

Infinitely generated pseudocompact modules for finite groups and Weiss' Theorem

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1 Introduction

In 1988, A. Weiss [36] proved one of the most beautiful theorems in integral representation theory: it characterizes the finitely generated \mathbb{Z}_p -permutation lattices for a finite p -group G in terms of information about the restriction to a normal subgroup N and the action of G/N on the N -invariants of the module (cf. also [37, §6], [28]). In 1993, Weiss generalized his own result to finite extensions of \mathbb{Z}_p , yielding the following theorem:

Theorem 1.1 ([38], [26, App. 1]). *Let R be a finite extension of \mathbb{Z}_p , let G be a finite p -group and let U be a finitely generated RG -lattice. Suppose there is a normal subgroup N of G such that*

- *the module U restricted to N is a free RN -module,*
- *the submodule of fixed points U^N is a permutation RG -module.*

Then U itself is a permutation RG -module.

The importance of this theorem is highlighted by its applications. Weiss originally proved it in order to show that every finite p -subgroup of augmentation-1 units in RG is conjugate to a subgroup of G [36]. Puig used it to show that if a block of the group algebra RF of a finite group F is stably Morita equivalent to a nilpotent block then it is nilpotent [26, Thm 8.2]. No alternative proofs of these results are known. We generalize Theorem 1.1 in two ways. First, we allow the coefficient ring R to be any complete discrete valuation ring in mixed characteristic (that is, with residue field of prime characteristic p and with field of fractions of characteristic 0). The second generalization is more important: we prove the result for all pseudocompact RG -lattices, possibly infinitely generated. What is important here is that the rank of the lattice is allowed to be infinite. Pseudocompact is just the correct generalization of profinite to the case when R is not profinite; we work with such modules because they are better behaved than abstract ones in the infinitely generated case.

The notion of permutation profinite modules was introduced by Mel'nikov in [25], where he studied their basic properties. As he pointed out, profinite permutation modules are important for the combinatorial theory of profinite groups and especially of pro- p groups. A different approach, which complements the approach of Mel'nikov in the study of the basic properties of profinite permutation modules, was introduced by the second author [33], where these modules were used to develop the cohomology theory of profinite groups along the lines followed for discrete groups, as in the book of Brown [8]. We note also that permutation profinite modules appear naturally in Galois Theory [2, Thm 1.3].

Before stating our main theorem for a finite p -group G we define a pseudocompact permutation RG -lattice U to be a pseudocompact module that is free over R and having a pointed compact G -invariant basis (several alternative definitions are given in detail and compared in Section 2.3).

Theorem 1.2. *Let R be a complete discrete valuation ring in mixed characteristic with residue field of characteristic p , let G be a finite p -group and let U be a pseudocompact RG -lattice. Suppose there is a normal subgroup N of G such that*

- the module U restricted to N is a free RN -module,
- the submodule of fixed points U^N is a permutation RG -module.

Then U itself is a permutation RG -module.

This theorem plays a central role in the proof of the pro- p version [39] of the theorem of Karras, Pietrowski, Solitar, Cohen and Scott [19, 12, 31], which states that a virtually free group acts on a tree with finite vertex stabilizers. Indeed, in the pro- p case Theorem 1.2 replaces Stallings' theory of ends, crucial in the proof of the original result. Recently, the greater generality of coefficient ring has yielded applications of Theorem 1.2 in the calculation of Picard groups of blocks of finite groups [6, 16, 21].

One might expect that the generalization to infinite rank lattices could be proved using a simple limit argument, but this does not seem to be the case. Instead, we need to recast the proof in a way that does not depend on expressing a module as a sum of indecomposable modules. In order to do this we look closely at the properties of various classes of infinitely generated modules; these results should be of independent interest.

We make extensive use of relative homological algebra, and in particular of the concepts of covers and precovers of a module by a module with desirable properties. We prove the existence of a large class of covers of pseudocompact modules (Theorem 6.2). We give an explicit description of the permutation cover of a pseudocompact module for a finite group when R is a complete discrete valuation ring (Theorem 6.13). We also generalize a related theorem of Cliff and Weiss (Theorem 7.2).

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2 Definitions, terminology and background

2.1 Algebras and modules

Our coefficient ring R will always be a commutative pseudocompact ring. In later sections we will require further structure on R , the main coefficient rings of interest to us being complete discrete valuation rings.

Let Λ be a pseudocompact R -algebra (we follow the treatments in [9] and [18]). Examples of particular interest are the completed group algebra $R[[G]]$ of a profinite group G , or later the group algebra RG of a finite group G . We consider the following categories of modules for Λ :

- $\Lambda\text{-Mod}^{\mathcal{C}}$: the category whose objects are pseudocompact left Λ -modules [9, §1]. Objects of this category are complete, Hausdorff topological Λ -modules U having a basis of open neighbourhoods of 0 consisting of submodules V for which U/V has finite length. In other words, the category of inverse limits of left Λ -modules of finite length over R .
- $\Lambda\text{-Mod}^{\mathcal{D}}$: the category whose objects are those topological Λ -modules having the discrete topology. Such modules are precisely the direct limits of left Λ -modules of finite length over R . We will call such modules *discrete* (they are called “locally finite” in [18]).
- $\Lambda\text{-Mod}^{\text{abs}}$: the category of abstract left Λ -modules

Morphisms in $\Lambda\text{-Mod}^{\mathcal{C}}$ and $\Lambda\text{-Mod}^{\mathcal{D}}$ are continuous Λ -module homomorphisms while morphisms in $\Lambda\text{-Mod}^{\text{abs}}$ are arbitrary Λ -module homomorphisms. The corresponding categories of right modules are denoted $\text{Mod}^{\mathcal{C}}\text{-}\Lambda$, $\text{Mod}^{\mathcal{D}}\text{-}\Lambda$ and $\text{Mod}^{\text{abs}}\text{-}\Lambda$. We include the decorations \mathcal{C} , \mathcal{D} , abs inconsistently, omitting them when the category is clear from the context. We occasionally write Hom^{cts} or Hom^{abs} to make explicit whether we consider continuous or arbitrary homomorphisms. A Λ -*lattice* is a Λ -module in $\Lambda\text{-Mod}^{\mathcal{C}}$, $\Lambda\text{-Mod}^{\text{abs}}$ or in $\Lambda\text{-Mod}^{\mathcal{D}}$ that is projective as an R -module in the corresponding category. Note that the way sums and products appear is not quite as one might expect from the abstract case – for example, free pseudocompact Λ -modules are direct products of copies of Λ (cf. [9, §1]).

Following [9], denote by E_R the dualizer of R : that is, E_R is the injective hull in $R\text{-Mod}^{\mathcal{D}}$ of the module

$$\bigoplus_{\mathfrak{m}} R/\mathfrak{m},$$

where \mathfrak{m} runs through the maximal ideals of R . The functor sending a module M (in $\Lambda\text{-Mod}^{\mathcal{C}}$ or $\text{Mod}^{\mathcal{D}}\text{-}\Lambda$) to the module of continuous R -module homomorphisms from M to E_R induces a Pontryagin duality between the categories $\Lambda\text{-Mod}^{\mathcal{C}}$ and $\text{Mod}^{\mathcal{D}}\text{-}\Lambda$ [9, Prop.2.3] so that any result in one category corresponds to a dual result in the other. We will at times use this observation without comment. Given a module M in either the pseudocompact or the discrete category, we denote its Pontryagin dual by M^* .

Denote by $\Lambda\text{-Proj}$ (resp. $\Lambda\text{-Inj}$) the full subcategories of (a given category of) left Λ -modules having as objects the projective modules (resp. injective modules). The category $\Lambda\text{-Mod}^{\mathcal{C}}$ has projective covers and the category $\Lambda\text{-Mod}^{\mathcal{D}}$ has injective hulls [18, Ch.II, Thm 2]. An arbitrary product of projective modules in $\Lambda\text{-Mod}^{\mathcal{C}}$ is projective and an arbitrary sum of injective modules in $\Lambda\text{-Mod}^{\mathcal{D}}$ is injective (the first statement follows from [9, Cor. 3.3] and the second is its dual).

