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Differential and finite-difference problems of active shielding

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Abstract

The paper presents the solution of a very important problem, which can be used for active noise shielding and vibration control. The problem of active shielding is related with shielding one domain from the influence of another one via a distribution of additional sources outside of the first domain. The general solution of the problem of active shielding in the differential form is obtained. The solution only requires the knowledge of the total field on the boundary of the shielded area. It does not need any additional information about the characteristics of the undesirable sources or the surrounding medium. The knowledge of the Green's function is not required either. The active shielding source terms in acoustics are obtained in the form of the potential of a simple layer. We show a correspondence between the active shielding solutions obtained in the analytical and finite-difference formulations. We also suggest an estimate of a space step in the finite-difference formulation.

Keywords: Active shielding; Noise control; Generalized solution; Simple-layer potential; General solution; Acoustics equations

1. Introduction

The problem of active shielding (AS) is related to shielding one domain from the influence of another one via a distribution of additional sources outside of the first domain. The solution of this problem can be applied to active noise shielding and vibration control. The problem of active noise control is a relatively new but extensively developed area in acoustics. In this problem it is assumed that either some internal or external area is shielded via the implementation of additional (secondary) sources or active shielding. This is the key difference from "passive" shielding where noise reduction is achieved via mechanical obstacles. In addition, the problem becomes much more complicated if some "friendly" sound is assumed to be in the protected area.

There are many publications devoted to the problem of AS, sound and vibration control. The first theoretical papers in this field by Jessel, Malyuzhinets and Fedoryuk appeared about 30 years ago [5,10,3], while the first publications related to some possible realistic implementation arose much later (see, e.g., [1,2]). The most comprehensive theoretical and practical reviews can be found in books [11,4,14]. As mentioned above in the standard approaches to AS, it is necessary to know the characteristics of "adverse" sources including their location. From a practical view, this information is often not available. To our knowledge, it was Malyuzhinets who first suggested a solution of the problem that only requires the information on the total sound (both "friendly" and "adverse") at the perimeter of the

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domain to be shielded. The method of the factorization of the "adverse" and "friendly" components was considered by Malyuzhinets and Fedoryuk. These works were based on the accessibility of the Green's function and were developed for the Helmholtz equation with constant coefficients. Tsynkov [15] derived the general single-layer and double-layer solutions of the problem for the Helmholtz equation in both continuous and finite-difference formulations. This solution is applicable to the linear analogue of the Helmholtz equation with variable coefficients. The general solution of the problem in a finite-difference formulation was obtained by Ryaben'kii [12,13]. The solution is based on the Difference Potential Method (DPM) [13] and applicable to arbitrary geometric configurations, medium properties, and boundary conditions. The optimization of the finite-difference solution [13,15] in application to the Helmholtz equation is studied in [7–9].

In the current paper, the solution of the AS problem is obtained in the general continuous formulation. In a way, this solution can be considered as a generalization of the solution [12,13] to the continuous case. In particular, the solution provides the general surface-layer AS for the Helmholtz equation (and its linear analogue with variable coefficients) and the acoustics equations without any specific limitations apart from linearity and the uniqueness of solution. The correspondence between the differential and finite-difference solutions is shown. The solution for the discrete case depends on a parameter, a space step, and its optimal value is evaluated.

2. General formulation of the active shielding problem

General mathematical formulation of the AS problem is presented in the following form. Let us assume that some field (sound) is described by the following boundary value problem in a domain $D^0 \subseteq \mathbb{R}^m$:

$$LW = F, \tag{1}$$

$$W \in \widetilde{U}_{D^0}. \tag{2}$$

Here, Eq. (1) is supposed to be linear; $\widetilde{U}_{D^0}$ is a linear subspace of the functions belonging to the space of smooth enough functions on $\overline{D^0}$ and satisfying some linear boundary conditions such that the only solution of the boundary value problem (1), (2), if $F = 0$, is trivial. It is worth noting that Eq. (1) can represent a system of equations, so that W and F can be vector-functions. The operator L is some linear differential operator. In particular, it can correspond to the operator of either the Helmholtz equation or the acoustics equations.

