tag{3.49}$$

Using the definitions of w_1 and w_2, which are respectively (3.6) and (3.7), we can write out a more explicit version of (3.49). This is

$$A_x(t\omega)^{u\alpha}_{r\nu} = g(x)^{\alpha\alpha'}\, g(x)_{\nu\nu'}\, g(x)_{d'j'}\, g(\gamma_{x,\omega}(t))^{kp} \left(\mathcal{I}^{\gamma_{x,\omega}}_{t,0}\right)^{j'}_{b} \left(\mathcal{I}^{\gamma_{x,\omega}}_{t,0}\right)^{p'}_{k} \Big(\left(U_2^{-1}(x,\omega)\right)^{d'}_{a'}$$
$$\left(\overline{U_2^{-1}(\dot\gamma_{x,\omega}(t))}\right)^{b}_{a} [P_{x,\omega}]^{a'\nu'}_{m'\alpha'}\, [P_{\dot\gamma_{x,\omega}(t)}]^{au}_{mr}\, (U_1(x,\omega))^{m'}_{p'} \left(\overline{U_1(\dot\gamma_{x,\omega}(t))}\right)^{m}_{p} - \left(U_2^{-1}(x,-\omega)\right)^{d'}_{a'}$$
$$\left(\overline{U_2^{-1}(-\dot\gamma_{x,\omega}(t))}\right)^{b}_{a} [P_{x,-\omega}]^{a'\nu'}_{m'\alpha}\, [P_{-\dot\gamma_{x,\omega}(t)}]^{au}_{mr}\, (U_1(x,-\omega))^{m'}_{p'} \left(\overline{U_1(-\dot\gamma_{x,\omega}(t))}\right)^{m}_{p} \Big).$$

A corresponding formula holds for $A'_x(t\omega)$, and we wish to estimate the difference of the two. To do this, it is sufficient to estimate the differences of the corresponding terms in the two formulas for $A_x(t\omega)$ and $A'_x(t\omega)$. If we note that the projections can be written in terms of g and g', we see that all of these differences are bounded in the C^2 norm by $C\epsilon$ by a combination of the hypotheses and (3.35), except for the difference in the parallel translation terms. To bound this last difference we note that for any vector η^b, $\left(\mathcal{I}^{\gamma_{x,\omega}}_{0,t}\right)^{j'}_{b} \eta^b$ satisfies the system of ODEs

$$\left(\frac{\partial\, \mathcal{I}^{\gamma_{x,\omega}}_{0,t}}{\partial t}\right)^{j'}_{b} \eta^b = \Gamma(\gamma_{x,\omega}(t))^{j'}_{kl}\, \dot\gamma_{x,\omega}(t)^k \left(\mathcal{I}^{\gamma_{x,\omega}}_{0,t}\right)^{l}_{b} \eta^b \quad \text{and} \quad \left(\mathcal{I}^{\gamma_{x,\omega}}_{0,0}\right)^{j'}_{b} \eta^b = \eta^b$$

where $\Gamma^{j'}_{kl}$ are the Christoffel symbols of the metric g. The same formula holds for the parallel translation with respect to the g' metric when the Christoffel symbols and geodesics are those of the g' metric. Therefore, by lemma 5, (3.35), the hypothesis that $\|g - g'\|_{C^4 S^2 M_1} < \epsilon$, and the definition of the Christoffel symbols,

$$\left\| \left(\left(\mathcal{I}^{\gamma_{x,\omega}}_{0,t}\right)^{j'}_{b} - \left(\mathcal{I}'^{\gamma'_{x,\omega}}_{0,t}\right)^{j'}_{b} \right) \eta^b \right\|_{C^3_{x,t,\omega}} < C\epsilon. \tag{3.50}$$

Since this holds for any vector η^b, and $\mathcal{I}^{\gamma_{x,\omega}}_{0,t} = \left(\mathcal{I}^{\gamma_{x,\omega}}_{t,0}\right)^{-1}$, this implies the needed estimate on the difference of the parallel translation factors, and so completes the proof.

□

3.3 *Generic injectivity for the linear problem*

The results of the previous section establish that the set of U_1, U_2, and g for which I_{U_1,U_2} is injective is open in the C^4 topology when the dimension is greater than 3. In dimension 3 the same is true if I_{U_1,U_2} is restricted to $L^2_\beta\tau^1_1(M)$. We would like to first know that this set is also nonempty, but in fact we will do much better than that in this section. We will show that I_{U_1,U_2} is injective for any real analytic U_1, U_2, and g. Indeed, we have the following theorem.

Theorem 15 *Suppose that (M, g) is a real analytic simple manifold, and U_1 and U_2 are real analytic. If the dimension of M is greater than 3, then I_{U_1,U_2} is injective. If M has dimension 3, then I_{U_1,U_2} is injective on the subspace of $C^3_\beta\tau^1_1(M)$ consisting of fields that satisfy the tangential boundary condition.*

Our proof will use analytic microlocal analysis, and as a primary reference on this topic we use [21]. A different approach to analytic microlocal analysis is also given in [29]. Since we are using analytic methods, for this section we must assume that M is a real analytic manifold (ie. that the transition maps are all real analytic). Our notation for the analytic wave front set of $f \in \mathcal{D}'\tau^1_1(M)$ will be $\mathrm{WF_a}(f)$. The main step in the proof of Theorem 15 is the proof of the following lemma.

Lemma 6 *Suppose that (M, g), U_1, and U_2 are as in Theorem 15, and $\xi_0 \in T^*M^{int}\backslash\{0\}$. In dimension greater than 3 let $f \in L^2_c\tau^1_1(M)$, and in dimension 3 assume that $f \in (L^2_\beta)_c\tau^1_1(M)$. If there is an open subset $V \subset \Omega^{\mathbb{R}}M$ such that $V \cap \xi_0^\perp \neq \emptyset$, and on the set of unit speed geodesics whose tangent vectors pass through V $I_{U_1,U_2}[f]$ is zero, then $\xi_0 \notin \mathrm{WF_a}(f)$.*

Remark 9: This result is actually more general than required for the proof of Theorem 15. Using this lemma we could show injectivity for the map I_{U_1,U_2} composed with restriction to a smaller set than all of $\partial_-\Omega^{\mathbb{R}}M$.

Remark 10: The method of proof used here was developed in [6] and [25], and is also used in [10].

Proof: Let ξ_0 and V be as in the statement of the theorem. Now take any $v' \in V \cap \xi_0^\perp$. Then by the hypothesis there exists a $v \in \partial_-\Omega^{\mathbb{R}}M$ such that the tangent vector of γ_v passes through v', and for every w in a neighborhood of v, $I_{U_1,U_2}[f](\gamma_w) = 0$. For the majority of the proof we will work in a set of coordinates in a neighborhood of γ_v which we will now introduce. Indeed, let us take a set of analytic coordinates $(w^1, \dots, w^{n-1})$ on $\Omega^{\mathbb{R}}_{\pi(v)}M$

centered at v whose domain is contained in the set of w for which $I_{U_1,U_2}[f](\gamma_w) = 0$. Then by the simplicity assumption $(w^1, \dots, w^{n-1}, t)$ provide analytic coordinates on a neighborhood of U of γ_v in M via the map

$$(w^1, \dots, w^{n-1}, t) \mapsto \exp(t(w^1, \dots, w^{n-1})).$$

By translating in the t coordinate we can also make $\pi(\xi_0) = 0$ in this coordinate system, and by rotating if necessary we may assume that $\xi_0 = \mathrm{d}w^{n-1}$. For some $\epsilon > 0$, t_1, and $t_2 \in \mathbb{R}$ the set

$$U' = \{(w^1, \dots, w^{n-1}, t) \,:\, |(w^1, \dots, w^{n-1})| < \epsilon \quad \text{and} \quad t_1 < t < t_2\}$$

is contained in the range of the coordinates, and the points $(w,t) = (w^1, \dots, w^{n-1}, t_j)$ for $j = 1$ or 2 both lie in a small neighborhood of ∂M which does not intersect the support of f. We will identify the set $U' \subset \mathbb{R}^n$ with the image of U' under the inverse coordinate map. Also, from now on we will denote these coordinates by $U' \ni x = (x', x^n) = (w, t)$.

Next we introduce a family of analytic coordinate systems, each providing coordinates on a subset of U'. Indeed, if we take any $\theta' = (\theta^1, \dots, \theta^{n-1}) \in \mathbb{R}^{n-1}$ sufficiently small then the map

$$(x', t) \mapsto \exp\left(t \left(\theta'^j \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^n}\bigg|_{(x',0)} \right)\right) \tag{3.51}$$

is the inverse of a coordinate map on a subset of U', is defined on $\{|x'| < 3\epsilon/4,\, t_1 \le t \le t_2\}$, and in the corresponding coordinates the points (x', t_j) for $j = 1$ or 2 lie in a small neighborhood of ∂M that does not intersect the support of f. Furthermore, since $V \subset \Omega^{\mathbb{R}} M$ is open, if θ' is in a possibly smaller neighborhood of 0 then for any $|x'| < 3\epsilon/4$

$$I_{U_1,U_2}[f]\left(\gamma_{\left(\theta'^j \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^n}\big|_{(x',0)}\right.} \right) = 0. \tag{3.52}$$

Here and in the remainder of this proof for any $\theta \in T^{\mathbb{R}} M \setminus \{0\}$, γ_θ denotes the *unit speed* geodesic with initial tangent vector parallel to θ. For convenience we will also use the notation $\theta = \left(\theta'^j \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^n}\big|_{(x',0)}\right.$ below.