Given a module M in $\Lambda\text{-Mod}^{\mathcal{C}}$, denote by $\text{Add}^{\mathcal{C}}(M)$ the full subcategory of $\Lambda\text{-Mod}^{\mathcal{C}}$ whose objects are summands of products of M . Denote by $\text{add}^{\mathcal{C}}(M)$ the full subcategory of $\text{Add}^{\mathcal{C}}(M)$ consisting of summands of finite products of M . Dually, given a module M in $\Lambda\text{-Mod}^{\mathcal{D}}$ we have the categories $\text{Add}^{\mathcal{D}}(M)$ of summands of direct sums of M and $\text{add}^{\mathcal{D}}(M)$ whose objects are summands of finite sums of M . If $M \in \Lambda\text{-Mod}^{\mathcal{C}}$, the category dual to $\text{Add}^{\mathcal{C}}(M)$ is $\text{Add}^{\mathcal{D}}(M^*)$.

Nakayama's Lemma holds in the category $R\text{-Mod}^{\mathcal{C}}$: if N is a closed submodule of the pseudocompact R -module M such that $N + \text{Rad}(R)M = M$, where $\text{Rad}(R)$ is the intersection of the maximal ideals of R , then $M = N$ [9, Lem. 1.4]. There is a dual version for $R\text{-Mod}^{\mathcal{D}}$.

It is convenient to be able to describe a Λ -lattice in terms of Λ -lattices of finite rank over R . We prove only a special case that we will require later.

Lemma 2.1. *Let G be a finite group, R a complete discrete valuation ring and U a pseudocompact RG -lattice. Then U can be expressed as the inverse limit of an inverse system of RG -lattices of finite rank with surjective homomorphisms.*

Proof. As an R -lattice, we can write $U = \prod_{i \in I} R$. For each cofinite subset J of I , let $U_J = \prod_{i \in J} R$ and $V_J = \bigcap_{g \in G} gU_J$. Then V_J is an RG -submodule of U . The obvious homomorphism

$$U/V_J \rightarrow \bigoplus_{g \in G} U/gU_J$$

is injective, so U/V_J is isomorphic as an R -module to a submodule of an R -lattice of finite rank, hence it is an RG -lattice of finite rank itself, since R is a discrete valuation domain.

Because $V_J \subseteq U_J$ and

$$U = \varprojlim_J U/U_J,$$

we have

$$U = \varprojlim_J U/V_J.$$

□

2.2 Tensor products, homomorphisms, Ext functors

The references in this section are sometimes incomplete, insofar as the proofs given are for profinite, rather than pseudocompact modules. What we mean is that the proof given in the reference also works for pseudocompact modules.

Let Λ, Γ be pseudocompact R -algebras, M a pseudocompact $\Lambda\text{-}\Gamma$ -bimodule (that is, M is a bimodule that is pseudocompact both as a left Λ -module and as a right Γ -module) and N a pseudocompact left Γ -module. The *completed tensor product* defined in [9, §2] is a pseudocompact R -module $M \widehat{\otimes}_{\Gamma} N$ together with a continuous bilinear map $M \times N \rightarrow M \widehat{\otimes}_{\Gamma} N$, written $(m, n) \mapsto m \widehat{\otimes} n$ satisfying the condition that $mg \widehat{\otimes} n = m \widehat{\otimes} gn$ for all $m \in M, n \in N, g \in \Gamma$ and universal with respect to this condition. The completed tensor product inherits

naturally the structure of a left Λ -module with multiplication given on pure tensors by $x(m\widehat{\otimes}n) := xm\widehat{\otimes}n$. If $M = \varprojlim M_i, N = \varprojlim_j N_j$ are expressions of M, N as inverse limits of Γ -modules of finite R -length, then

$$M\widehat{\otimes}_\Gamma N = \varprojlim_{i,j} M_i \otimes N_j$$

[27, Lem. 5.5.1]. The completed tensor product commutes with inverse limits in both variables [27, Lem. 5.5.2]. It is not, in general, isomorphic to the abstract tensor product, but we do have the following result.

Proposition 2.2. *If either M or N is finitely presented as a Γ -module, or if both M and N are finitely generated as Γ -modules, then the natural map $M \otimes_\Gamma N \rightarrow M\widehat{\otimes}_\Gamma N$ is an isomorphism.*

Proof. The result for finitely presented modules is [14, Lem. 1.1]. When both modules are finitely generated, one need only observe that the left-most vertical map in the proof of [14, Lem. 1.1 (iii)] is still surjective, so the same proof works. \square

Note that the assertion in [9, Lem. 2.1 (ii)] that the tensor products $\widehat{\otimes}$ and \otimes coincide when only one of the modules involved is finitely generated is false in general [14, Corol. 2.4].

If M is a topological Λ - Γ -bimodule and N is a topological left Λ -module, $\text{Hom}_\Lambda(M, N)$ denotes the left Γ -module of continuous Λ -module homomorphisms from M to N and is given the compact-open topology. If M, N are pseudocompact, then $\text{Hom}_\Lambda(M, N) = \varprojlim_i \text{Hom}_\Lambda(M, N_i)$, where $N = \varprojlim N_i$ and each N_i has finite length over R . The compact-open topology coincides with the topology obtained by declaring each $\text{Hom}_\Lambda(M, N_i)$ to be discrete and giving $\text{Hom}_\Lambda(M, N)$ the inverse limit topology. In particular, $\text{Hom}_\Lambda(M, N)$ commutes with products in the second variable. We also have that $\text{Hom}_\Lambda(M, N)$ commutes with finite products in the first variable. If M is finitely generated as a Λ -module and N is either pseudocompact or discrete, then every abstract homomorphism from M to N is continuous. If M is finitely generated and N is pseudocompact then $\text{Hom}_\Lambda(M, N)$ is pseudocompact as an R -module. Furthermore:

Lemma 2.3. *If M is a finitely generated pseudocompact Λ -module, then $E = \text{End}_\Lambda(M)$ is a pseudocompact R -algebra.*

Proof. Write $\Lambda = \varprojlim \Lambda/I_i$ with each Λ/I_i of finite length over R . Then $M/\text{Cl}(I_i M)$ has finite length over R (where Cl denotes topological closure) and hence so does $\text{End}_\Lambda(M/\text{Cl}(I_i M))$. Let J_i denote the two sided ideal $\{\rho \in E \mid \rho(M) \subseteq \text{Cl}(I_i M)\}$ of E . The map $E/J_i \rightarrow \text{End}_\Lambda(M/\text{Cl}(I_i M))$ is injective, so each E/J_i has finite length over R . But $E = \varprojlim E/J_i$ and hence E is pseudocompact. \square

Let X be a pseudocompact Λ - Γ -bimodule that is finitely generated as a left Λ -module. The functor

$$X\widehat{\otimes}_\Gamma - : \Gamma\text{-Mod}^c \rightarrow \Lambda\text{-Mod}^c$$

is left adjoint to the functor

$$\text{Hom}_\Lambda(X, -) : \Lambda\text{-Mod}^c \rightarrow \Gamma\text{-Mod}^c :$$

given a pseudocompact left Γ -module A and a pseudocompact left Λ -module $C = \varprojlim C_i$ with each C_i of finite length, we have

$$\begin{aligned} \text{Hom}_\Gamma(A, \text{Hom}_\Lambda(X, \varprojlim C_i)) &\cong \text{Hom}_\Gamma(A, \varprojlim \text{Hom}_\Lambda(X, C_i)) \\ &\cong \varprojlim \text{Hom}_\Gamma(A, \text{Hom}_\Lambda(X, C_i)) \\ &\cong \varprojlim \text{Hom}_\Lambda(X\widehat{\otimes}_\Gamma A, C_i) \\ &\cong \text{Hom}_\Lambda(X\widehat{\otimes}_\Gamma A, \varprojlim C_i), \end{aligned}$$

where the second to last isomorphism is [9, Lem. 2.4] (though note that the side conventions there are different). If Ψ, Φ are other pseudocompact algebras, A is a pseudocompact Γ - Φ -module and C is a pseudocompact Λ - Ψ -bimodule, then the above isomorphisms are isomorphisms of Φ - Ψ -bimodules [5, Thm 4.2].