Consider now some bounded domain $D \equiv D^+$ such that $\overline{D} \subset D^0$. It is worth noting that the domain D can be a disconnected domain. It is assumed that the domain D has a smooth boundary Γ. The sources at the right-hand side can be situated both in D and outside of D:

$$\begin{aligned} &F = f^+ + f^-, \\ &\operatorname{supp} f^+ \subset \overline{D}, \\ &\operatorname{supp} f^- \subset D^- \stackrel{\text{def}}{=} D^0 \backslash \overline{D}. \end{aligned} \tag{3}$$

Here, f^+ is the source of a "friendly" field (sound), while f^- is the source of an "adverse" field (noise).

Suppose that we know the value $W_\Gamma \stackrel{\text{def}}{=} W_{|\Gamma}$ of the function W on the boundary Γ of D. It is to be noted that only this information is assumed to be available. In particular, the distribution of the sources F at the right-hand side is unknown. The AS problem is reduced to seeking additional sources G in $\overline{D^-}$ such that the solution of problem

$$\begin{aligned} &LW^{(g)} = F + G, \\ &\operatorname{supp} G \subset \overline{D^-}, \end{aligned} \tag{4}$$

$$W^{(g)} \in \widetilde{U}_{D^0} \tag{5}$$

coincides with the solution W of problem (1), (2) on the subdomain D if $F = f^+$. It is to be noted that an "obvious" solution $F = -f^-$ is not appropriate here because the distribution of f^- is unknown.

3. General solution of finite-difference active shielding problem

Following the DPM [13], let us introduce some grid M^0 in D^0. We introduce subsets of grid M^0 as follows: $M^+ \stackrel{\text{def}}{=} M^0 \cap \overline{D}$, $M^- \stackrel{\text{def}}{=} M^0 \backslash M^+$. Suppose that Eq. (1) is approximated on some stencil by equation

$$L_h W^{(h)}{}_{|m} \stackrel{\text{def}}{=} \sum_n a_{mn} W_n^{(h)} = F^{(h)}{}_{|m}, \quad m \in M^0. \tag{6}$$

Eq. (6) is completed by a boundary condition approximating the continuous boundary condition (2):

$$W^{(h)} \in U_{D^0}^{(h)}, \tag{7}$$

where $W_{D^0}^{(h)}$ is a linear space that is a counterpart of the space $\widetilde{U}_{D^0}$.

The extensions of the sets M^0, M^+, M^- by the stencil we denote as N^0, N^+, N^-, respectively. It is clear that the sets N^+ and N^- intersect each other. Their intersection is regarded as the grid boundary γ of the domain D: $\gamma = N^+ \cap N^-$. The grid boundary γ is split into two nonintersecting subsets: $\gamma = \gamma^+ \cup \gamma^-$, where $\gamma^+ = \gamma \cap M^+$ and $\gamma^- = \gamma \cap M^-$.

Now we wish to find the finite-difference solution of the AS problem (4), (5).

It is given by the following theorem [12]:

$$G_h = -L_h V^{(h)}{}_{|M^-}, \quad m \in M^-, \tag{8}$$

$$G_h = 0, \quad m \in M^+, \tag{9}$$

where $V^{(h)}$ is an arbitrary function such that

$$V^{(h)} \in U_{D^0}^{(h)}, \quad V_\gamma^{(h)} = W_\gamma^{(h)}. \tag{10}$$

The proof of this theorem can be found in [13]. It is important to emphasize that the solution given by Eq. (8) is obtained for the general statement of the problem formulated in the previous section and is not based on the knowledge of the Green's function.

It is clear that the function $V^{(h)}$ in (8) is not unique. A particular case of this function corresponds to $V^{(h)}{}_{|N^0\backslash\gamma} = 0$. In this case the AS source term is only situated on a minimal possible support.

4. General solution of the differential problem

Now suppose that the field W is described on either $\mathbb{R}^m$ or $\mathbb{R}^{m+1}$ by the following linear boundary value problem defined on $D^0 = D^+ \cup D^-$:

$$LW \stackrel{\text{def}}{=} \sum_1^m A^i \frac{\partial W}{\partial x^i} + R = F, \tag{11}$$

$$W \in U_{D^0},$$

$$F = f^- + f^+,$$

$$\operatorname{supp} f^+ \subset \overline{D^+}, \qquad \operatorname{supp} f^- \subset D^-, \tag{12}$$

where $\{x^i\}$ $(i = 1, \dots, m)$ is some nonsingular coordinate system; W, $R \stackrel{\text{def}}{=} BW$ and F are vector-functions taking values in $\mathbb{R}^p$; A^i, B are $p \times p$ matrices such that $A^i = A^i(\mathbf{x})$, $B = B(\mathbf{x}) \in C^\infty(\mathbb{R}^m)$.