Now we introduce a sequence of cutoff functions $\chi_N \in C_c^\infty(\mathbb{R}^{n-1})$ with support in $B_{3\epsilon/4}(0)$, equal to 1 on a neighborhood of $x' = 0$, and satisfying the estimates

$$|\partial_{x'}^\alpha \chi_N(x')| \le (C\,N)^{|\alpha|} \quad \text{for all multi-indices } \alpha \text{ with } |\alpha| \le N \tag{3.53}$$

for some constant $C > 0$. It is possible to construct such a sequence of functions, see [29]. By (3.52), for every $\xi \in \mathbb{C}^n$ and $h > 0$, we have for θ' in a small neighborhood of 0

$$e^{\frac{i}{h}((x',1)\cdot\xi)}\chi_N(x')\, I_{U_1,U_2}[f](\gamma_\theta) = 0.$$

Choosing some basis for $T_{\pi(v)}M$, we can express the integrals defining the components $I_{U_1,U_2}[f](\gamma_\theta)_{kb}$ with respect to the coordinates (x', x^n) and the chosen basis, and write out as in (3.1) the formula

$$0 = e^{\frac{i}{h}((x',1)\cdot\xi)}\chi_N(x') \int_{t_1}^{t_2} R(\dot\gamma_\theta(t))_{ab}\, [P_{\dot\gamma_\theta(t)}\, f](\gamma_\theta(t))^a_m\, Q(\dot\gamma_\theta(t))^m_k\, \mathrm{d}t.$$

Now we integrate this formula with respect to x' to obtain for each k and b

$$0 = \int_{B_{3\epsilon/4}(0)} \int_{t_1}^{t_2} e^{\frac{i}{h}((x',1)\cdot\xi)}\chi_N(x')\, R(\dot\gamma_\theta(t))_{ab}\, [P_{\dot\gamma_\theta(t)}\, f](\gamma_\theta(t))^a_m\, Q(\dot\gamma_\theta(t))^m_k\, \mathrm{d}t\, \mathrm{d}x'. \qquad (3.54)$$

As described in the previous paragraph, for each θ' (x', t) provides an analytic coordinate system on a subset of U', and furthermore the support of the integrand in (3.54) is contained in domain of these coordinates. Therefore we may make an analytic coordinate change in (3.54), which depends on θ' in an analytic manner, to obtain

$$0 = \int_{U'} e^{\frac{i}{h}((\tilde{x}'(x,\theta'),1)\cdot\xi)}\, \chi_N(\tilde{x}'(x,\theta'))\, R(v(x,\theta'))_{ab}\, [P_{v(x,\theta')}\, f](x)^a_m\, Q(v(x,\theta'))^m_k\, J(x,\theta')\, \mathrm{d}x \qquad (3.55)$$

where $\tilde{x}'(x,\theta')$ is an analytic function of x and θ' which satisfies

$$\tilde{x}'((x',0),\theta') = x' \quad \text{and} \quad \frac{\partial \tilde{x}'}{\partial x^n}((x',0),\theta') = -\theta'. \qquad (3.56)$$

$v(x,\theta')$ is also an analytic function of x and θ' which satisfies

$$v((x',0),\theta') = \theta,$$

and $J(x,\theta')$ is the Jacobian of the change of variables which is a positive analytic function. The identity (3.54) holds for any θ' sufficiently small and all $\xi \in \mathbb{C}^n$. We will now choose θ' as a function of ξ in a specific way to find a new identity which holds for any ξ in a small complex neighborhood of $(0, \ldots, 1, 0) = e^{n-1}$, and in which θ' has been eliminated.

Indeed, define

$$\theta'(\xi) = \left(\xi_1, \dots, \xi_{n-2}, -\frac{\xi_n + \sum_{k=1}^{n-2} \xi_k^2}{\xi_{n-1}} \right). \tag{3.57}$$

We will identify ξ_0 with e^{n-1} since $\xi_0 = \mathrm{d}x^{n-1}$. The vector valued function $\theta'(\xi)$ is well-defined and analytic in a neighborhood of ξ_0, and satisfies

$$\theta'(\xi_0) = 0, \quad (\theta'(\xi), 1) \cdot \xi = 0 \quad \text{for all } \xi.$$

Let us write $\psi(x, \xi) = ((\tilde{x}'(x, \theta'(\xi)), 1) \cdot \xi)$ and $\tilde{\chi}_N(x, \xi) = \chi_N(\tilde{x}'(x, \theta'(\xi)))$, and then plug $\theta'(\xi)$ into (3.55) to obtain

$$0 = \int_{U'} e^{\frac{i}{h}\psi(x,\xi)} \, \tilde{\chi}_N(x, \xi) \, R(\tilde{v}(x, \xi))_{ab} \, [P_{\tilde{v}(x,\xi)} \, f](x)^a_m \, Q(\tilde{v}(x, \xi))^m_k \, \tilde{J}(x, \xi) \, \mathrm{d}x \tag{3.58}$$

for all ξ in a complex neighborhood of ξ_0. Since $\tilde{\chi}_N(x, \xi)$ is a composition of χ_N with an analytic function for each N, $\tilde{\chi}_N$ still satisfies an inequality like (3.53) where now the derivatives are taken with respect to x and ξ.

A simple calculation shows that in Euclidean case, where the geodesics are all straight lines, $\psi(x, \xi) = x \cdot \xi$ is the usual phase function, and we can further observe that by (3.56), even in our present more general case the function $\psi(x, \xi)$ satisfies $\psi_x(0, \xi) = \xi$, and so

$$\psi_{x\xi}(0, \xi) = \mathrm{Id}. \tag{3.59}$$

Again by (3.56) we can also see that

$$\psi_{\xi\xi}(0, \xi) = 0. \tag{3.60}$$

To apply the analytic microlocal theory, we must study the critical points of this phase function ψ with respect to ξ. Following [6] we now continue to establish a technical lemma about the gradient of ψ with respect to ξ that will be required for this study.

Lemma 7 *There exists a $\delta > 0$ such that*

$$\psi_\xi(x, \xi) \neq \psi_\xi(y, \xi)$$

whenever $x = (x', x^n) \neq y$, $|x'| < \delta$, $|y| < \delta$, and $|\xi - \xi_0| < \delta$ where ξ may be complex.

The proof of essentially the same lemma can be found in [6], although the phase function there is slightly different. Nonetheless, I include the proof here for completeness.

Proof of lemma: First note that $\tilde{x}'(x,\theta')$ is defined implicitly by

$$x = \exp\left(t\left(\theta'^j \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^n}\bigg|_{(\tilde{x}',0)}\right)\right).$$

Differentiating this equation with respect to θ'^j and evaluating at $\theta'(\xi_0) = 0$ we obtain

$$\frac{\partial\left(\exp\left(x^n\left(\theta'^j \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^n}\big|_{(x',0)}\right)\right)\right)}{\partial\theta'^j}\Bigg|'_{\theta'=0} = -\frac{\partial\tilde{x}'}{\partial\theta'^j}(x,0) \tag{3.61}$$

where the prime on the left hand side indicates that we are only taking the first $n-1$ components. By the simplicity assumption the exponential map is everywhere a diffeomorphism, and so the vectors given by the left hand side of (3.61) must be linearly independent, and therefore they must form a basis for $\mathbb{R}^{n-1}$.

Keeping this in mind, let us first assume that $y = (y', 0)$, $x = (y', x^n)$, and $\xi = \xi_0 = e_{n-1}$. With such a y, $\psi_\xi(y, \xi_0) = (y',\, 1)$. On the other hand

$$\frac{\partial\psi}{\partial\xi^k}(x,\xi_0) = \frac{\partial\tilde{x}'^{n-1}}{\partial\theta'^j}(x,0)\,\frac{\partial\theta'^j}{\partial\xi^k}(\xi_0) + (\tilde{x}'(x,0),\, 1)_k.$$

Calculating the derivatives of $\theta'(\xi)$ from (3.57) and noting that $\tilde{x}'(x,0) = y'$, this last equation may be rewritten as

$$\psi_\xi(x,\xi_0) = \left(\frac{\partial\tilde{x}'^{n-1}}{\partial\theta'^1}(x,0),\, \ldots,\, \frac{\partial\tilde{x}'^{n-1}}{\partial\theta'^{n-2}}(x,0),\, 0,\, -\frac{\partial\tilde{x}'^{n-1}}{\partial\theta'^{n-1}}(x,0)\right) + (y',\, 1).$$

Thus if $\psi_\xi(x,\xi_0) = \psi_\xi(y,\xi_0)$, then $\frac{\partial\tilde{x}'^{n-1}}{\partial\theta'^j}(x,0) = 0$ for $j = 1$ to $n-1$. However this is impossible since as shown in the previous paragraph the vectors $\frac{\partial\tilde{x}'}{\partial\theta'^j}(x,0)$ must form a basis for $\mathbb{R}^{n-1}$ and thus they cannot all have a zero $(n-1)$st component.

Now, since $\psi_{x\xi}(0,\xi) = \mathrm{Id}$, the map $x \mapsto \psi_\xi(x,\xi)$ is a local diffeomorphism near $x = 0$, and so there is a $\delta > 0$ such that if $|x|, |y| < \delta$ and $x \neq y$, then $\psi_\xi(x,\xi) \neq \psi_\xi(y,\xi)$. On the other hand, suppose $x = (0, x^n)$ for $|x^n| > \delta$. Then by what has already been shown, $\psi_\xi(x,\xi_0) \neq \psi_\xi(0,\xi_0)$, and so by continuity this is still true when $|x'| < \delta$, $|y| < \delta$, and $|\xi - \xi_0| < \delta$ for some possibly smaller $\delta > 0$. Repeating this argument for a finite number of values of x^n and taking the smallest constants thus obtained, the proof of the lemma is complete.

□

Now suppose that ϵ was originally chosen to be less than δ from the previous lemma, and so the lemma actually holds for any $x \in U'$.