If H is a closed subgroup of the profinite group G and V is a pseudocompact $R[[H]]$ -module, the induced $R[[G]]$ -module $V \uparrow^G$ is defined to be

$$V \uparrow^G := R[[G]] \widehat{\otimes}_{R[[H]]} V,$$

where $R[[G]]$ is treated as an $R[[G]]$ - $R[[H]]$ -bimodule in the obvious way. Given a pseudocompact $R[[G]]$ -module U , the restriction $U \downarrow_H$ (that is, U treated as an $R[[H]]$ -module) can be expressed as $U \downarrow_H = \text{Hom}_{R[[G]]}(R[[G]], U)$, where

again $R[[G]]$ is being treated as an $R[[G]]$ - $R[[H]]$ -bimodule in the obvious way. It follows from the adjunction above that induction is left adjoint to restriction. When H is open in G , induction is also right adjoint to restriction. To see this, we need only note that, since H is open, restriction is anyway left adjoint to the coinduction functor $\text{Hom}_{R[[H]]}(R[[G]], -)$. But since $R[[G]]$ is finitely presented as an $R[[H]]$ -module, every functor involved coincides with its abstract counterpart. We may thus apply [35, Lem. 6.3.4]. Let H, L be closed subgroups of the profinite group G with either H or L open. The Mackey decomposition formula applies. That is, given a pseudocompact $R[[H]]$ -module V , we have

$$V \uparrow^G \downarrow_L \cong \bigoplus_{g \in L \backslash G / H} {}^g V \downarrow_{gH \cap L} \uparrow^L,$$

where g runs through a (finite) set of double coset representatives of $L \backslash G / H$ and ${}^g V$ is the $R[[gH]]$ -module V with action $ghg^{-1} \cdot v = hv$.

Given pseudocompact Λ -modules M, N , denote by $\text{Ext}_\Lambda^i(M, N)$ the i th right derived functor of $\text{Hom}_\Lambda(-, N)$ applied to M . It can be calculated via a projective resolution for M in the usual way. Since Hom_Λ commutes with finite products in the first variable and arbitrary products in the second, it follows that $\text{Ext}_\Lambda^i(M, N)$ also commutes with finite products in M and arbitrary products in N . However, it does not commute with arbitrary inverse limits in the second variable. Since $\Lambda\text{-Mod}^{\mathcal{C}}$ is abelian and has enough projectives, the functor Ext_Λ^1 classifies extensions as one would hope [35, Vis. 3.4.6].

Dually, in $\Lambda\text{-Mod}^{\mathcal{D}}$, denote by $\text{Ext}_\Lambda^i(M, N)$ the i th right derived functor of $\text{Hom}_\Lambda(M, -)$ applied to N . By Pontryagin duality,

$$\text{Ext}_{\Lambda\text{-Mod}^{\mathcal{C}}}^i(M, N) \cong \text{Ext}_{\text{Mod}^{\mathcal{D}}\text{-}\Lambda}^i(N^*, M^*).$$

In what follows, when we say that Λ is finitely generated over R , we mean finitely generated as an R -module. Examples of particular interest are $\Lambda = RG$ for a finite group G .

Lemma 2.4. *Suppose Λ is finitely generated over R . If M, N are pseudocompact Λ -modules with M finitely generated, and if either R is noetherian or M is a lattice, then*

$$\text{Ext}_{\Lambda\text{-Mod}^{\mathcal{C}}}^i(M, N) \cong \text{Ext}_{\Lambda\text{-Mod}^{\text{abs}}}^i(M, N).$$

Proof. Our conditions guarantee that there exists a projective resolution of M in $\Lambda\text{-Mod}^{\mathcal{C}}$ with each module finitely generated. This is also a projective resolution of M by projective modules in $\Lambda\text{-Mod}^{\text{abs}}$. But $\text{Hom}_{\Lambda\text{-Mod}^{\mathcal{C}}}(P, N) = \text{Hom}_{\Lambda\text{-Mod}^{\text{abs}}}(P, N)$ whenever P is finitely generated (that is, every abstract homomorphism $P \rightarrow N$ is continuous), and so the groups obtained are the same. \square

Lemma 2.5. *Suppose that R is noetherian and Λ is finitely generated over R . For $M, N \in \Lambda\text{-Mod}^{\mathcal{D}}$ we have*

$$\text{Ext}_{\Lambda\text{-Mod}^{\mathcal{D}}}^i(M, N) \cong \text{Ext}_{\Lambda\text{-Mod}^{\text{abs}}}^i(M, N).$$

Proof. By taking the Pontryagin dual of a free resolution of the pseudocompact module N^* , we obtain an injective resolution of N by modules that are sums of modules of the form $\Lambda^* = \text{Hom}_{R\text{-Mod}}^{\text{cts}}(\Lambda, E_R)$. Since Λ is finitely generated over R , we have $\text{Hom}_{R\text{-Mod}}^{\text{cts}}(\Lambda, E_R) = \text{Hom}_{R\text{-Mod}}^{\text{abs}}(\Lambda, E_R)$. The injective hull in $R\text{-Mod}^{\text{abs}}$ of the module R/\mathfrak{m} (\mathfrak{m} maximal ideal of R) is locally finite, since every element is annihilated by some power of \mathfrak{m} by [24, Thm 18.4] and R/\mathfrak{m}^n has finite length over R for any $n \in \mathbb{N}$ because R is noetherian. Hence the injective hulls of R/\mathfrak{m} in the abstract and the discrete categories coincide. Furthermore, R being noetherian implies that a direct sum of injective modules in the abstract category is injective [24, Thm 18.5]. We conclude that E_R is injective in $R\text{-Mod}^{\text{abs}}$.

Now, the functor $\text{Hom}_{R\text{-Mod}}^{\text{abs}}(\Lambda, -) : R\text{-Mod}^{\text{abs}} \rightarrow \Lambda\text{-Mod}^{\text{abs}}$ is a right adjoint by the theory of abstract modules, and hence by [35, Prop. 2.3.10], Λ^* is injective in $\Lambda\text{-Mod}^{\text{abs}}$.

Putting all this together, the above injective resolution of N in $\Lambda\text{-Mod}^{\mathcal{D}}$ is also an injective resolution of N in $\Lambda\text{-Mod}^{\text{abs}}$. Furthermore, the hom groups obtained by applying $\text{Hom}_{\Lambda\text{-Mod}^{\mathcal{D}}}(M, -)$ and $\text{Hom}_{\Lambda\text{-Mod}^{\text{abs}}}(M, -)$ to this injective resolution are the same and so the Ext groups are the same. \square

As usual, when G is a finite group we denote by $H^n(G, -)$ the n th right derived functor of the fixed point functor $(-)^G$. We require only the following basic version of the Eckmann-Shapiro Lemma, which can be proved in the usual way given that the induction and coinduction functors coincide for finite groups: When H is a subgroup of G and V is a pseudocompact RH -module, then $H^n(H, V) \cong H^n(G, V \uparrow^G)$ for all $n \geq 1$.

2.3 Permutation modules

Let G be a profinite group and R a commutative pseudocompact ring. Let $(X, *)$ be a pointed profinite G -space (that is, X is a profinite space on which G acts continuously and such that $g* = * \forall g \in G$). Recall (cf. [25, 1.7]) that the free pseudocompact R -module on $(X, *)$ is a pseudocompact R -module $R[[X, *]]$ together with a continuous map $\iota : (X, *) \rightarrow R[[X, *]]$ sending $*$ to 0 and satisfying the following universal property:

Given any pseudocompact R -module M and continuous map $\beta : X \rightarrow M$ sending $*$ to 0, there is a unique continuous homomorphism of R -modules $\beta' : R[[X, *]] \rightarrow M$ such that $\beta'\iota = \beta$.