Let us consider the solution of problem (11), (12) in the weak sense (see, e.g., [6,16]). Suppose that U_{D^0} is the completion of the linear space $\widetilde{U}_{D^0}$ in some Sobolev norm, so that $\widetilde{U}_{D^0}$ is dense in U_{D^0}. The space U_{D^0} contains, in particular, piece-wise smooth functions. We say that a function $W \in U_{D^0}$ is a generalized solution of boundary value problem (11), (12) if $(LW, \Phi) = (F, \Phi)$ for any test function $\Phi(D^0) \in C_0^\infty(D^0)$. Here, (f, Φ) denotes the linear functional associated with a given generalized function f.

Suppose that the right-hand side F belongs to some space Y_F of "admissible" functions such that problem (11), (12) has a unique generalized solution. It is also supposed that if $F = f^+ + f^-$, then $f^+ \in Y_F$, $f^- \in Y_F$ and vice versa. This means that a generalized solution of problem (11), (12) exists if either $F = f^+$ or $F = f^-$. To avoid possible uncertainties, we also make a natural assumption that for any non-zero density the appropriate simple-layer potential solution is unique and cannot be trivial in D^+.

Then the solution of the AS problem is given by the following theorem.

Theorem. *The general solution of the AS problem formulated for* (11), (12) *is as follows*:

$$G = A_n W_\Gamma \delta(\Gamma) + LW^{(v)} \equiv G_0(\Gamma) + LW^{(v)}, \tag{13}$$

$$\operatorname{supp} W^{(v)} \subset D^-, \quad W^{(v)} \in U_{D^0}, \tag{14}$$

$$A_n = \sum_1^m A^i n_i, \tag{15}$$

where $W^{(v)}$ is an arbitrary function satisfying conditions (14), **n** *is the external normal vector to the boundary Γ, n_i $(i = 1, \dots, m)$ are its covariant coordinates in the coordinate system $\{x^i\}$.*

Proof. To prove the statement of the problem, let us consider the following four auxiliary boundary-value problems:

$$1^\circ \quad LW^+ = f^+, \quad W^+ \in U_{D^0}. \tag{16}$$

$$2^\circ \quad LW^- = f^-, \quad W^- \in U_{D^0}. \tag{17}$$

$$3^\circ \quad LW^{(0)} = G_0(\Gamma), \quad W^{(0)} \in U_{D^0}. \tag{18}$$

$$4^\circ \quad LW^{(1)} = LW^{(v)}, \quad W^{(1)} \in U_{D^0}. \tag{19}$$

The solution of problem (18) is the following:

$$W^{(0)}(\mathbf{x}) = \begin{cases} -W^-, & \text{if } \mathbf{x} D^+, \\ W^+, & \text{if } \mathbf{x} \subset D^-. \end{cases} \tag{20}$$

Thus, the function $W^{(0)}(\mathbf{x})$ is a simple-layer potential with the discontinuity on the boundary Γ of $[W^{(0)}]_{|\Gamma} = (W^+ + W^-)_{|\Gamma}$.

Relation (20) can be proved as follows. First, the matrix A_n in (14) is invariant. In order to prove this, let us rewrite Eq. (11) in another coordinate system $\{\xi^i\}$ $(i = 1, \dots, m)$:

$$LW = \sum_1^m \overline{A}^j \frac{\partial W}{\partial \xi^j} + \overline{R} = F,$$

where

$$\overline{A}^j = \sum_1^m \frac{\partial \xi^j}{\partial x^i} A^i.$$

On the other hand, the covariant coordinates of the normal vector in the new coordinate system are given by:

$$\bar{n}_k = \sum_1^m \frac{\partial x^i}{\partial \xi^k} n_i \quad (k = 1, \dots, m).$$

Then,

$$\overline{A}_n = \sum_1^m \overline{A}^j \bar{n}_j = \sum_1^m \sum_1^m \frac{\partial \xi^j}{\partial x^k} \frac{\partial x^i}{\partial \xi^j} A^i n_k = \sum_1^m A^k n_k.$$

Thus, without the loss of generality we can further assume that the coordinate system $\{x^i\}$ is Cartesian.