Take an $\epsilon' > 0$ such that $\epsilon' < \epsilon/2$, and let χ be the characteristic function of the complex ball of radius ϵ' centered at 0. Then for any real vector $\eta \in B_{\epsilon/2}(\xi_0)$, multiply (3.58) by $\chi(\xi - \eta)\, e^{\frac{i}{h}(\frac{i}{2}(\xi-\eta)^2 - \psi(y,\eta))}$, and integrate the result over $\xi \in \mathbb{R}^n$ to obtain

$$0 = \int_{B_{\epsilon'}(\eta)} \int_{U'} e^{\frac{i}{h}\Psi(x,y,\xi,\eta)}\, \tilde{\chi}_N(x,\xi)\, R(\tilde{v}(x,\xi))_{ab}\, [P_{\tilde{v}(x,\xi)}\, f](x)^a_m\, Q(\tilde{v}(x,\xi))^m_k\, \tilde{J}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi. \tag{3.62}$$

Here and in the remainder of the proof we are assuming that $|y| < \epsilon/2$ and so lemma 7 applies. The phase function $\Psi(x,y,\xi,\eta)$ is given by

$$\Psi(x,y,\xi,\eta) = \psi(x,\xi) - \psi(y,\xi) + \frac{i}{2}(\xi-\eta)^2. \tag{3.63}$$

Let us now analyze the critical points of $\xi \mapsto \Psi(x,y,\xi,\eta)$. Taking the gradient of Ψ with respect to ξ we have

$$\Psi_\xi(x,y,\xi,\eta) = \psi_\xi(x,\xi) - \psi_\xi(y,\xi) + i\,(\xi - \eta). \tag{3.64}$$

From this we can observe that when $x = y$, Ψ has a unique critical point at $\xi_c(x,x,\eta) = \eta$. Using (3.60) we have

$$\Psi_{\xi\xi}(0,0,\xi,\eta) = i\,\mathrm{Id},$$

and therefore, by the implicit function theorem, Ψ still has a unique (necessarily complex when $x \neq y$) critical point $\xi_c(x,y,\eta)$ for x and y in a sufficiently small neighborhood of 0. Also, the function $\xi_c(x,y,\eta)$ is analytic in all of its variables, and from (3.64) we may calculate

$$(\xi_c)_x(x,y,\eta)|_{x=y} = i\,\psi_{x\xi}(x,\eta). \tag{3.65}$$

Let us take $C > 2$ sufficiently large so that when $|y| < \epsilon/C$ and $|x - y| < \epsilon/C$, the function $\xi_c(x,y,\eta)$ is still defined. Finally, note that by lemma 7, on the set $|x-y| \geq \epsilon/C$ $\Psi(x,y,\xi,\eta)$ has no real critical point with respect to ξ.

Now we will split the integration in (3.62) with respect to x into two pieces. We will first consider the piece where $|x-y| \geq \epsilon/C$:

$$I_{|x-y|\geq\epsilon/C} = \int_{B_{\epsilon'}(\eta)} \int_{U'\setminus\{|x-y|<\epsilon/C\}} e^{\frac{i}{h}\Psi(x,y,\xi,\eta)}\, \tilde{\chi}_N(x,\xi)\, R(\tilde{v}(x,\xi))_{ab} \\ \times [P_{\tilde{v}(x,\xi)}\, f](x)^a_m\, Q(\tilde{v}(x,\xi))^m_k\, \tilde{J}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi. \tag{3.66}$$

As mentioned above, $\xi \mapsto \Psi(x,y,\xi,\eta)$ has no real critical point on this set, and so $\Psi_\xi(x,y,\xi,\eta)$ must be bounded below on this domain of integration. Thus we may use the standard integration by parts trick to bound this portion of the integral. Indeed, we have

$$L_\xi e^{\frac{i}{h}\Psi(x,y,\xi,\eta)} = \frac{h\,\overline{\Psi}_\xi \cdot \partial_\xi}{i|\Psi_\xi|^2} e^{\frac{i}{h}\Psi(x,y,\xi,\eta)} = e^{\frac{i}{h}\Psi(x,y,\xi,\eta)},$$

and so if we perform integration by parts with respect to ξ N times in (3.66) we get

$$I_{|x-y|\geq\epsilon/C} = \int_{B_{\epsilon'}(\eta)} \int_{U'\setminus\{|x-y|<\epsilon/C\}} e^{\frac{i}{h}\Psi(x,y,\xi,\eta)}\, L^t_\xi\Big(\tilde{\chi}_N(x,\xi)\, R(\tilde{v}(x,\xi))_{ab} \\ \times [P_{\tilde{v}(x,\xi)}\, f](x)^a_m\, Q(\tilde{v}(x,\xi))^m_k\, \tilde{J}(x,\xi)\Big)\, \mathrm{d}x\, \mathrm{d}\xi + \mathcal{O}(e^{-\frac{C_1}{h}})$$

for some constant $C_1 > 0$. Here we have also used the fact that on the boundary of integration in ξ, where $|\xi - \eta| = \epsilon'$, $e^{\frac{i}{h}\Psi(x,y,\xi,\eta)}$ is exponentially decaying as $h \to 0^+$. Now we use the fact that all functions of ξ in the integrand satisfy estimates of the form (3.53) where the derivatives are taken with respect to ξ, and so

$$\Big|L^t_\xi\left(\tilde{\chi}_N(x,\xi)\, R(\tilde{v}(x,\xi))_{ab}\, [P_{\tilde{v}(x,\xi)}\, f](x)^a_m\, Q(\tilde{v}(x,\xi))^m_k\, \tilde{J}(x,\xi)\right)\Big| \leq (C_2\, N\, h)^N$$

for some new constant $C_2 > 0$. Therefore

$$|I_{|x-y|\geq\epsilon/C}| \leq C_2(C_2\, N\, h)^N + e^{-\frac{C_1}{h}} \tag{3.67}$$

where C_2 may now be larger.

Next we analyze the part of the integral from (3.62) where $|x-y| < \epsilon/C$:

$$h^{1/2}\, I_{|x-y|<\epsilon/C} = h^{1/2} \int_{B_{\epsilon'}(0)} \int_{\{|x-y|<\epsilon/C\}} e^{\frac{i}{h}\Psi(x,y,\xi,\eta)}\, \tilde{\chi}_N(x,\xi)\, R(\tilde{v}(x,\xi))_{ab} \\ \times [P_{\tilde{v}(x,\xi)}\, f](x)^a_m\, Q(\tilde{v}(x,\xi))^m_k\, \tilde{J}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi. \tag{3.68}$$

By taking C sufficiently large and ϵ' sufficiently small we can ensure that on the domain of integration in (3.68) $\tilde{\chi}_N(x,\xi) = 1$ for all N, and thus (3.68) actually becomes

$$h^{1/2}\, I_{|x-y|<\epsilon/C} = h^{1/2} \int_{B_{\epsilon'}(0)} \int_{\{|x-y|<\epsilon/C\}} e^{\frac{i}{h}\Psi(x,y,\xi,\eta)}\, R(\tilde{v}(x,\xi))_{ab} \times [P_{\tilde{v}(x,\xi)}\, f](x)^a_m\, Q(\tilde{v}(x,\xi))^m_k\, \tilde{J}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi. \tag{3.69}$$

Now we apply the complex method of stationary phase [21, Theorem 2.8], and remark 2.10 from [21] to the integration with respect to ξ in (3.69). This yields

$$h^{1/2} I_{|x-y|<\epsilon/C} = \int_{\{|x-y|<\epsilon/C|\}} e^{\frac{i}{h}\Psi(x,y,\xi_c(x,y,\eta),\eta)}\, \tilde{P}(x,y,\eta;\, h)^m_{akb} \times f(x)^a_m\, \mathrm{d}x + \mathcal{O}(e^{-\frac{C_3}{h}}) \tag{3.70}$$

where $C_3 > 0$ is another constant and we have combined all the factors in the integrand into a single array of functions $\tilde{P}(x,y,\eta;\, h)^m_{akb}$ which is obtained by the formula in remark 2.10 of [21]. In particular $\tilde{P}(x,y,\eta;\, h)$ is, in the language of [21], an indexed array of classic analytic symbols of order 0. The array of principal symbols, or the zero order terms of the formal asymptotic expansion of $\tilde{P}$, is

$$\sigma_{\tilde{P}}(x,y,\eta)^m_{akb} = R(\tilde{v}(x,\xi_c(x,y,\eta)))_{sb}\, (P_{\tilde{v}(x,\xi_c(x,y,\eta))})^{ms}_{at} Q(\tilde{v}(x,\xi_c(x,y,\eta)))^t_k\, \tilde{J}(x,\xi_c(x,y,\eta)).$$

Furthermore

$$\sigma_{\tilde{P}}(0,0,\xi_0)^m_{akb} = R(v')_{sb} (P_{v'})^{ms}_{at}\, Q(v')^t_k. \tag{3.71}$$

Now we put the estimates for the two parts of the integral back together. From (3.62) $h^{1/2}(I_{|x-y|<\epsilon/C} + I_{|x-y|\geq\epsilon/C}) = 0$, and so using (3.67) and (3.70) we get

$$\left| \int_{\{|x-y|<\epsilon/C|\}} e^{\frac{i}{h}\Phi(x,y,\eta)}\, \tilde{P}(x,y,\eta;\, h)^m_{akb}\, f(x)^a_m\, \mathrm{d}x \right| \leq C_2\, h^{1/2} (C_2\, N\, h)^N + \mathcal{O}(e^{-\frac{C_4}{h}}) \tag{3.72}$$

for a new constant $C_4 > 0$ where $\Phi(x,y,\eta) = \Psi(x,y,\xi_c(x,y,\eta),\eta)$. The integral in (3.72) does not depend on N, and so we are free to choose it however we wish. In particular we can choose N as a function of h by taking $N = \text{floor}\left((e\, C_2\, h)^{-1}\right)$, and then

$$(C_2\, N\, h)^N \leq e^{-N} = e\, e^{-N-1} \leq e\, e^{-\frac{1}{e\, C_2\, h}}.$$

Therefore the right hand side of (3.72) can actually be rewritten as $\mathcal{O}(e^{-\frac{C_5}{h}})$. As a final adjustment we make the change of variables

$$(x, y, \eta) \mapsto (x, y, \zeta) = (x, y, \psi_x(y, \eta))$$

in (3.72). This is a valid change of variables by the inverse function theorem and (3.59) for x and y in a small enough neighborhood of 0. Therefore, after possibly taking C larger it may be applied to (3.72). Now we check that this new phase function $\Phi(x, y, \zeta)$ satisfies the hypotheses required in the definition of the analytic wave front set given in [21]. Indeed

$$\Phi(x, x, \zeta) = 0 \quad \text{and} \quad \Phi_x(x, x, \zeta) = \psi_x(x, \eta) = \zeta. \tag{3.73}$$

To finish showing that Φ is an appropriate phase function we need to prove that

$$\mathrm{Im}(\Phi(x, y, \zeta)) > C' \, |x - y|^2. \tag{3.74}$$

In order to show (3.74), we expand $\Phi(x, y, \zeta)$ in the x variable about $y = x$. Doing this we obtain, using also (3.65),

$$\Phi(x, y, \zeta) = \zeta \cdot (x - y) + \frac{1}{2}(x - y)^t \left(\psi_{xx}(y, \eta) + i\, \psi_{x\xi}(y, \eta) \cdot \psi_{\xi x}(y, \eta)\right)(x - y) + \mathcal{O}(|x - y|^3).$$

From (3.59) we may therefore conclude that when y is restricted to a small enough neighborhood of 0, and for $|x-y|$ sufficiently small, (3.74) holds. Therefore after possibly making C larger (3.74) holds on the domain of integration for (3.72).