Definition 2.6 (cf. [25, Def. 1.8]). *Let G be a profinite group and R a commutative pseudocompact ring. Let $(X, *)$ be a pointed profinite G -space. The corresponding pseudocompact permutation module for G is the R -module $R[[X, *]]$ with action from G given by the action of G on X and the universal property of $R[[X, *]]$.*

Note that the corresponding map ι is thus G -equivariant. The definition can be succinctly stated as follows: a pseudocompact permutation $R[[G]]$ -module is a pseudocompact $R[[G]]$ -module having a pointed profinite G -invariant R -basis. By construction, a permutation module is free as an R -module, so in particular is a lattice.

Lemma 2.7. *The permutation $R[[G]]$ -module $R[[X, *]]$ is determined by the following universal property:*

Given any pseudocompact $R[[G]]$ -module M and any continuous G -equivariant map $\beta : X \rightarrow M$ sending $$ to 0, there is a unique continuous $R[[G]]$ -module homomorphism $\beta' : R[[X, *]] \rightarrow M$ such that $\beta'\iota = \beta$.*

Proof. That there exists a unique R -module homomorphism is the universal property of $R[[X, *]]$ as an R -module. Using the universal property again, we see that this map is an RG -module homomorphism. \square

We give several alternative definitions of a permutation module for a profinite group G , which in general are not equivalent. Given a profinite group G and an (unpointed) profinite G -space X , the corresponding discrete permutation module $F(X)$ is defined in [33] to be the module of continuous functions $X \rightarrow T$, where T is the maximal submodule in $R\text{-Mod}^{\mathcal{D}}$ of the injective hull of R as an abstract R -module. The pseudocompact permutation module $R[[X]]$ in the sense of [33] is the Pontryagin dual of $F(X)$. This is the same as saying that $R[[X]]$ is the pseudocompact $R[[G]]$ -module having profinite R -basis X . Note that the modules $R[X]$ and $R[(X \cup \{*\}, *)]$ are clearly isomorphic when X is finite. A profinite G -space X (resp. profinite pointed G -space $(X, *)$) can be expressed as the inverse limit of an inverse system of finite (resp. finite pointed) G -sets X_i (resp. $(X_i, *)$) [27, Lem. 5.6.4]. We have that $R[[X]] \cong \varprojlim R[X_i]$ and $R[[X, *]] = \varprojlim R[(X_i, *)]$. A strict pseudocompact permutation module for G is a module isomorphic to one of the form $\prod_{j \in J} R[[G/H_j]]$, where J is a set and each H_j is a closed subgroup of G .

Proposition 2.8. *Let R be a commutative local pseudocompact ring with residue class field of characteristic p . Let G be a finite p -group and let U be a pseudocompact RG -module. The following are equivalent:*

1. U is a permutation module,
2. U has a profinite G -invariant R -basis,
3. U is a strict pseudocompact permutation module,
4. U is an inverse limit of finite rank permutation modules.

Proof. Given a module of the form $\prod_{j \in J} R[G/H_j]$, choose for each $j \in J$ the basis of left cosets of H_j in G , and let X be the union of these bases (a discrete set) compactified at the point $*$. One now checks that $\prod_{j \in J} R[G/H_j] \cong R[[X, *]]$ by observing that the universal property of Lemma 2.7 follows from the universal property of the product. Thus 3 implies 1. The equivalence of 3 and 4 follows from [23, Thm 2.2] in case R is profinite (note that [23, Cor. 2.3] is not correct as stated, since the module $R[G/H]$ need not be indecomposable when the residue class field of R is not of characteristic p). The same proof works for pseudocompact R (see final remark of §3). As the G -space X (resp. pointed G -space $(X, *)$) is the inverse limit of finite G -sets (resp. finite pointed G -sets), 2 (resp. 1) implies 4. We show that 3 implies 2. As G is finite, we may suppose that $U = \prod_J R[G/H]$ for some fixed subgroup H of G and discrete set J . The result is obvious when J is finite so suppose that J is infinite. Let J^* be the one-point compactification of J with point at infinity $*$. Consider the (unpointed) profinite G -space $G/H \times J^*$, where G acts

only on the left factor, in the obvious way. We obtain

$$\begin{aligned}
R[[G/H \times J^*]] &\cong \varprojlim_{\substack{F \subseteq J \\ F \text{ finite}}} R[G/H \times (* \cup F)] \\
&\cong \varprojlim_{\substack{F \subseteq J \\ F \text{ finite}}} R[G/H] \times \prod_F R[G/H] \\
&\cong R[G/H] \times \prod_J R[G/H] \\
&\cong U \quad (\text{since } J \text{ is infinite}).
\end{aligned}$$

□

Even for finite groups we can not, in general, suppose that the module $R[G/H]$ is indecomposable, and indeed the class of permutation modules need not be closed under summands. We define an *R-permutation module* to be a summand of a permutation module. For a profinite group G , denote by $\text{Perm}(G)$ the module $\prod_{H \leq G} R[[G/H]]$ (subgroups of profinite groups are always taken to be closed). A strict pseudocompact R -permutation module for G is an object of $\text{Add}^c(\text{Perm}(G))$. The second author showed [33, Cor. 3.21] that every R -permutation module for a finite group is strict.

Lemma 2.9. *Let G be a finite group and suppose that $|G|$ is not a zero divisor of R . If V is a pseudocompact R -permutation module then $H^1(H, V) = 0$ for all $H \leq G$.*

Proof. By the Eckmann-Shapiro Lemma, additivity and the Mackey decomposition formula, we need only check that $H^1(H, V) = 0$ for V a trivial pseudocompact RH -module. But $H^1(H, V)$ is isomorphic to group homomorphisms from H to V . If ρ is such a homomorphism then for any $h \in H$ we have

$$0 = \rho(1) = \rho(h^{|G|}) = |G|\rho(h) \implies \rho(h) = 0$$

since $|G|$ is not a zero-divisor. □

Permutation modules of the form $R[[G/H]]$ can be written as $R \uparrow_H^G$, the trivial RH -module induced up to G . A larger class of modules of interest to us is the class of *monomial* modules (called generalized permutation modules in [36]), namely the category of continuous direct summands of direct products of modules of the form $V \uparrow_H^G$, where V is an R -rank 1 RH -lattice. When G is finite and R is an integral domain there are only finitely many isomorphism classes of lattices of R -rank 1 (because R has only finitely many $|G|$ th roots of unity), so the category of monomial modules is $\text{Add}^c(M)$ with $M = \bigoplus_{H \leq G} V_H \uparrow_H^G$, where V_H runs through the set of isomorphism classes of R -rank 1 RH -lattices. Notice that monomial modules are preserved under induction and restriction.

3 Change of category

Recall that $\Lambda\text{-Mod}^c$ has exact inverse limits [18, Ch.IV, Thm 3] and $\Lambda\text{-Mod}^D$ has exact direct limits. Let M be a finitely generated object of $\Lambda\text{-Mod}^c$ and $E = \text{End}_\Lambda(M)$, a pseudocompact algebra by Lemma 2.3. Treating M as a left E -module in the obvious way the actions of Λ and E are compatible, in so far as $\rho(\lambda m) = \lambda \rho(m)$ for $\rho \in E, \lambda \in \Lambda, m \in M$. In what are perhaps more familiar terms, this amounts to saying that M is a $\Lambda\text{-}E^{\text{op}}$ -bimodule. Consider the following functors:

$$\begin{aligned}
U &= \text{Hom}_\Lambda(M, -) : \Lambda\text{-Mod}^c \rightarrow \text{Mod}^c\text{-}E \\
V &= - \widehat{\otimes}_E M : \text{Mod}^c\text{-}E \rightarrow \Lambda\text{-Mod}^c.
\end{aligned}$$

Lemma 3.1. *The functor V is left adjoint to U .*

Proof. See the discussion after Lemma 2.3. □

Proposition 3.2. *Restricting the domain and codomain of U and V to $\text{Add}^c(M)$ and $\text{Proj}^c\text{-}E$ yields mutually inverse equivalences of categories.*