In this case for any regular function X we are able to introduce functions $X^i \stackrel{\text{def}}{=} A^i X$ $(i = 1, \dots, m)$. Then, we have

$$\sum_1^m \frac{\partial}{\partial x^i} X^i \stackrel{\text{def}}{=} \nabla \mathbf{X}. \tag{21}$$

Next, having introduced the conjugate operator

$$L^* \stackrel{\text{def}}{=} -\sum_1^m A^{i\,\mathrm{T}} \frac{\partial}{\partial x^i} + B^{\mathrm{T}}, \tag{22}$$

we obtain

$$\begin{aligned}
\big(LW^{(0)}, \Phi\big) &= \big(W^{(0)}, L^*\Phi\big) = \int\limits_{\mathbb{R}^m} W^{(0)} L^*\Phi \,\mathrm{d}\mathbf{x} = \int\limits_{D^-} W^{(0)} L^*\Phi \,\mathrm{d}\mathbf{x} + \int\limits_{D^+} W^{(0)} L^*\Phi \,\mathrm{d}\mathbf{x} \\
&= -\int\limits_{D^-} W^{(0)} \sum_1^m \frac{\partial}{\partial x^i}\big(A^{i\,\mathrm{T}}\Phi\big)\,\mathrm{d}\mathbf{x} - \int\limits_{D^+} W^{(0)} \sum_1^m \frac{\partial}{\partial x^i}\big(A^{i\,\mathrm{T}}\Phi\big)\,\mathrm{d}\mathbf{x} + (R, \Phi) \\
&= -\int\limits_{D^-} \nabla\big(\mathbf{W}^{(0)}\Phi\big)\,\mathrm{d}\mathbf{x} + \int\limits_{D^-} \nabla\mathbf{W}^{(0)}\Phi \,\mathrm{d}\mathbf{x} - \int\limits_{D^+} \nabla\big(\mathbf{W}^{(0)}\Phi\big)\,\mathrm{d}\mathbf{x} + \int\limits_{D^+} \nabla\mathbf{W}^{(0)}\Phi \,\mathrm{d}\mathbf{x} + (R, \Phi) \\
&= \int\limits_{\mathbb{R}^m} \big\{\nabla\mathbf{W}^{(0)} + R\big\}\Phi \,\mathrm{d}\mathbf{x} = \int\limits_{\Gamma} A_n W_\Gamma \Phi_\Gamma \,\mathrm{d}\mathbf{x} + \int\limits_{\mathbb{R}^m} \big\{LW^{(0)}\big\}\Phi \,\mathrm{d}\mathbf{x} = \big(A_n W_\Gamma \delta(\Gamma), \Phi\big).
\end{aligned}$$

Here, $\{LW\}$ denotes the regular part of the operator L, $\mathbf{W}_n^{(0)} \equiv \mathbf{W}^{(0)} \cdot \mathbf{n} = A_n W^{(0)}$.

Problem (19) obviously has the following solution

$$W^{(1)} = W^{(v)}. \tag{23}$$

Then, the general solution of problem (4), (5) is represented by the following sum:

$$W^{(g)} = W^- + W^+ + W^{(0)} + W^{(1)}. \tag{24}$$

Thus, if

$$G(\mathbf{x}) = A_n W_\Gamma \delta(\Gamma) + LW^{(v)}, \tag{25}$$

then on D^+: $W^{(g)} = W^+$.

Conversely, if there is any other AS source $\widetilde{G}$, it is to be a partial case of $LW^{(v)}$. In order to prove this statement, let us consider the following problem:

$$LV = F + \widetilde{G}, \quad V \in U_{D^0}. \tag{26}$$

From Eq. (26) we obtain:

$$\begin{aligned}
(\widetilde{G}, \Phi) &= \big(L(V - W), \Phi\big) = (V - W, L^*\Phi) = \int\limits_{\mathbb{R}^m} (V - W) L^*\Phi \,\mathrm{d}\mathbf{x} \\
&= -\int\limits_{D^+} W^- L^*\Phi \,\mathrm{d}\mathbf{x} - \int\limits_{D^-} (V - W) \frac{\partial}{\partial x^i}\big(A^{i\,\mathrm{T}}\Phi\big)\,\mathrm{d}\mathbf{x} + \int\limits_{D^-} B(V - W)\Phi \,\mathrm{d}\mathbf{x} \\
&= \int\limits_{\Gamma} \mathbf{W}_n^- \Phi_\Gamma \,\mathrm{d}\mathbf{x} + \int\limits_{\Gamma} (\mathbf{V}_n - \mathbf{W}_n)\Phi_\Gamma \,\mathrm{d}\mathbf{x} + \int\limits_{D^-} L(V - W)\Phi \,\mathrm{d}\mathbf{x} \\
&= \int\limits_{\Gamma} \mathbf{W}_n \Phi_\Gamma \,\mathrm{d}\mathbf{x} + \int\limits_{\Gamma} \mathbf{Z}_n \Phi_\Gamma \,\mathrm{d}\mathbf{x} + \int\limits_{D^-} LZ \cdot \Phi \,\mathrm{d}\mathbf{x} = \big(\{LZ\} + G_0(\Gamma) + \mathbf{Z}_n \delta(\Gamma), \Phi\big),
\end{aligned}$$

here $Z = V - W - W^+$, $\mathbf{Z} = \mathbf{V} - \mathbf{W} - \mathbf{W}^+$, $\mathbf{Z}_n \equiv \mathbf{Z} \cdot \mathbf{n} = A_n Z(\Gamma)$.

Let us show that $\mathbf{Z}_n = 0$. If $\widetilde{G}_0 = LZ + G_0(\Gamma) + \mathbf{Z}_n\delta(\Gamma)$ is an AS solution, then $\widetilde{G}_1 = G_0(\Gamma) + \mathbf{Z}_n\delta(\Gamma)$ is to be an AS solution also. It immediately follows from the fact that supp $\{LZ\} \subset \overline{D^-}$. On the other hand, $Z(\Gamma) = \beta(\Gamma) W_\Gamma$, where $\beta(\Gamma)$ is some surface function defined on Γ, since the value W_Γ is only known.

Assume that $\widetilde{G}_1$ is an AS solution. Consider the following two problems:

$$LW_0 = F + G_0, \quad W_0 \in U_{D^0} \tag{27}$$

and

$$LW_1 = F + \widetilde{G}_1, \quad W_1 \in U_{D^0}. \tag{28}$$

By subtracting (27) from (28), we have:

$$LW_2 = \beta(\Gamma) A_n W_\Gamma \delta(\Gamma), \quad W_2 \in U_{D^0}, \tag{29}$$

where $W_2 = W_1 - W_0$.

In (29), $W_2 \equiv 0$ on D^+ for any distribution of W_Γ. Hence, $\beta(\Gamma) = 0$. $\square$

From the theorem it follows that the solution of the AS problem is not unique. The first term at the right-hand side of (13) represents a simple layer with the density of $\overline{G}_0 \equiv A_n W_\Gamma$ which is the necessarily component of the AS solution; whereas the second term can be variable and represents the volume distribution of AS additional (free) sources. In particular, if $W^{(v)} \equiv 0$, we obtain the AS having the minimal support Γ: $G = G_0 \equiv A_n W_\Gamma \delta(\Gamma)$. It should be noted that the solution of the AS problem is obtained here in the general differential form for a wide class of differential equations with variable coefficients under some general natural assumptions. It is worth noting that after some appropriate modifications a similar AS solution can be used for shielding the domain D^- from the domain D^+. In particular, the simple-layer AS solution $G_0(\Gamma)$ is simply obtained by the change of sign since the normal direction $\mathbf{n}$ is to be altered.

Let us now consider a few examples of possible applications of the obtained results.

Example 1. In the case of the Helmholtz equation

$$\Delta\phi + k^2\phi = s \tag{30}$$

we rewrite it as the system of first-order equations:

$$\nabla\mathbf{a} + k^2\phi = s, \qquad \nabla\phi - \mathbf{a} = 0. \tag{31}$$

In $\mathbb{R}^3$, we have:

$$W = (a_1, a_2, a_3, \phi)^{\mathrm{T}}, \tag{32}$$

where a_i $(i = 1, 2, 3)$ are the coordinates of the vector $\mathbf{a}$. Hence,

$$A_n = \begin{pmatrix} n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & n_2 \\ 0 & 0 & 0 & n_3 \end{pmatrix} \tag{33}$$

and

$$G_0(\Gamma) = (a_n, \phi n_1, \phi n_2, \phi n_3)^{\mathrm{T}} \delta(\Gamma), \tag{34}$$

where $a_n = \mathbf{a} \cdot \mathbf{n}$.