If $f(x)^a_m$ and $\tilde{P}(x, y, \eta)^m_{akb}$ were merely scalars then (3.72) would be sufficient to show that $(x_0, \xi_0) \notin \mathrm{WF_a}(f)$. However in this case we must consider a system of equations like (3.72) which we obtain by varying v'. Indeed, since V is open and $V \cap \xi_0^\perp \neq \emptyset$, we may find a basis $\{v_j\}_{j=1}^{n-1}$ for $\xi_0^\perp$ contained entirely in $V \cap \xi_0^\perp$ and repeat the above analysis with v' replaced by v_j for each j. Furthermore, by choosing $\{v_j\}_{j=1}^{n-1}$ sufficiently close together we can be sure that $(v_i + v_j)/\|v_i + v_j\| \in \xi_0^\perp \cap V$ for every i and j, and we add these $(n-1)(n-2)/2$ extra vectors to $\{v_j\}_{j=1}^{n-1}$ to obtain a set of vectors $\{v_j\}_{j=1}^{n(n-1)/2}$. Doing this we have from (3.72) a system of equations

$$\left| \int_{\{|x-y|<\epsilon/C|\}} e^{\frac{i}{h}\Phi_j(x,y,\eta)} \tilde{P}_j(x, y, \eta;\, h)^m_{akb}\, f(x)^a_m \, \mathrm{d}x \right| = \mathcal{O}(e^{-\frac{C_j}{h}}) \tag{3.75}$$

where $\Phi_j(x, y, \eta)$ are all phase functions satisfying (3.73) and (3.74), and $\tilde{P}_j(x, y, \eta;\, h)^m_{akb}$ are classical analytic symbols of order 0 with principal symbols satisfying, according to (3.71),

$$\sigma_{\tilde{P}_j}(0, 0, \xi_0)^m_{akb} = R(v_j)_{sb}(P_{v_j})^{ms}_{at}\, Q(v_j)^t_k.$$

The reader should note that we are using the terminology of [21] here. Suppose that $f \in (T^1_1)^{\mathbb{C}}_{x_0} M$ satisfies $\sigma_{P_j}(0, 0, \xi_0)^m_{akb} f^a_m = 0$ for every j. If we recall the definitions of R and Q, and use the fact that U_1 and U_2 are invertible, we see that $P_{v_j} f = 0$ for every j. We will now show that in dimension at least 4 this implies that $f = 0$. Indeed, let us express the components of $f^\flat$ with respect to the basis $\{v_1,\, \ldots, v_{n-1},\, \xi_0^\sharp\}$ as f_{ij}. Then we have for all $1 \leq j,\, k,\, l \leq n-1$ the following equations

$$0 = \big((P_{v_j} f)\big)^\flat(v_k,\, v_l) = f_{kl} - \langle v_j, v_k\rangle_{g(x_0)} f_{jl} - \langle v_j, v_l\rangle_{g(x_0)} f_{kj} + \langle v_j, v_l\rangle_{g(x_0)} \langle v_j, v_k\rangle_{g(x_0)} f_{jj}, \tag{3.76}$$

$$0 = \big((P_{v_j} f)\big)^\flat(\xi_0^\sharp,\, v_l) = f_{nl} - \langle v_j, v_l\rangle_{g(x_0)} f_{nj},$$

$$0 = \big((P_{v_j} f)\big)^\flat(v_k,\, \xi_0^\sharp) = f_{kn} - \langle v_j, v_k\rangle_{g(x_0)} f_{jn},$$

and

$$0 = \big((P_{v_j} f)\big)^\flat(\xi_0^\sharp,\, \xi_0^\sharp) = f_{nn}.$$

The last three of these equations together with the fact that $-1 < \langle v_j, v_l\rangle_{g(x_0)} < 1$ imply that f_{nj} and $f_{jn} = 0$ for all j. Thus it only remains to show that $f_{kl} = 0$ for $1 \leq k,\, l \leq n-1$. From the conditions that $P_{v_j} f = 0$ for $j = n$ to $n(n-1)/2$, we obtain for all $1 \leq k,\, l \leq n-1$ that

$$0 = \langle \big((P_{v_k+v_l} f)\big)^\flat v_k,\, v_l\rangle_{g(x_0)} = \frac{1}{4}\langle f(v_k - v_l),\, v_l - v_k\rangle_{g(x_0)} = \frac{1}{4}(f_{kl} + f_{lk} - f_{kk} - f_{ll}). \tag{3.77}$$

These equations together with (3.76) form a system of linear equations for the components f_{jk} of $f^\flat$. We will now show that the only solution is $f_{jk} = 0$ for all $1 \leq j, k \leq n-1$.

First we deal with the symmetric part of $f^\flat$. Note that when we set $k = l$ in (3.76), we get

$$0 = f_{kk} + \langle v_j, v_k\rangle^2_{g(x_0)} f_{jj} - \langle v_j, v_k\rangle_{g(x_0)}(f_{jk} + f_{kj}). \tag{3.78}$$

By subtracting the corresponding equation with j and k switched we get

$$(1 - \langle v_j, v_k\rangle^2_{g(x_0)}) f_{kk} = (1 - \langle v_j, v_k\rangle^2_{g(x_0)}) f_{jj} \Rightarrow f_{jj} = f_{kk}$$

since $|\langle v_j, v_k\rangle_{g(x_0)}| < 1$. Thus all the diagonal entries of f are equal. Therefore (3.77) shows that $f_{jj} = \frac{1}{2}(f_{kl} + f_{lk})$ for any j, k, and l. Plugging this back into (3.78) then shows that

$$0 = (1 + \langle v_j, v_k\rangle^2_{g(x_0)} - 2\langle v_j, v_k\rangle_{g(x_0)})f_{jj} = (1 - \langle v_j, v_k\rangle_{g(x_0)})^2 f_{jj},$$

and since $\langle v_j, v_k\rangle_{g(x_0)} < 1$, this implies that the symmetric part of f is zero.

Now we show that the antisymmetric part of $f = 0$. Suppose that the i, j, and k in (3.76) are all distinct (note that this portion of the proof fails in dimension 3 because in that case there are only two vectors, v_1 and v_2, in the basis for $\xi_0^\perp$). Then we may cyclically permute the three indices in (3.76) and subtract the resulting equations pairwise to obtain

$$\begin{aligned}
(f_{kl} - f_{lk}) - \langle v_j, v_k\rangle_{g(x_0)}(f_{jl} - f_{lj}) - \langle v_j, v_l\rangle_{g(x_0)}(f_{kj} - f_{jk}) &= 0,\\
-\langle v_j, v_l\rangle_{g(x_0)}(f_{kl} - f_{lk}) + \langle v_l, v_k\rangle_{g(x_0)}(f_{jl} - f_{lj}) + (f_{kj} - f_{jk}) &= 0,\\
\langle v_j, v_k\rangle_{g(x_0)}(f_{kl} - f_{lk}) - (f_{jl} - f_{lj}) - \langle v_l, v_k\rangle_{g(x_0)}(f_{kj} - f_{jk}) &= 0.
\end{aligned}$$

If we consider this as a system of equations for $f_{kl} - f_{lk}$, $f_{jl} - f_{lj}$, and $f_{jk} - f_{kj}$, and then compute the determinant of the coefficient matrix we get

$$1 - (\langle v_l, v_k\rangle^2_{g(x_0)} + \langle v_j, v_k\rangle^2_{g(x_0)} + \langle v_j, v_l\rangle^2_{g(x_0)}) + 2\langle v_j, v_k\rangle_{g(x_0)}\langle v_j, v_l\rangle_{g(x_0)}\langle v_l, v_k\rangle_{g(x_0)}.$$

The fact that v_j, v_k, and v_l are linearly independent implies that this quantity is not zero. Therefore the antisymmetric part of f is zero, and we have completed the proof that $f = 0$. This shows that our system of equations (3.75) provides an elliptic system near $(0, 0, \xi_0)$ in dimension greater than 3, in the sense that the principal symbol at $(0, 0, \xi_0)$ admits a left inverse. In order to have an elliptic system in dimension 3, we must add another equation corresponding to the condition that $\mathrm{d}_\beta(f) = 0$. We will now proceed to introduce this extra equation.