Proof. This is more or less standard and the usual proof goes through (see, for instance, [1, Lem. 29.4]). The only difference is that we must check that the functors U and V commute with products rather than sums. But this follows from observations in §2.2. □

With M still a finitely generated module in $\Lambda\text{-Mod}^{\mathcal{C}}$, we may dualize the functors above to obtain

$$\begin{aligned} U' &= \text{Hom}_{\Lambda}(M, (-)^*)^* = \text{Hom}_{\Lambda}(-, M^*)^* : \text{Mod}^{\mathcal{D}}\text{-}\Lambda \rightarrow E\text{-Mod}^{\mathcal{D}} \\ V' &= ((-)^* \widehat{\otimes}_E M)^* = \text{Hom}_E(M, -) : E\text{-Mod}^{\mathcal{D}} \rightarrow \text{Mod}^{\mathcal{D}}\text{-}\Lambda. \end{aligned}$$

Lemma 3.3. *The functor V' is right adjoint to U' .*

Lemma 3.4. *The functor V commutes with direct limits and the functor V' commutes with inverse limits.*

Proof. These are formal properties of adjoint functors, see for instance [35, Thm 2.6.10]. \square

Again, by duality, restricting the domain and codomain, we obtain an equivalence of categories between $\text{Add}^{\mathcal{D}}(M^*)$ and $E\text{-Inj}^{\mathcal{D}}$.

These equivalences allow one to deduce properties of $\text{Add}(M)$ from already understood properties of $E\text{-Proj}^{\mathcal{C}}$ or $\text{Inj}^{\mathcal{D}}\text{-}E$. For example, from [18, Ch.III, Cor.1 to Thm 3] it follows that every module in $\text{Add}^{\mathcal{C}}(M)$ is a product of indecomposable summands of M . Furthermore, from [18, Ch.IV, Thm.2] this decomposition is essentially unique. Indeed, $\text{Add}^{\mathcal{C}}(M)$ has the *exchange property*:

Proposition 3.5 ([18, Ch. IV, Prop. 8]). *Let X be a continuous direct summand of the pseudocompact Λ -module $Y = \prod_{i \in I} Y_i$, where each Y_i is an indecomposable summand of the finitely generated pseudocompact Λ -module M . There exists a subset J of I such that*

$$Y = X \oplus \prod_{i \in J} Y_i.$$

Lemma 3.6 ([9, Lem. A.4]). *The functor $V : \text{Mod}^{\mathcal{C}}\text{-}E \rightarrow \Lambda\text{-Mod}^{\mathcal{C}}$ commutes with inverse limits.*

Proposition 3.7 ([9, Cor. 3.3], [18, Ch.II, Cor.1 to Thm 2]). *An inverse limit of projective modules in $\Lambda\text{-Mod}^{\mathcal{C}}$ is again a projective module.*

Theorem 3.8. *Let Λ be a pseudocompact R -algebra and M a finitely generated pseudocompact left Λ -module. The inverse limit in $\Lambda\text{-Mod}^{\mathcal{C}}$ of modules in $\text{Add}^{\mathcal{C}}(M)$ is again an object of $\text{Add}^{\mathcal{C}}(M)$.*

Proof. The proof of [23, Thm 2.2] works in this greater generality, but this also follows easily from the results above: an inverse system in $\text{Add}^{\mathcal{C}}(M)$ yields, by applying U , an inverse system of projective modules, whose limit is projective by Proposition 3.7. Applying V to this limit we obtain an object of $\text{Add}^{\mathcal{C}}(M)$. But V commutes with inverse limits by Lemma 3.6, and hence this module is the inverse limit of the original inverse system. \square

Remark: This result is false without the topology. In [4], Bergman shows that any abstract module can be expressed as an inverse limit of injective modules. This is most striking when Λ is self-injective (for instance, when R is a field and Λ is the group algebra of a finite group) [3, §1.6]. In this case projective modules are the same as injective modules, so that every abstract module is an inverse limit of projective modules.

Remark: Let R be a finite unramified extension of the p -adic integers \mathbb{Z}_p and let G be a finite group. When G is cyclic of order p or p^2 it is known that there are only finitely many isomorphism classes of finitely generated indecomposable RG -lattices [15, Thm 33.7, Cor. 33.3a]. It follows that for general finite G there are finitely many isomorphism classes of indecomposable finitely generated RG -lattices that are projective relative to subgroups of order p or p^2 (for the definition of relative projectivity see after Lemma 6.11, for further details see for example [3, §3.6], or [22] for profinite modules). Denote by M_G the sum of (a representative of each isomorphism class of) the indecomposable RG -lattices projective relative to cyclic subgroups of order p or p^2 . If G is cyclic of order p or p^2 , the category $\text{add}(M_G)$ coincides with the category of finitely generated RG -lattices. As pseudocompact RG -lattices are inverse limits of finite rank RG -lattices by Lemma 2.1, Theorem 3.8 tells us that compact RG -lattices are summands of products of indecomposables, hence products of indecomposables by the exchange property (Proposition 3.5). For general G , a compact RG -lattice L that is projective relative to cyclic subgroups of order p or p^2 is a direct summand of

$$\prod_{\substack{C \leq G \\ C \cong C_p \text{ or } C_{p^2}}} L \downarrow_C \uparrow^G$$

and hence (by the theorem) is an object of $\text{Add}(M_G)$.

Remark: Every profinite G -set X can be expressed as an inverse limit of finite G -sets $X = \varprojlim X_i$ [27, Lem. 5.6.4(a)] and hence $R[[X]] = \varprojlim R[X_i]$. Theorem 3.8 now yields a different proof of the result of [33, Corol. 3.21]: a pseudocompact permutation module can be expressed as a summand of a product of modules isomorphic to $R[G/H]$ ($H \leq G$).

4 Other limits

We have observed that $\Lambda\text{-Mod}^{\mathcal{C}}$ has exact inverse limits and dually that $\Lambda\text{-Mod}^{\mathcal{D}}$ has exact direct limits. It is interesting to note that $\Lambda\text{-Mod}^{\mathcal{C}}$ also has direct limits and $\Lambda\text{-Mod}^{\mathcal{D}}$ has inverse limits (though of course, they are not exact) [18, Ch.II, Cor. 2 to Thm 1]. Inverse limits in $\Lambda\text{-Mod}^{\mathcal{D}}$ are obtained by taking the inverse limit in $\Lambda\text{-Mod}^{\text{abs}}$, giving it the discrete topology and then taking the submodule generated by elements annihilated by an open ideal of Λ . Direct limits in $\Lambda\text{-Mod}^{\mathcal{C}}$ are described in [14, §2]. Note that these limits do not in general commute with restriction to subgroups.

By definition, $\Lambda = \varprojlim \Lambda/I_j$ for some set of open ideals I_j of Λ with each Λ/I_j an R -algebra of finite length over R . Given such an ideal I and a module M (compact, discrete torsion or abstract), denote by $T^I M$ the set $\{m \in M \mid Im = 0\}$. Note that $T^I \varprojlim^{\mathcal{D}} = \varprojlim^{\mathcal{D}} T^I$. Let TM denote the union of the $T^I M$ as I varies over the open ideals of Λ .

Proposition 4.1. *A direct limit in $\Lambda\text{-Mod}^{\mathcal{C}}$ of projective modules in $\Lambda\text{-mod}^{\mathcal{C}}$ is again projective. An inverse limit in $\Lambda\text{-Mod}^{\mathcal{D}}$ of injective modules in $\Lambda\text{-mod}^{\mathcal{D}}$ is again injective.*

Proof. We prove the second statement. First suppose that Λ has finite length over R . An inverse system in $\text{add}^{\mathcal{D}}(\Lambda^*)$ can be treated as an inverse system in $\text{add}^{\mathcal{C}}(\Lambda^*)$, and so by Theorem 3.8 it has an inverse limit in $\text{Add}^{\mathcal{C}}(\Lambda^*)$. The module obtained by giving this module the discrete topology coincides with the inverse limit in $\Lambda\text{-Mod}^{\mathcal{D}}$ because all these modules are torsion.