Coming back to the Helmholtz equation for the variable ϕ, we obtain

$$\Delta\phi + k^2\phi = s + q_0, \tag{35}$$

where the shielding function q_0 is as follows:

$$q_0 = \delta(\Gamma)\frac{\partial\phi}{\partial\mathbf{n}} + \nabla\big(\delta(\Gamma)\phi\mathbf{n}\big) \tag{36}$$

or

$$q_0 = \delta(\Gamma)\frac{\partial\phi}{\partial\mathbf{n}} + \frac{\partial\delta(\Gamma)\phi}{\partial\mathbf{n}}. \tag{37}$$

This solution coincides with the solution derived in [15]. As mentioned above, the solution is applicable to the linear analogue of the Helmholtz equation with variable coefficients.

Example 2. It should be noted that the result of the theorem is valid for nonstationary problems. As an example, consider the following 1D nonstationary problem. Suppose that the domain $D \equiv D^+$ corresponds to $x < 0$ and the field W is described by the following problem:

$$W_t + AW_x = f^- + f^+, \quad W \in U_{D^0}, \tag{38}$$

where we assume that the location of the sources f^- and f^+ is stationary. Next, consider the following boundary value problems:

$$W_t^- + AW_x^- = f^-, \quad W^- \in U_{D^0}, \tag{39}$$

and

$$W_t^+ + AW_x^+ = f^+, \quad W^+ \in U_{D^0}. \tag{40}$$

Then, the following vector G_0 is a solution of the AS problem:

$$G_0 = AW(0,t)\delta(0). \tag{41}$$

Indeed, the generalized solution of equation

$$W_t^{(0)} + AW_x^{(0)} = G_0 \tag{42}$$

is given by

$$W^{(0)} = \begin{cases} -W^-, & \text{if } x < 0, \\ W^+, & \text{if } x > 0 \end{cases} \tag{43}$$

and $[W^{(0)}(t)]_{x=0} = W^+(0,t) + W^-(0,t)$.

Example 3. Next, let us consider the acoustics equations:

$$p_t + \rho c^2 \mathbf{u}_x = \rho c^2 q_{\text{vol}}, \qquad \rho u_t + \nabla p = \mathbf{f}_{\text{vol}}, \tag{44}$$

where q_{vol} is the volume velocity per unit volume and $\mathbf{f}_{\text{vol}}$ is the force per unit volume [11]. In this case, we have

$$W = (u_1, u_2, u_3, p)^{\mathrm{T}}, \tag{45}$$

where u_j $(j = 1,2,3)$ are the coordinates of the velocity $\mathbf{u}$.

It turns out that the matrix A_n coincides with the corresponding matrix (33) of the Helmholtz equation.

As a result,

$$q_{\text{vol}} = \mathbf{u} \cdot \mathbf{n}_{|\Gamma}\delta(\Gamma), \quad \mathbf{f}_{\text{vol}} = p_{|\Gamma}\mathbf{n}\delta(\Gamma). \tag{46}$$

Example 4. It is possible to derive the counterpart of the general continuous solution in a discrete space.

In $\mathbb{R}^3$, let us introduce some orthogonal coordinate system $\{\eta^i\}$ $(i = 1,2,3)$ related to the boundary Γ in such a way that the coordinate η^3 is directed along the external normal to Γ. Eq. (11), written in the coordinate system $\{\eta^i\}$ at some point (k,l,m), can be approximated as follows:

$$A_n \frac{W_{k,l,m+1/2} - W_{k,l,m-1/2}}{\eta^3_{k,l,m+1} - \eta^3_{k,l,m}} + \widetilde{R}_{k,l,m+1/2} = F_{k,l,m}, \tag{47}$$

where $\widetilde{R}_{k,l,m+1/2}$ includes the terms with finite-difference derivatives in the other two directions. Assume that we shield the area $m < 0$, then the boundary γ is single-layer: $(k,l,-1/2)$.