Let $\chi_0 \in C_c^\infty(U')$ be a smooth cut-off function that is equal to 1 on a neighborhood of $x_0 = 0 \in U'$. Then, since $\mathrm{d}_\beta(f) = 0$ we have

$$h\, e^{\frac{i}{h}\Phi_1(x,y,\zeta)}\, \chi_0(x)\, \mathrm{d}_\beta(f)(x) = 0$$

where Φ_1 is one of the phase functions from (3.75). By integrating this equality in the x variable we obtain

$$\int h\, e^{\frac{i}{h}\Phi_1(x,y,\zeta)}\, \chi_0(x)\, \mathrm{d}_\beta(f)(x)\mathrm{d}x = 0,$$

and then integration by parts yields

$$\int e^{\frac{i}{h}\Phi_1(x,y,\zeta)}\Bigg(i\frac{\chi_0}{\sqrt{\det(g)}}\Big[\frac{\partial\Phi_1}{\partial x^3}(g_{1k}f_2^k - g_{2k}f_1^k) + \frac{\partial\Phi_1}{\partial x^2}(g_{3k}f_1^k - g_{1k}f_3^k) \\ + \frac{\partial\Phi_1}{\partial x^1}(g_{2k}f_3^k - g_{3k}f_2^k)\Big] + h\,D(x)_a^m f_m^a(x)\Bigg)\mathrm{d}x = 0 \tag{3.79}$$

where $D(x)_a^m$ is calculated from derivatives of χ_0 and the metric g. We consolidate the factors in the integrand into one classical analytic symbol and thus rewrite the previous formula as

$$\int e^{\frac{i}{h}\Phi_1(x,y,\zeta)}\chi_0(x)\tilde{D}(x,y,\zeta;h)_m^a f(x)_a^m \mathrm{d}x = 0$$

where the array of principal symbols of $\tilde{D}$ at $(0,0,\xi_0)$ is given by

$$\sigma_{\tilde{D}}(0,0,\xi_0)[f] = \frac{i}{\sqrt{\det(g)}}\left(g_{2k}f_1^k - g_{1k}f_2^k\right) \tag{3.80}$$

where we are once again expressing the components of f and g with respect to the basis $\{v_1,\, v_2,\, \xi_0^\sharp\}$. To show that the addition of the extra equation creates an elliptic system we must show that, for $f \in (T_1^1)^{\mathbb{C}}_{x_0}$, $\sigma_{\tilde{D}}(0,0,\xi_0)[f] = 0$ and $P_{v_j}f = 0$ for each j as above implies that $f = 0$. The argument to show that the symmetric part of f is zero still holds in this case, so we only need to show that the antisymmetric part is zero. Using the previous notation this means that we must show $f_{21} - f_{12} = 0$, but by (3.80) this is exactly equivalent to $\sigma_{\tilde{D}}(0,0,\xi_0)[f] = 0$. Therefore the system provided by the extra equation in dimension 3 is elliptic in the same sense as before.

In order to finish the proof we now must generalize a result from [21] to the case of systems of operators. This generalization has already been done in [25] and [10], and applied to different systems of operators. I will repeat the arguments given there and apply them in the present situation. We first combine the systems of operators derived so far in the proof as

$$\int_{|x-y|<C} e^{\frac{i}{h}\Phi_j(x,\beta)}\mathbf{A}(x,\beta;h)_{mkbj}^a\, f(x)_a^m \mathrm{d}x = \mathcal{O}(e^{-\frac{C}{h}}) \tag{3.81}$$

where C is a new constant, and following [21] we write $\beta = (y,\zeta)$. $\mathbf{A}(x,\beta;h)_{mkbj}^a$ is an array of classic analytic symbol of order 0 made up entirely of the symbols $\tilde{P}_j$ from (3.75) in the case of dimension greater than 3. In dimension 3 the symbol $\tilde{D}$ from (3.79) is also added. The key feature of $\mathbf{A}(x,\beta;h)^a_{mkbj}$ is that its principal symbol is injective as a map from $(T^1_1)_{x_0}M$ to $\big((T_2)_{\pi(v)}M\big)^{n-1}$ in the case of greater than 3 dimensions and to $\big((T_2)_{\pi(v)}M\big)^{n-1}\times\mathbb{C}$ in the case of three dimensions. Thus these maps have left inverses.

Now, following [21], [25], and [10] we define a system of ΨDO's in the complex domain

$$\mathrm{Op}(\mathbf{A})[f](y)_{kbj} = \iint e^{\frac{i}{h}\left(\Phi_j(y,\beta)-\overline{\Phi_j(x,\beta)}\right)}\mathbf{A}(x,\beta;h)^a_{mkbj}\, f(x)^m_a \mathrm{d}x\,\mathrm{d}\beta. \tag{3.82}$$

These operators have different phase functions Φ_j, but using the trick of Kuranishi (see [21, Remark 4.3]) we may make an appropriate series of changes of variables to change them all to the same phase function Φ without changing the principal symbols. Therefore we may construct a parametrix for $\mathrm{Op}(\mathbf{A})$ and use this parametrix to express $\mathbf{Id}\, e^{\frac{i}{h}\Phi}$ (where $\mathbf{Id} : (T^1_1)_{x_0}M \to (T^1_1)_{x_0}M$ is the identity map) as a superposition of the $\mathbf{A}^a_{mkbj}\, e^{\frac{i}{h}\Phi}$ modulo an exponentially decreasing function. Following now the same argument as is given for proposition 6.2 in [21], but with matrix valued symbols, we have

$$\int_{|x-y|<C} e^{\frac{i}{h}\Phi(x,\beta)}\mathbf{Id}[f](x)\,\mathrm{d}x = \mathcal{O}(e^{-C/h}),$$

possibly with yet another new constant C, for every $\beta=(y,\zeta)$ in a neighborhood of $(0,\xi_0)$. This proves that (x_0,ξ_0) is not in $\mathrm{WF}_{\mathrm{a}}(f)$.

□

Proof of Theorem 15 First we consider the case of dimension greater than 3. Assume that the hypotheses are all satisfied and $f\in L^2\tau^1_1(M)$ is in the kernel of I_{U_1,U_2}. Let M_1 be as in Theorem 12. Then, by Theorem 12, when we extend f as zero on $M_1\setminus M$, the resulting function is smooth on all of M_1, and still in the kernel of I_{U_1,U_2} acting now on $L^2\tau^1_1(M_1)$. Now by lemma 6 applied on M_1 the analytic wavefront set of f is empty, and therefore f is analytic. Since f vanishes on $M_1\setminus M$ this implies that $f=0$, and therefore proves that I_{U_1,U_2} is injective.

Now let us turn to the case of 3 dimensions. Once again assume that the hypotheses are all met, and that $f\in C^3_\beta\tau^1_1(M)$ is in the kernel of I_{U_1,U_2}, and that f satisfies the tangential boundary condition. Let M_1 be as in Theorem 13. Then by Theorem 13 when we extend f as zero on $M_1\setminus M$, the resulting tensor field is smooth on all of M_1, and satisfies $\mathrm{d}_\beta(f)=0$

on M_1. Therefore lemma 6 implies that the analytic wavefront set of f is empty, and just as above this implies that $f = 0$. This completes the proof.

$\square$

Chapter 4

THE NONLINEAR PROBLEM

We will now return to the fully nonlinear problem of recovering a tensor field $f \in \tau_1^1(M)$ from its polarization data $(U|_{\partial_+\Omega^{\mathbb{R}}M}$. Our main goal will be to establish so-called local injectivity for generic metrics for the nonlinear map $f \mapsto (U|_{\partial_+\Omega^{\mathbb{R}}M}$, and to accomplish this we use the stability and injectivity results for the linear problem established in Chapter 3. The precise meaning of this local injectivity will be stated later in Section 4.3. Our method here mostly fits into a general approach to the linearization of nonlinear inverse problems presented in [26].

In order to apply the results of Chapter 3 we must first deal with the issue of extending the semi-basic tensor fields U_1 and U_2 given respectively by (1.9) for some f_1 and $f_2 \in \tau_1^1(M)$ to the larger manifold M_1. Note that U_1 and U_2 are only defined from (1.9) on $\Omega^{\mathbb{R}}M \setminus T\partial M$, and so it is not clear that U_1 and U_2 can be extended to semi-basic fields on the larger manifold $\Omega^{\mathbb{R}}M_1$. In fact it is not possible in general to make such an extension since the derivatives of U_1 and U_2 may be unbounded near $T\partial M$. We will avoid this issue by replacing U_1 and U_2 with another pair of semi-basic tensor fields $\tilde{U}_1$ and $\tilde{U}_2 \in \beta_1^1(\Omega^{\mathbb{R}}M_1)$ such that (1.21) still holds. These fields are obtained by solving (1.9) on a larger manifold. A second related issue arises since we would like to establish uniqueness results for generic metrics g' obtained by perturbations near an analytic metric g. As noted in remark 8, when the metric is changed the set $\Omega^{\mathbb{R}}M$ also changes, and so we actually need to consider U_1 and $U_2 \in \beta_1^1(T^{\mathbb{R}}M_1 \setminus \{0\})$ as in Theorems 12 and 13. Section 4.1 deals with these issues of extending U_1 and U_2.

In Section 4.2 we consider the recovery of a field f only at the boundary from its polarization data. Finally in Section 4.3 we present and prove our main results concerning local injectivity.

4.1 Extending U_1 and U_2

Suppose that (M, g) is a simple manifold. Also suppose that we have an extension M_1 of M as described in Section 3.2 and let M_2 be an extension of M_1 accomplished in the same way as the extension from M to M_1. In particular we are assuming that (M_2, g) is still a simple manifold, and that $M \Subset M_1 \Subset M_2$. The metric g will be the reference around which we will perturb in order to obtain results for "generic" metrics.

Now let f_1 and $f_2 \in \tau_1^1(M)$. By [17] it is possible to define a linear and continuous extension mapping $E : \tau_1^1(M) \to (\tau_1^1)_c(M_2)$ so that all tensor fields in the range of E have support contained within a given compact set K such that $M \Subset K \Subset M_2$. We replace (1.9) with

$$HU(\xi) = [(P_\xi f)(x)]U(\xi) \quad \text{on } T^{\mathbb{R}}M_2 \setminus \{\{0\} \cup T\partial M_2\}, \quad U|_{\partial_- T^{\mathbb{R}}M_2} = E. \tag{4.1}$$

Here $\partial_- T^{\mathbb{R}}M_2$ is the space of inward pointing tangent vectors not necessarily having unit length. In this case (1.10) holds where η is still given by (1.6), but with $\xi \in \partial_- T^{\mathbb{R}}M_2$ rather than just $\partial_- \Omega^{\mathbb{R}}M$.