In $\text{Add}^{\mathcal{C}}(\Lambda^*)$ our module has the form $\bigoplus I_j \widehat{\otimes}_R V_j$, where I_j runs through the indecomposable summands of Λ^* and V_j is a free module in $R\text{-Mod}^{\mathcal{C}}$. But each I_j is of finite length, hence finitely presented, so that $I_j \widehat{\otimes} V_j \cong I_j \otimes V_j$ by Proposition 2.2. If we ignore the topology, V_j is a free abstract module, by [10, Thm 3.3]. Thus, the module is in $\text{Add}^{\mathcal{D}}(\Lambda^*) = \text{Inj}^{\mathcal{D}}\text{-}\Lambda$.

For the general case, write $\Lambda = \varprojlim \Lambda/I_i$. Denote by $X = \varprojlim X_j$ our module, where each X_j is injective. Then

$$T^{I_i} \varprojlim^{\mathcal{D}} X_j = \varprojlim^{\mathcal{D}} T^{I_i} X_j.$$

But each $T^{I_i} X_j$ is injective as a Λ/I_i -module, so by the special case above $\varprojlim_j^{\mathcal{D}} T^{I_i} X_j$ is in $\text{Inj}^{\mathcal{D}}\text{-}\Lambda/I_i$. Now take \varinjlim_{I_i} , observing that by [18, Ch.2, Cor. 1 to Thm 1] a direct limit of modules injective over Λ/I_i is itself injective over Λ . \square

Theorem 4.2. *A direct limit in $\Lambda\text{-Mod}^{\mathcal{C}}$ of modules in $\text{add}^{\mathcal{C}}(M)$ is in $\text{Add}^{\mathcal{C}}(M)$. An inverse limit in $\Lambda\text{-Mod}^{\mathcal{D}}$ of modules in $\text{add}^{\mathcal{D}}(M^*)$ is in $\text{Add}^{\mathcal{D}}(M^*)$.*

Proof. We prove the first statement. Let $X = \varinjlim X_i$ be a direct limit of modules in $\text{add}^{\mathcal{C}}(M)$. Then by Proposition 3.2 there are projective right E -modules P_i for which

$$X = \varinjlim X_i = \varinjlim V(P_i) = V(\varinjlim P_i),$$

where the last equality is from Lemma 3.4. Hence $X \in \text{Add}^{\mathcal{C}}(M)$ by Propositions 4.1 and 3.2. \square

5 Flatness

We say that a module X in $\Lambda\text{-Mod}^{\mathcal{C}}$ is *flat* if the functor $-\widehat{\otimes}_{\Lambda} X$ is exact on $\text{Mod}^{\mathcal{C}}\text{-}\Lambda$. A module X in $\Lambda\text{-Mod}^{\mathcal{C}}$ is flat if, and only if, it is projective. This follows from the existence of projective covers, and so the usual proof for perfect rings (see eg. [20, Thm 24.25]) carries through.

We consider other criteria for flatness. Observe that a direct limit of projective modules need not be projective, but by Proposition 4.1 a direct limit in $\Lambda\text{-Mod}^{\mathcal{C}}$ of finitely generated projectives is again projective. The converse is false: a projective module need not be a direct limit of finitely generated projectives. This is easily seen in the dual case, by taking $\Lambda = R = k$, a finite field. The sum $\bigoplus_{n \in \mathbb{N}} k$ of countably many copies of k (an injective object of $\text{Mod}^{\mathcal{D}}\text{-}k$) is countable, hence not an inverse limit of finite dimensional vector spaces by [27, Prop. 2.3.1(b)].

On the other hand, if we allow summands, the problem vanishes: If X is an object of $\text{Inj}^{\mathcal{D}}\text{-}\Lambda$ then $X = \bigoplus_{j \in J} I_j$ with each I_j an indecomposable injective. Consider $Y = \prod_{j \in J} I_j$ in the category of *abstract* Λ -modules. Then $X \subseteq TY \subseteq Y$. Because Y is an inverse limit of finitely cogenerated injectives, it follows that TY is an inverse limit in $\text{Ind}^{\mathcal{D}}\text{-}\Lambda$. The inclusion $X \rightarrow TY$ splits because X is injective.

6 Add(M) precovers and covers

It is convenient to follow the approach to covers and precovers given in [17]. In this section, M denotes a finitely generated module in $\Lambda\text{-Mod}^{\mathcal{C}}$.

Definition 6.1 (cf. [17, §5]). *Let X be an object of $\Lambda\text{-Mod}^{\mathcal{C}}$. The continuous homomorphism $\rho : P \rightarrow X$ is an Add(M)-precover of X if*

- $P \in \text{Add}(M)$,
- *Given any continuous homomorphism $\alpha : S \rightarrow X$ with S in $\text{Add}(M)$, there exists a continuous homomorphism $\gamma : S \rightarrow P$ such that $\rho\gamma = \alpha$.*

An Add(M)-precover $\rho : P \rightarrow X$ is an Add(M)-cover of X if every continuous homomorphism $\gamma : P \rightarrow P$ such that $\rho\gamma = \rho$ is an automorphism of P .

The Add(M)-cover of X , if it exists, is clearly unique up to isomorphism. It is also easily checked that the Add(M)-cover of X is a direct summand of any Add(M)-precover of X , and that an Add(M)-precover is a split-surjection if, and only if, $X \in \text{Add}(M)$. It follows that an Add(Λ)-cover corresponds to the usual notion of a projective cover – that is, an Add(Λ)-cover $\rho : P \rightarrow X$ of X is precisely a surjective homomorphism from the projective Λ -module P with P minimal with respect to direct sum decompositions (cf. [3, §1.5]).

Theorem 6.2. *Let M be a finitely generated Λ -module and let X be any pseudocompact Λ -module. The Add(M)-cover of X exists.*

Proof. Denote by ε, η the counit and unit of the adjunction of Lemma 3.1. By results mentioned there, restricting the domain and codomain of ε to $\text{Add}(M)$, $\text{Proj}(E)$ yields a natural isomorphism. Since E is pseudocompact by Lemma 2.3, $\text{Mod}^{\mathcal{C}}\text{-}E$ has projective covers.

Let $\rho : P \rightarrow U(X)$ be the projective cover of $U(X)$. Then $V(P)$ is an Add(M)-module and

$$\varepsilon_X \circ V(\rho) : V(P) \rightarrow VU(X) \rightarrow X$$

is the Add(M)-cover of X . We sketch the argument. Given $\alpha : S \rightarrow X$ with $S \in \text{Add}(M)$, there is a homomorphism $\gamma : U(S) \rightarrow U(X)$ such that $\rho\gamma = U(\alpha)$ because ρ is a precover. The diagram

$$\begin{array}{ccc} & & V(P) \\ & \nearrow^{V(\gamma)\varepsilon_S^{-1}} & \downarrow \varepsilon_X V(\rho) \\ S & \xrightarrow{\alpha} & X \end{array}$$

commutes:

$$\begin{aligned} \varepsilon_X V(\rho) V(\gamma) \varepsilon_S^{-1} &= \varepsilon_X V(\rho\gamma) \varepsilon_S^{-1} \\ &= \varepsilon_X VU(\alpha) \varepsilon_S^{-1} \\ &= \alpha \varepsilon_S \varepsilon_S^{-1} \\ &= \alpha, \end{aligned}$$

hence $\varepsilon_X V(\rho)$ is a precover. To check it is a cover, consider γ completing the following diagram:

$$\begin{array}{ccc} V(P) & \xrightarrow{\gamma} & V(P) \\ & \searrow \varepsilon_X V(\rho) & \swarrow \varepsilon_X V(\rho) \\ & & X \end{array}$$

One checks that $\eta_P^{-1}U(\gamma)\eta_P : P \rightarrow P$ is an isomorphism using the counit-unit equations and the fact that ρ is a cover. Hence γ too is an isomorphism. \square

From the construction it follows that if X is finitely generated over Λ , then so is its Add(M)-cover. In the following definition, $H \leq_G G$ indicates that H runs through a set of representatives of the classes of G -conjugates of subgroups of G .