It is possible to see that the following function satisfies the general finite-difference solution (8)–(10):

$$G^{(h)}_{k,l,0} = A_n \frac{W_{k,l,-1/2}}{\eta^3_{k,l,0} - \eta^3_{k,l,-1}}. \tag{48}$$

Hence,

$$G^{(h)}_{k,l,0} = \frac{(\mathbf{u} \cdot \mathbf{n}, pn_1, pn_2, pn_3)^{\mathrm{T}}_{|k,l,-1/2}}{\eta^3_{k,l,0} - \eta^3_{k,l,-1}}. \tag{49}$$

For the sake of simplicity let us set $h = \eta^3_{k,l,0} - \eta^3_{k,l,-1}$. Then we obtain a finite-difference representation of source terms (46):

$$q^{(h)}_{\mathrm{vol}} = \frac{1}{h}\mathbf{u}\cdot\mathbf{n}_{|\gamma}, \qquad \mathbf{f}^{(h)}_{\mathrm{vol}} = \frac{1}{h}p_{|\gamma}\mathbf{n}. \tag{50}$$

The optimal value of the space step h can be found from the following analysis. Let us consider the domain D with the boundary Γ and the simple layer having the density $\overline{G}_0(\Gamma)$. In reality, it is impossible to realize the delta-function source term exactly. Therefore, we approximate the surface source $f_s(\Gamma) = \overline{G}_0(\Gamma)\delta(\Gamma)$ by the volume source f_v with the density $\overline{G}_0(\Gamma)/h$ distributed on the layer $\sigma(\Gamma)$ of thickness Δ along the surface Γ.

The simple-layer potential W_s is as follows:

$$W_s(\mathbf{x}) = Gr * f_s(\Gamma) = \int\limits_{\Gamma} Gr(\mathbf{x}|\mathbf{y}_\Gamma)\overline{G}_0(\mathbf{y}_\Gamma)\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2, \tag{51}$$

where Gr is the Green's function of problem (1), (2). We assume here that the convolutions $Gr * f_s$ and $Gr * f_v$ exist.

Suppose that $\Delta \ll V_D^{1/3}$, where V_D is the volume of the domain D. Then, the volume source f_v yields the following volume potential:

$$\begin{aligned} W_v(\mathbf{x}) &= \int\limits_D Gr(\mathbf{x}|\mathbf{y})f_v(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int\limits_D Gr(\mathbf{y}|\mathbf{x})f_v(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &\approx \frac{\Delta}{h}\int\limits_\Gamma Gr(\mathbf{y}_\Gamma|\mathbf{x})\overline{G}_0(\mathbf{y}_\Gamma)\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2 = \frac{\Delta}{h}\int\limits_\Gamma Gr(\mathbf{x}|\mathbf{y}_\Gamma)\overline{G}_0(\mathbf{y}_\Gamma)\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2. \end{aligned} \tag{52}$$

Here, in the volume integral, we first integrate in the normal direction neglecting by the variation of the Green's function across Δ. In (52), the principle of reciprocity is used: $Gr(\mathbf{x}|\mathbf{y}) = Gr(\mathbf{y}|\mathbf{x})$. Thus, if we require the equality between W_s and W_v, then we obtain $h \approx \Delta$. Hence, the optimal space step in the finite-difference AS solution equals the thickness of the shielding layer. This result seems to be quite expectable, because on one hand h should be as small as possible, but on the other hand, it should be not less than the thickness Δ. The latter statement follows from the fact that the AS discrete source is single-layer.

If the Green's function $Gr(\mathbf{y}|\mathbf{x})$ is regular in the vicinity of the boundary Γ and $\Delta \ll |\mathbf{x}|$, it is possible to show that the potentials W_s and W_v coincide with each other modulo $\mathrm{O}(h^2)$. Indeed, in the volume integral of the Green's function we have:

$$\int\limits_{-\Delta/2}^{\Delta/2} Gr(\mathbf{y}|\mathbf{x})\,\mathrm{d}\eta^3 = \Delta\cdot Gr(\mathbf{y}_\Gamma|\mathbf{x}) + \mathrm{O}(\Delta^3). \tag{53}$$

In this analysis we do not consider any discretization of the AS terms along the boundary Γ, which would correspond to a fine mesh in the directions η^1 and η^2. The discretization in these directions is beyond the current consideration.

5. Conclusion

The general solution of the AS problem in the differential form has been obtained. The solution only requires the knowledge of the total field on the boundary of the shielded area. It does not require any additional information about the characteristics of the undesirable sources or the surrounding medium. The knowledge of the Green's function is not required either. The AS source terms in acoustics have been explicitly obtained in the invariant form of a simple layer. The correspondence between the AS solutions obtained in the analytical and finite-difference formulations has been obtained. The evaluation of the space step in the finite-difference formulation has been done.

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