Now define $\tilde{U}_1$ and $\tilde{U}_2 \in \beta_1^1(T^{\mathbb{R}}M_2 \setminus \{\{0\} \cup T\partial M_2\})$ by solving (4.1) with f replaced respectively by either $E[f_1]$ or $E[f_2]$ on M_2. Certainly $\tilde{U}_1$ and $\tilde{U}_2$ restrict to smooth semi-basic tensor fields in $\beta_1^1(T^{\mathbb{R}}M_1 \setminus \{0\})$, and thus we will be able to apply the results from Chapter 3 to the operator $I_{\tilde{U}_1, \tilde{U}_2}$.

The main task in this section is to prove that $\tilde{U}_1$ and $\tilde{U}_2$ have the properties given in the following lemma. We first recall the definition of the annulus $(\Omega_a^b)^{\mathbb{R}}M_1 = \{(x, v) \in T^{\mathbb{R}}M_1 \,|\, a < \|v\|_g < b\}$ where $0 < a < b$.

Lemma 8 *The tensor fields $\tilde{U}_1$ and $\tilde{U}_2$ defined above posses the following properties.*

- *For every $v \in T^{\mathbb{R}}M_1 \setminus \{0\}$, $\tilde{U}_1(v), \tilde{U}_2(v) : T_{\pi(v)}M_1 \to T_{\pi(v)}M_1$ are invertible and satisfy (1.11).* (4.2)

- *If γ is any geodesic between points in ∂M_1 of length l, and $0 \leq t_1 \leq t_2 \leq l$, then (1.22) holds where M is replaced by M_1.* (4.3)

- *For fixed f_1 there exists an $\epsilon > 0$ such that if $\|f_1 - f_2\|_{C^3\tau_1^1(M)} < \epsilon$ then*

$$\|\tilde{U}_2 - \tilde{U}_1\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < C\|f_1 - f_2\|_{C^3\tau_1^1(M)} \tag{4.4}$$

for some constant $C > 0$ which may depend on a, b, g, and f_1, but does not depend on f_2.

Proof: The first two statements of the lemma, (4.2) and (4.3), are consequences of results from Chapter 1 applied on the larger manifold M_2. Indeed (4.2) follows from Lemma 1 and (4.3) follows from the derivation of the main identity in section 1.4. Now we turn to the proof of (4.4).

For any given $\xi \in \partial_-(\Omega_a^b)^{\mathbb{R}}M_2$ and $\eta_0 \in T_{\pi(\xi)}M_2$, let $\eta_i(\gamma_\xi(s), \xi)$ ($i = 1$ or 2) be the solution of (1.6) on M_2 corresponding to either f_1 or f_2. Using global coordinates on M_2 we also define $\eta_i(s, \xi)$ to be the vector $\eta_i(\gamma_\xi(s), \xi)$ expressed with respect to the coordinates. Then (1.6) becomes

$$\begin{aligned} \frac{\partial \eta_i^j}{\partial s}(s, \xi) &= \left([(P_{\dot{\gamma}_\xi(s)} f)(\gamma_\xi(s)]_k^j + \Gamma(\gamma_\xi(s))_{lk}^j \, \dot{\gamma}_\xi(s)^l \right) \eta_i^k(s, \xi) \\ &= G_i(s, \eta_i(s, \xi), \xi)^j \end{aligned} \tag{4.5}$$

and

$$\eta_i^j(0, \xi) = \eta_0^j.$$

Here the $\Gamma(x)_{lk}^j$ are the Christoffel symbols of the metric g. Now, in order to estimate $U_i(x, v)$ for any $(x, v) \in (\Omega_a^b)^{\mathbb{R}}M_1$ we modify (1.10) to get

$$\eta_i \left(l(x, -v), \left(\mathcal{I}_{0,-l(x,-v)}^{\gamma_{x,v}} \right)_a^b v^a \right)^j = \tilde{U}_i(x, v)_c^j \left(\mathcal{I}_{-l(x,-v),0}^{\gamma_{x,v}} \right)_d^c \eta_0^d. \tag{4.6}$$

Here $l(x, v)$ gives the positive endpoint of the maximally extended geodesic $\gamma_{x,v}$ in M_2. From this last equation we see that $\|\tilde{U}_1(x, v) - \tilde{U}_2(x, v)\|_{\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)}$ may be bounded if we can bound the difference $\eta_1(s, \xi) - \eta_2(s, \xi)$ for every $\xi \in \partial_-(\Omega_a^b)^{\mathbb{R}}M_2$ and $s \in \mathbb{R}$ such that $\gamma_\xi(s) \in M_1$. To do this we use lemma 5 and the fact that

$$G_1(s, \eta, \xi)^j - G_2(s, \eta, \xi)^j = [(P_{\gamma_\xi(s)}(f_1 - f_2)(\gamma_\xi(s))]_k^j \, \eta^k.$$

If we assume a priori that f_2 is close to f_1, then $\eta_2(s, \xi)$ will be bounded, and so with the hypotheses this implies that $\|\eta_1(s, \xi) - \eta_2(s, \xi)\| < C\|f_1 - f_2\|_{C\tau_1^1(M)}$ uniformly for the required values of s and ξ where $C > 0$ does not depend on f_2. The norm on the left hand side of this estimate could be any norm on $\mathbb{C}^n$.

Next, differentiating (4.5) with respect to either ξ or s we may obtain ODEs satisfied by the derivatives of the left hand side of (4.6). Using this we can apply the same analysis as above to bound derivatives of $\tilde{U}_1(x,v) - \tilde{U}_2(x,v)$, except we also require bounds on the corresponding derivatives of $f_1 - f_2$. This proves the result.

□

The final part of lemma 8 shows how the solution of (4.1) behaves when f is perturbed, but we also want to see this behavior under perturbations of the metric g. Thus, we now suppose that $g' \in S_2M_2$ is another metric on M_2. If g' is sufficiently close to g in $C^4S_2M_2$, then by Theorem 14 (M_2, g') is still a simple manifold, and we will always assume that g' is such a metric. Let $f \in \tau_1^1(M)$ be one of the two tensor fields from above (either f_1 or f_2), and let $\tilde{U}$ also be the corresponding semi-basic tensor field defined from (4.1). If $\tilde{U}'$ is defined from (4.1) with g replaced by g', then we have the following lemma.

Lemma 9 *For a fixed metric g, there is an $\epsilon > 0$ such that whenever $\|g - g'\|_{C^4S_2M_2} < \epsilon$, for every $A > 0$ there is a constant C such that*

$$\|\tilde{U} - \tilde{U}'\|_{C^3\beta_1^1((\Omega_a^b)^{\mathbb{R}}M_1)} < C\|g - g'\|_{C^4S_2M_2}$$

for every f with $\|f\|_{C^3\tau_1^1(M)} < A$.

Proof: As in the proof of (4.4) above, this result follows essentially from lemma 5. Throughout we will use primes to indicate objects corresponding to the metric g', while unprimed objects will be those corresponding to g. As above we use (4.6) to estimate the difference $\tilde{U} - \tilde{U}'$. Indeed, working in global coordinates and using (4.6) we have

$$\begin{aligned}
&\eta\left(l(x,-v), \left(\mathcal{I}^{\gamma_{x,v}}_{0,-l(x,-v)}\right)^b_a v^a\right)^j - \eta'\left(l'(x,-v), \left(\mathcal{I}'^{\gamma'_{x,v}}_{0,-l'(x,-v)}\right)^b_a v^a\right)^j \\
&\quad + \tilde{U}'(x,v)^j_c \left(\mathcal{I}'^{\gamma'_{x,v}}_{-l'(x,-v),0} - \mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0}\right)^c_d \eta_0^d \\
&\qquad = \left(\tilde{U}(x,v) - \tilde{U}'(x,v)\right)^j_c \left(\mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0}\right)^c_d \eta_0^d.
\end{aligned} \tag{4.7}$$

We will estimate each of the lines in (4.7) separately, but in order to do this we first need an estimate of the difference $l(x,v) - l'(x,v)$.

We now proceed to establish this estimate. We will use the fact from the proof of Theorem 3 that both l and l' can be defined implicitly by (2.3) (now of course ρ is a defining function for ∂M_2). Suppose now that for every $s \in [0,1]$ we define the metric g_s by

$$g_s = sg + (1-s)g' \tag{4.8}$$

so that $g_0 = g'$ and $g_1 = g$. When ϵ is small enough every g_s will still be a simple metric on M_2. Now let $\exp^{\mathrm{s}}$ denote the exponential map corresponding to each s, and let l^s be the corresponding positive function on $(\Omega_a^b)^{\mathbb{R}} M_1$ defined as in (2.3) by

$$\rho(\exp_x^s(l^s v)) = 0 \tag{4.9}$$

where ρ is a defining function for ∂M_2. As in the proof of Theorem 3, by the implicit function theorem, $l^s(x,v)$ is a smooth function of s, x, and v. Furthermore, we may calculate the derivative of $l^s(x,v)$ with respect to s from (4.9). Indeed

$$\frac{\partial l^s}{\partial s}(x,v) = -\frac{1}{\mathrm{d}\rho\left(\dot{\gamma}^s_{x,v}(l^s(x,v)v)\right)}\, \mathrm{d}\rho\left(\frac{\partial (\exp_x^s)}{\partial s}(l^s(x,v)v)\right). \tag{4.10}$$

By examining the ODE defining the exponential map, we can bound the second term on the right hand side of (4.10) by $C\|g - g'\|_{C^1 S_2 M_2}$ for any $(x,v) \in (\Omega_a^b)^{\mathbb{R}} M_1$ where the constant C does not g'. The first term on the right side of (4.10) can be also be bounded, using also the simplicity assumption, uniformly for any $(x,v) \in (\Omega_a^b)^{\mathbb{R}} M_1$. Therefore since $l^1 = l$ and $l^0 = l'$, the mean value theorem shows that $\|l - l'\|_{C((\Omega_a^b)^{\mathbb{R}} M_1)} < C\|g - g'\|_{C^1 S_2 M_2}$. Differentiating (4.10) we can similarly show that $\|l - l'\|_{C^3((\Omega_a^b)^{\mathbb{R}} M_1)} < C\|g - g'\|_{C^4 S_2 M_2}$.