Definition 6.3. Let G be a finite group and X a pseudocompact RG -module. A permutation precover of X is an $\text{Add}(M)$ -precover of X , where

$$M = \prod_{H \leq G} R[G/H].$$

The permutation cover of X is the corresponding $\text{Add}(M)$ -cover.

Let R be pseudocompact integral domain. A monomial precover (or generalized permutation precover) of X is an $\text{Add}(M)$ -precover of X , where

$$M = \prod_{H \leq G} V_H \uparrow_H^G$$

and V_H runs through the set of isomorphism classes of R -rank 1 RH -lattices. The monomial cover of X is the corresponding $\text{Add}(M)$ -cover.

Both the permutation and monomial covers of X exist by Theorem 6.2.

We present some useful properties of permutation and monomial precovers that we will require in Section 8. Following Samy Modeliar [29], we say that an RG -module homomorphism $\rho : V \rightarrow U$ is *supersurjective* if the induced homomorphism $\rho^H : V^H \rightarrow U^H$ is surjective for every subgroup H of G . Similarly, ρ is *monomial supersurjective* if the induced homomorphism $\text{Hom}_{RG}(W \uparrow_H^G, V) \rightarrow \text{Hom}_{RG}(W \uparrow_H^G, U)$ is surjective for every R -rank 1 RH -lattice W . Note that monomial supersurjective homomorphisms are supersurjective.

Lemma 6.4. 1. A homomorphism from a permutation module is a permutation precover if, and only if, it is supersurjective.

2. Let $\pi : X \twoheadrightarrow Y$ be supersurjective and $f : L \rightarrow Y$ a homomorphism from a permutation module L . Then f lifts to a homomorphism $\tilde{f} : L \rightarrow X$ such that $\pi \tilde{f} = f$.

Proof. 1. Given a subgroup H of G , we have isomorphisms of functors

$$\text{Hom}_{RG}(R \uparrow_H^G, -) \cong \text{Hom}_{RH}(R, (-) \downarrow_H) \cong (-)^H.$$

Suppose first that $\rho : P \rightarrow U$ is a permutation precover. Via the above isomorphisms, an element $u \in U^H$ is $\alpha(H)$ for a homomorphism $\alpha : R[G/H] \rightarrow U$. This homomorphism factors as $\alpha = \rho\gamma$ for a homomorphism $\gamma : R[G/H] \rightarrow P$ and hence $\gamma(H)$ is an element of P^H mapping onto u via ρ^H , showing that ρ^H is surjective.

Suppose now that ρ is supersurjective. An R -permutation module is a finite sum of isotypic components so it suffices to check that we can lift a homomorphism $\alpha : X \rightarrow U$ where X is a direct summand of $\prod R[G/H]$ for some subgroup H of G . Let $\iota : X \rightarrow \prod R[G/H]$, $\pi : \prod R[G/H] \rightarrow X$ be the splitting maps. By the functor isomorphisms above and supersurjectivity, there is a homomorphism $\gamma : \prod R[G/H] \rightarrow P$ such that $\rho\gamma = \alpha\pi$, and now $\gamma\iota$ is the required lifting of α .

2. Let $\alpha : P \rightarrow X$ be a permutation precover of X . Then $\pi\alpha$ is supersurjective, hence a permutation precover of Y . It follows that there exists a $\gamma : L \rightarrow P$ such that $\pi\alpha\gamma = f$, and now $\alpha\gamma = \tilde{f}$ is the required lift. \square

In just the same way we have:

Lemma 6.5. Let R be a pseudocompact integral domain.

1. A homomorphism from a monomial module is a monomial precover if, and only if, it is monomial supersurjective.

2. Let $\pi : X \twoheadrightarrow Y$ be monomial supersurjective and $f : L \rightarrow Y$ a homomorphism from a monomial module L . Then f lifts to a homomorphism $\tilde{f} : L \rightarrow X$ such that $\pi \tilde{f} = f$.

Lemma 6.6. If M is an R -permutation module and $\rho : L \rightarrow M$ is supersurjective, then ρ splits.

Proof. Let $\gamma : C \rightarrow L$ be a permutation precover of L . The composition $\rho\gamma$ is supersurjective and hence a permutation precover of the R -permutation module M . Thus $\rho\gamma$ splits, hence so does ρ . \square

A lattice L in $RG\text{-Mod}^C$ is said to be *coflasque* if $H^1(H, L) = 0$ for all $H \leq G$.

Proposition 6.7. *Let $0 \rightarrow L \rightarrow M \xrightarrow{\rho} N \rightarrow 0$ be a short exact sequence of RG -lattices. Suppose that N is an R -permutation module and that L is coflasque. Then the sequence splits.*

Proof. For any subgroup H of G the sequence $0 \rightarrow L^H \rightarrow M^H \xrightarrow{\rho^H} N^H \rightarrow 0$ is exact because $H^1(H, L) = 0$. Thus ρ is supersurjective and hence split by Lemma 6.6. \square

Corollary 6.8. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of RG -modules with L, N both R -permutation modules. If $|G|$ is not a zero divisor of R , then the sequence splits.*

Proof. By Lemma 2.9, L is coflasque, so the result is immediate from the above proposition. \square

We will give a formula for the permutation cover of a pseudocompact RG -module using methods similar to those developed by Samy Modeliar [30], when R is a complete discrete valuation ring whose residue class field has characteristic p and G is a finite group. Let π be a prime element of R . Given a pseudocompact RG -module X and a subgroup H of G , consider the map

$$\begin{aligned} \mathrm{Tr}_H^G : X^H &\rightarrow X^G \\ x &\mapsto \sum_{g \in G/H} gx \end{aligned}$$

where g runs through a set of left coset representatives of G/H . This is a special case of the map defined in [3, Def. 3.6.2] and the properties given there apply. Using [3, Lem. 3.6.3] one sees that if $|G/H|$ is coprime to p , then Tr_H^G is surjective.

Definition 6.9. *Given a p -subgroup P of G and a pseudocompact RG -module X , define*

$$\begin{aligned} X^{[P]} &:= \frac{X^P}{\sum_{Q < P} \mathrm{Tr}_Q^P(X^Q) + \pi X^P} \\ X^{\llbracket P \rrbracket} &:= \frac{X^P}{\sum_{\substack{P < Q \\ Q \text{ } p\text{-group}}} X^Q + \sum_{Q < P} \mathrm{Tr}_Q^P(X^Q) + \pi X^P} \end{aligned}$$

wherein the symbol $<$ indicates strict inclusion.

Note that in $X^{[P]}$ and $X^{\llbracket P \rrbracket}$ we quotient out by πX^P , so that both are $(R/\pi)[N_G(P)/P]$ -modules. The module $X^{[P]}$ is studied in [7] and references therein, while $X^{\llbracket P \rrbracket}$ is inspired by [30, Def. 5]. An RG -module homomorphism $f : X \rightarrow Y$ sends X^P into Y^P and the denominator of $X^{[P]}$ into the denominator of $Y^{[P]}$, and hence induces a natural map $f^{[P]} : X^{[P]} \rightarrow Y^{[P]}$. Similarly, f induces a natural map $f^{\llbracket P \rrbracket} : X^{\llbracket P \rrbracket} \rightarrow Y^{\llbracket P \rrbracket}$.

Remark: Note that $X^{[P]} \downarrow_{P/P} = (X \downarrow_P)^{[P]}$. Such a strong statement cannot be made for $X^{\llbracket P \rrbracket}$ but we do at least have that $X^{\llbracket P \rrbracket} = (X \downarrow_{N_G(P)})^{\llbracket P \rrbracket}$ because if Q is a p -subgroup of G properly containing P , then so is $Q \cap N_G(P) = N_Q(P)$.