Now we return to estimating the left hand side of (4.7). By the argument used to establish (3.50) at the end of the proof of Theorem 14, we already have that

$$\left\| \left(\left(\mathcal{I}_{0,t}^{\gamma_{x,v}}\right)^j_b - \left(\mathcal{I}'^{\gamma'_{x,v}}_{0,t}\right)^j_b \right) \eta^b \right\|_{C^3_{x,t,v}} < C\|g - g'\|_{C^4 S_2 M_2}$$

where the norm on the left is over $(x,v) \in (\Omega_a^b)^{\mathbb{R}} M_1$ and t in the domain of both $\gamma_{x,v}$ and $\gamma'_{x.v}$. Since the determinants of the parallel translations are bounded below uniformly, and

$\mathcal{I}^{\gamma_{x,v}}_{t,0} = (\mathcal{I}^{\gamma_{x,v}}_{0,t})^{-1}$, we also obtain

$$\left\| \left(\left(\mathcal{I}^{\gamma_{x,v}}_{t,0} \right)^j_b - \left(\mathcal{I}'^{\gamma'_{x,v}}_{t,0} \right)^j_b \right) \eta^b \right\|_{C^3_{x,t,v}} < C \|g - g'\|_{C^4 S_2 M_2}.$$

Additionally, using (4.5) and (4.6) as well as lemma 5, we have $\|\tilde{U}'(x,v)\|_{C^3 \beta^1_1((\Omega^b_a)^{\mathbb{R}} M_1)} < C \|f\|_{C^3 \tau^1_1(M_1)}$ (by for example comparing (4.5) for f with (4.5) for the zero tensor field) were C may depend on g and ϵ. Putting all this together and assuming that f satisfies an a priori bound as given in the hypothesis, this establishes the desired bound on the second line of (4.7).

Finally, we estimate the first line in (4.7) by using (4.5), and the previous results established in this proof. Using (4.7) for every vector η_0, we thus obtain that for $(x,v) \in (\Omega^b_a)^{\mathbb{R}} M_1$

$$\| \left(\tilde{U}(x,v) - \tilde{U}'(x,v) \right)^j_c \left(\mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0} \right)^c_d \eta^d_0 \|_{C^3_{x,v}} < C \|g - g'\|_{C^4 S_2 M_2}.$$

Since $\mathcal{I}^{\gamma_{x,v}}_{0,-l(x,-v)}$ is uniformly bounded in $C^3_{x,v}$ for $(x,v) \in (\Omega^b_a)^{\mathbb{R}} M_1$, this proves the result.

□

As a warmup for the full inverse problem, in the next section we will examine the problem of recovering a tensor field f on ∂M from its polarization data.

4.2 Recovery at the boundary

The problem of recovering $f \in \tau^1_1(M)$ on ∂M from its polarization data is considered in [9]. Some of the results in the present section follow from the main results found there, however there are a few differences. First, we consider here only the case of a simple manifold, while [9] applies to CNT manifolds. Second, [9] provides an explicit method for recovering the full jet of f on ∂M from the polarization data, while here we only prove that this full jet is uniquely determined from the polarization data, but do not give a recovery procedure. The 3 dimensional results are also slightly different. In particular we consider here the case when f satisfies $\mathrm{d}_\beta(f) = 0$ and the tangential boundary condition with respect to a perturbed metric g'. This is necessary for our main result in section 4.3.

Theorem 16 *Let (M, g) be a simple manifold of dimension greater than 3, and let f_1 and $f_2 \in \tau^1_1(M)$ be smooth tensor fields with the same polarization data. Then f_1 and f_2 agree to infinite order on ∂M. In dimension 3, the same statement is true if we assume that $\mathrm{d}_\beta(f_1 - f_2) = 0$ on M, and that $f_1 - f_2$ satisfies the tangential boundary condition. Furthermore, there is an $\epsilon > 0$ such that whenever $g' \in S_2M_2$ satisfies $\|g - g'\|_{S_2M_2} < \epsilon$, if $\mathrm{d}_\beta(f_1 - f_2) = 0$ and $f_1 - f_2$ satisfies the tangential boundary condition with respect to g', then f_1 and f_2 agree to first order on ∂M.*

Proof: Let f_1 and f_2 be as in the statement of the theorem, and let $\tilde{U}_1$ and $\tilde{U}_2 \in \beta^1_1(T^{\mathbb{R}}M_1 \setminus \{0\})$ be the corresponding semi-basic tensor fields introduced in section 4.1. Now define $f \in L^2\tau^1_1(M_1)$ by setting

$$f(x) = \begin{cases} f_1(x) - f_2(x) & \text{if } x \in M \\ 0 & \text{if } x \in M_1 \setminus M. \end{cases} \tag{4.11}$$

We will consider the x-ray transform $I_{\tilde{U}_1,\tilde{U}_2}[f]$ on M_1.

Let $v \in \partial_-\Omega^{\mathbb{R}}M_1$, and let γ_v denote the geodesic in M_1 with initial data $\dot{\gamma}(0) = v$. Suppose that γ_v has length l. If γ_v does not pass through the interior of M, then we easily see from the definition of f that $I_{\tilde{U}_1,\tilde{U}_2}[f](v) = 0$. Thus, suppose that γ_v does pass through the interior of M. Then since ∂M is convex and (M_1, g) is simple there must be unique times t_1 and t_2 with $0 \le t_1 \le t_2 \le l$ when γ_v enters and exits M respectively. Then by (1.22), which holds according to (4.3), we have for any η_0 and $\zeta \in T_{\pi(v)}(M_1)$

$$\begin{aligned} &\Big\langle \Big(\mathcal{I}^{\gamma_v}_{t_2,0}\, \tilde{U}_2^{-1}(\xi(t_2))\, \tilde{U}_1(\xi(t_2))\, \mathcal{I}^{\gamma_v}_{0,t_2} - \mathcal{I}^{\gamma_v}_{t_1,0}\, \tilde{U}_2^{-1}(\xi(t_1))\, \tilde{U}_1(\xi(t_1))\, \mathcal{I}^{\gamma_v}_{0,t_1}\Big)\, \eta_0, \zeta\Big\rangle_{g(y)} \\ &\qquad = \int_{t_1}^{t_2} \Big\langle \tilde{U}_2^{-1}\left[P_{\xi(s)}\,(f_1 - f_2)\right](\gamma_v(s))\, \tilde{U}_1\, \mathcal{I}^{\gamma_v}_{0,s}\, \eta_0,\, \mathcal{I}^{\gamma_v}_{0,s}\, \zeta\Big\rangle_{g(\gamma_v(s))} \mathrm{d}s \\ &\qquad = \int_0^l \Big\langle \tilde{U}_2^{-1}\left[P_{\xi(s)}\,(f_1 - f_2)\right](\gamma_v(s))\, \tilde{U}_1\, \mathcal{I}^{\gamma_v}_{0,s}\, \eta_0,\, \mathcal{I}^{\gamma_v}_{0,s}\, \zeta\Big\rangle_{g(\gamma_v(s))} \mathrm{d}s \\ &\qquad = I_{\tilde{U}_1,\tilde{U}_2}[f](v)(\eta_0, \zeta). \end{aligned} \tag{4.12}$$

where, as in (1.22), $\xi(s) = \dot{\gamma}_v(s)$. We will show that when f_1 and f_2 have the same polarization data the first line of (4.12) is zero.

In order to do this, let us consider how $U_i(\xi(s))$ and $\tilde{U}_i(\xi(s))$ are related for $t_1 \le s \le t_2$. Here and in the rest of this paragraph i may be either 1 or 2. Recall that U_i is given by solving (1.9) on M, while $\tilde{U}_i$ is obtained by solving (4.1) on M_2 with f replaced by $E[f]$. Take any $\eta_0 \in T_{\gamma_v(t_1)}M$ and note that both $U_i(\xi(s))\,\mathcal{I}^{\gamma_v}_{t_1,s}\,\eta_0$ and $\tilde{U}_i(\xi(s))\,\mathcal{I}^{\gamma_v}_{0,s}\,\mathcal{I}^{\gamma_v}_{t_1,0}\,\tilde{U}_i^{-1}(\xi(t_1))\,\eta_0$ solve (1.6) where the initial data is taken at t_1 rather than 0. Therefore, by the uniqueness of solutions to (1.6) we obtain that

$$U_i(\xi(s))\,\mathcal{I}^{\gamma_v}_{t_1,s} = \tilde{U}_i(\xi(s))\,\mathcal{I}^{\gamma_v}_{0,s}\,\mathcal{I}^{\gamma_v}_{t_1,0}\,\tilde{U}_i^{-1}(\xi(t_1)) \tag{4.13}$$

for $t_1 \leq s \leq t_2$. If f_1 and f_2 have the same polarization data, then $U_1(\xi(t_2)) = U_2(\xi(t_2))$. Using (4.13) this can be written in terms of $\tilde{U}_1$ and $\tilde{U}_2$ as

$$\tilde{U}_1(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2}\,\tilde{U}_1^{-1}(\xi(t_1)) = \tilde{U}_2(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2}\,\tilde{U}_2^{-1}(\xi(t_1)). \tag{4.14}$$

Finally, using (4.14) to simplify the first line of (4.12) we obtain

$$0 = \left\langle \mathcal{I}^{\gamma_v}_{t_2,0}\,\tilde{U}_2^{-1}(\xi(t_2))\left(\tilde{U}_1(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2} - \tilde{U}_1(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2}\right)\mathcal{I}^{\gamma_v}_{0,t_1}\,\eta_0, \zeta\right\rangle_{g(y)} =$$

$$\left\langle \mathcal{I}^{\gamma_v}_{t_2,0}\,\tilde{U}_2^{-1}(\xi(t_2))\left(\tilde{U}_1(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2} - \tilde{U}_2(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2}\,\tilde{U}_2^{-1}(\xi(t_1))\,\tilde{U}_1(\xi(t_1))\right)\mathcal{I}^{\gamma_v}_{0,t_1}\,\eta_0, \zeta\right\rangle_{g(y)}$$

$$= I_{\tilde{U}_1,\tilde{U}_2}[f](v)(\eta_0, \zeta).$$

Therefore $I_{\tilde{U}_1,\tilde{U}_2}[f] = 0$. Thus far all steps in the proof apply in any dimension, but now we must apply either Theorem 12 or Theorem 13 to complete the proof. In either case we have that f vanishes to infinite order on ∂M. By the definition of f this implies that f_1 and f_2 agree to infinite order on ∂M. Finally, the last statement of the theorem follows from lemma 4.

□

Remark 11: Note that the result of Theorem 16 implies that $f_1 = f_2$ if f_1 and f_2 have the same polarization data and are assumed to be real-analytic.

4.3 Local invertibility of the non-linear problem

We now turn to the full inverse problem and establish local invertibility and stability for generic simple metrics and near a generic set of tensor fields $f \in \tau^1_1(M)$. We now restate and prove Theorem 1 given in the introduction.

Theorem *Assume that* (M, g) *is a real-analytic simple manifold of dimension greater than* 3 *with real-analytic metric* g*. If* $\hat{f} \in \tau_1^1(M)$ *is real-analytic, then there exists an* $\epsilon > 0$ *such that whenever* $g' \in S_2(M)$ *is another metric on* M *and* f_1*,* $f_2 \in \tau_1^1(M)$ *are such that*

$$\|g - g'\|_{C^4 S_2(M)} < \epsilon, \quad \text{and} \quad \|\hat{f} - f_i\|_{C^3 \tau_1^1(M)} < \epsilon \quad \text{for } i = 1 \text{ and } 2, \tag{4.15}$$

if the polarization data of f_1 *and* f_2 *with respect to the metric* g' *are the same then* $f_1 = f_2$ *(this is the meaning of local injectivity). Furthermore, there is a stability estimate for such* f_1 *and* f_2

$$\|f_1 - f_2\|_{L^2 \tau_1^1(M)} \leq C \|U_2' - U_1'\|_{H^1 \beta_1^1((\partial_+ \Omega^{\mathbb{R}})' M)} \tag{4.16}$$

for some constant $C > 0$*. If the dimension is* 3*, then the statement of local injectivity still holds if we also assume that* $\mathrm{d}_\beta(f_1 - f_2) = 0$ *and that* $f_1 - f_2$ *satisfies the tangential boundary condition with respect to the metric* g*. The stability estimate also holds if* f_1 *and* f_2 *are further restricted to have support within a given compact set* $K \Subset M$ *where the constant* C *in (4.16) may then depend on the set* K*.*

Proof: Let (M, g), $\hat{f}$, f_1, and f_2 be as in the statement of the theorem. Also let $\hat{U}$, $\tilde{\hat{U}}$, U_1, $\tilde{U}_1$, U_2 and $\tilde{U}_2$ denote the respective semi-basic tensor fields described in section 4.1 corresponding respectively to $\hat{f}$, f_1 and f_2, and define $f \in L^2 \tau_1^1(M)$ just as in the proof of Theorem 16 by (4.11). Further, the same objects defined with respect to g' will be denoted with a prime. In the 3 dimensional case note that $f \in L^2_\beta(M)$ since by assumption $\mathrm{d}_\beta(f_1 - f_2) = 0$ on the interior of M.

For the next paragraph we consider only the case of dimension greater than 3. We first assume that f_1 and f_2 have the same polarization data. Then, as already established in the proof of Theorem 16

$$I_{\tilde{U}_1', \tilde{U}_2'}[f] = 0. \tag{4.17}$$

Now we see that the results on the linear problem of inverting $I_{\tilde{U}_1, \tilde{U}_2}$ from Chapter 3 will play an important role here. Indeed, we will now show that $I_{\tilde{U}_1, \tilde{U}_2}$ is injective. In order to do this we must first use (4.4) and lemma 9. Taken together these imply that for ϵ sufficiently small

$$\|\tilde{\hat{U}} - \tilde{U}_i'\|_{C^3 \beta_1^1((\Omega_a^b)^{\mathbb{R}} M_1)} < C(\|\hat{f} - f_i\|_{C^3 \tau_1^1(M)} + \|g - g'\|_{C^4 S_2 M_2}) \tag{4.18}$$

for a constant $C > 0$ that does not depend on f_i or g'. Now note that by Theorem 15 $I_{\tilde{\tilde{U}},\tilde{\tilde{U}}}$ is injective on $L^2\tau_1^1(M)$, since $\tilde{\tilde{U}}$ is analytic, if the dimension is greater than 3. Therefore if ϵ is taken to be small enough, by Theorem 12 we have the stability estimate

$$\|f\|_{L^2\tau_1^1(M)} \leq C\|\mathcal{N}_{\tilde{U}_1',\tilde{U}_2'}[f]\|_{H^1\tau_1^1(M_1)}. \tag{4.19}$$

where $\mathcal{N}_{\tilde{U}_1',\tilde{U}_2'}$ is with respect to the g' metric. By (4.17) this implies that $f = 0$, or $f_1 = f_2$ when the polarization data are the same. This proves the local injectivity part of the theorem in dimension greater than 3. We next prove the local injectivity in dimension 3.

In the case of dimension 3, (4.17) and (4.18) from the previous paragraph still apply. The difficulty in the rest of the proof is that Theorem 15 only shows that $I_{\tilde{\tilde{U}},\tilde{\tilde{U}}}$ is injective on $C_\beta^3\tau_1^1(M)$, which is not a closed subspace of $L_\beta^2\tau_1^1(M)$ and so the stability estimate (3.18) may not hold. We avoid this difficulty however by noting that when $\mathrm{d}_\beta(f_1 - f_2) = 0$ on M and $f_1 - f_2$ satisfies the tangential boundary condition with respect to g, and ϵ is small enough, by Theorem 16 $f_1 - f_2$ vanishes to first order on ∂M. Therefore f is actually in $L_\beta^2\tau_1^1(M_1)$ (with respect to g), and has support contained in M. Now we introduce an intermediate manifold $M_{1/2}$ such that $M \Subset M_{1/2} \Subset M_1$. The subspace $\mathcal{L}$ of $L_\beta^2\tau_1^1(M_{1/2})$ consisting of tensor fields having support contained in M is a closed subspace of $L_\beta^2\tau_1^1(M_{1/2})$, and by Theorem 12 and Theorem 15 applied on the manifold $M_{1/2}$, $I_{\tilde{U}',\tilde{U}'}$ is injective on $\mathcal{L}$. Therefore, by Theorem 13 (3.18) holds for tensor fields in $\mathcal{L}$, and so (4.19) also holds there if ϵ is small enough. Since f is in $\mathcal{L}$, this proves that $f = 0$, or $f_1 = f_2$.

Now we move to the proof of the local stability estimate (4.16). As above, we first work in the case of dimension greater than 3. Assuming that ϵ is sufficiently small, most of the argument in the previous paragraph still holds, and we may still establish (4.19). By Theorem 11 this implies

$$\|f\|_{L^2\tau_1^1(M)} \leq C\|I_{\tilde{U}_1',\tilde{U}_2'}[f]\|_{H^1\beta_2((\partial_-\Omega^{\mathbb{R}})'M_1)} \tag{4.20}$$

We no longer have that the right hand side is zero, but we still have (4.12). Using (4.13) with $s = t_2$ we may rewrite (4.12) partially in terms of U_1' and U_2' as follows

$$I_{\tilde{U}_1',\tilde{U}_2'}[f](v)(\eta_0,\zeta) = \Big\langle \mathcal{I}^{\gamma_v}_{t_2,0}\,(\tilde{U}_2')^{-1}(\xi(t_2))\,\Big(\tilde{U}_1'(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2}\,(\tilde{U}_1')^{-1}(\xi(t_1)) - \tilde{U}_2'(\xi(t_2))\,\mathcal{I}^{\gamma_v}_{t_1,t_2}\,\tilde{(}U_2')^{-1}(\xi(t_1))\Big)\,\tilde{U}_1'(\xi(t_1))\,\mathcal{I}^{\gamma_v}_{0,t_1}\,\eta_0,\zeta\Big\rangle_{g(y)}$$

$$= \Big\langle \mathcal{I}^{\gamma_v}_{t_2,0}\,(\tilde{U}_2')^{-1}(\xi(t_2))\,\big(U_1'(\xi(t_2)) - U_2'(\xi(t_2))\big)\,\mathcal{I}^{\gamma_v}_{t_1,t_2}\,\tilde{U}_1'(\xi(t_1))\,\mathcal{I}^{\gamma_v}_{0,t_1}\,\eta_0,\zeta\Big\rangle_{g(y)}.$$

By (4.4) and the hypothesis that $\|f' - f_i\|_{C^3\tau^1_1(M)} < \epsilon$, for ϵ small enough the terms $\tilde{U}_2^{-1}(\xi(t_2))$ and $\tilde{U}_1(\xi(t_1))$ are bounded and have bounded derivatives, and so this last identity together with (4.20) shows that

$$\|f_1 - f_2\|_{L^2\tau^1_1(M)} = \|f\|_{L^2\tau^1_1(M)} \le C\|U_2' - U_1'\|_{H^1\beta^1_1((\partial_+\Omega^{\mathbb{R}})'M)}.$$

This completes the proof of the stability estimate for dimension greater than 3.

In dimension 3 we once again have the problem that $I_{\tilde{\tilde{U}},\tilde{\tilde{U}}}$ is only injective on $C^3_\beta\tau^1_1(M)$, which is not a closed subspace of $L^2_\beta\tau^1_1(M)$. However, if we restrict to consider the space $\mathcal{L}$ of tensor fields in $L^2_\beta\tau^1_1(M)$ having support within a fixed compact set $K \Subset M$, then by Theorems 13 and 15 $I_{\tilde{\tilde{U}},\tilde{\tilde{U}}}$ is injective on $\mathcal{L}$. Therefore by Theorem 13 we have (4.19) for $f \in \mathcal{L}$, and the remainder of the proof follows as in the higher dimensional case.

□

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