Lemma 6.10. *The homomorphism $f : M \rightarrow N$ in $RG\text{-Mod}^C$ is supersurjective if, and only if, $f^{\llbracket P \rrbracket} : M^{\llbracket P \rrbracket} \rightarrow N^{\llbracket P \rrbracket}$ is surjective for every p -subgroup P of G .*

Proof. The forward implication is clear. If f^H is not surjective for some subgroup H of G , one easily checks using the surjectivity of Tr_P^H that f^P is also not surjective for P any Sylow p -subgroup of H . Thus f is supersurjective if, and only if, f^P is surjective for every p -subgroup of G . Note further that if f^P is surjective for every subgroup P contained in a fixed Sylow p -subgroup S of G , then f is supersurjective (since $M^{gP} = gM^P$). Thus we restrict once and for all to a Sylow p -subgroup S of G .

Given a p -subgroup P of S , we will show that

$$f^{\llbracket P \rrbracket} \text{ surjective } \forall P \implies f^{[P]} \text{ surjective } \forall P \implies f^P \text{ surjective } \forall P.$$

We prove the second implication by induction on the order of P . The base case $P = 1$ is Nakayama's Lemma. Fix a p -subgroup P of S and suppose by induction that f^Q is surjective for every proper subgroup Q of P . As $f^{[P]}$ is surjective, we can write a given $n \in N^P$ as $f(m) + \sum_{Q < P} \mathrm{Tr}_Q^P(n_Q) + \pi x$ for some $m \in M^P$, $n_Q \in N^Q$ and $x \in N^P$. But $n_Q = f(m_Q)$ for some $m_Q \in M^Q$ by induction, and hence

$$n - \pi x = f(m) + \sum_{Q < P} \mathrm{Tr}_Q^P(f(m_Q)) = f(m + \sum_{Q < P} \mathrm{Tr}_Q^P(m_Q)) \in f(M^P).$$

This shows that f^P is surjective modulo π , so that f^P is surjective by Nakayama's Lemma. We prove the first implication by induction on the index of P in S , the case $P = S$ being trivial. Consider the cokernel C of $f^{[P]}$. Then $C^{\llbracket P \rrbracket}$ is the cokernel of $f^{\llbracket P \rrbracket}$, so it is 0 by hypothesis. We will show that $C_{N_S(P)} = 0$, which implies that $C = 0$ since taking coinvariants by a p -group cannot kill a non-zero module. We must check that

$$\sum_{\substack{P < Q \\ Q \text{ } p\text{-group}}} C^Q \leq \sum_{Q < P} \text{Tr}_Q^P(C^Q)$$

in $C_{N_S(P)}$. Consider $c \in C^Q$. We may suppose without loss of generality that $|Q : P| = p$. By induction, $c = \sum \text{Tr}_T^Q(d_T)$ with $d_T \in C^T$ (T a proper subgroup of Q). Fix such a $d = d_T$ and suppose without loss of generality that $|Q : T| = p$. Whenever $A \leq B$ are subgroups, denote by $\text{Res}_A^B : C^B \rightarrow C^A$ the inclusion. The Mackey formula yields

$$\text{Res}_P^Q \text{Tr}_T^Q(d) = \sum_{q \in P \backslash Q/T} \text{Tr}_{P \cap qT}^P q \text{Res}_{P^q \cap T}^T d.$$

If $P \neq T$ then, since both are normal in Q , no conjugate of T is equal to P and hence the corresponding element is induced from proper subgroups of P as required. We are left with the case where $P = T$. The sum simplifies to

$$\text{Res}_P^Q \text{Tr}_T^Q(d) = \sum_{q \in Q/P} qd.$$

But $Q \subseteq N_S(P)$ and hence $\sum_{q \in Q/P} qd$ is equal to pd in $C_{N_S(P)}$, so is 0. \square

Lemma 6.11. *Let G be a finite group and P, Q p -subgroups of G . If P is not G -conjugate to Q , then*

$$R[G/P]^{\llbracket Q \rrbracket} = 0.$$

Proof. By the remark before Lemma 6.10 and the Mackey decomposition, we may restrict to $N_G(Q)$ and hence suppose that Q is normal in G . There are two cases:

- $P \cap Q < Q$. The module $R[G/P] \downarrow_Q$ is a direct sum of modules of the form $R[Q/{}^gP \cap Q]$. But ${}^gP \cap Q < Q$ and hence $R[G/P]^{\llbracket Q \rrbracket} = 0$ by the remark, so that $R[G/P]^{\llbracket Q \rrbracket} = 0$ also.
- $P \cap Q = Q$, so that $Q < P$. For any $g \in G$ the basis element gP is in $R[G/P]^{gP}$, hence $gP = 0$ in $R[G/P]^{\llbracket Q \rrbracket}$ because $Q < {}^gP$.

\square

Here and in Section 8 we make use of the concept of relative projectivity. Recall that an RG -module U is projective relative to the subgroup H if any continuous epimorphism of RG -modules $V \rightarrow U$ that splits as an RH -module homomorphism, splits as an RG -module homomorphism. This is equivalent to saying that U is a direct summand of an RH -module induced up to G , or to saying that the identity map of U can be written as $\text{Tr}_H^G(\alpha) := \sum_{x \in G/H} x\alpha x^{-1}$ for some continuous RH -endomorphism α of U (the proof of [3, Prop. 3.6.4] goes through for pseudocompact modules). Dually, a module U is injective relative to the subgroup H if any continuous monomorphism of RG -modules $U \rightarrow V$ that splits as an RH -module homomorphism, splits as an RG -module homomorphism.

If U is an indecomposable finitely generated RG -module, a subgroup Q of G minimal with respect to the condition that U is relatively Q -projective is called a *vertex* of U . The vertices of U are p -subgroups and are conjugate in G [3, Prop. 3.10.2]. Furthermore, there is an indecomposable finitely generated RQ -module S (unique up to conjugation in $N_G(Q)$) such that $U \mid S \uparrow^G$, called a *source* of U (the notation $X \mid Y$ indicates that the module X is isomorphic to a direct summand of Y). If U has source the trivial module R , we call U a *trivial source module*. Given a finitely generated indecomposable $R[N_G(P)]$ -lattice E with vertex P , denote by $M(P, E)$ its Green correspondent [3, Sec. 3.12]. Thus $M(P, E)$ is an indecomposable RG -lattice with vertex P such that $M(P, E) \mid E \uparrow^G$ and $E \mid M(P, E) \downarrow_{N_G(P)}$. Furthermore, $M(P, E)$ is the only summand of $E \uparrow^G$ with vertex P and E is the only summand of $M(P, E) \downarrow_{N_G(P)}$ with vertex P , the other summands of $E \uparrow^G$ being projective relative to strictly smaller subgroups and the other summands of $M(P, E) \downarrow_{N_G(P)}$ being projective relative to proper conjugates of P [3, Thm 3.12.2]. More generally, when E is a product of finitely generated indecomposable $R[N_G(P)]$ -modules with vertex P , denote by $M(P, E)$ the product of the corresponding Green correspondents. Note that E is still a

(continuous) summand of $M(P, E) \downarrow_{N_G(P)}$ and $M(P, E)$ is still a (continuous) summand of $E \uparrow^G$ – this can be seen by direct calculation with the product topology.

Let E be a product of indecomposable trivial source $R[N_G(P)]$ -modules with vertex P and let the RG -module $M(P, E)$ be as above. Let $j : M(P, E) \rightarrow E \uparrow^G$ be a split inclusion and define further the split inclusion $s : E \rightarrow E \uparrow^G \downarrow_{N_G(P)}$ by $s(e) = 1 \otimes e$.

Lemma 6.12. 1. *The maps $j^{[P]}, s^{[P]}$ are isomorphisms.*

2. *If M is an RG -module and $f : E \rightarrow M \downarrow_{N_G(P)}$ is an $R[N_G(P)]$ -module homomorphism, there exists an RG -module homomorphism $\hat{f} : M(P, E) \rightarrow M$ such that*

$$\hat{f}^{[P]} = f^{[P]} \gamma,$$

where γ is an isomorphism from $M(P, E)^{[P]}$ to $E^{[P]} = E/\pi E$.

Proof. 1. We show the result for s , the argument for j being similar. Let $E \uparrow^G \downarrow_{N_G(P)} = s(E) \oplus Y$. To prove the claim, it is sufficient to show that $Y^{[P]} = 0$. By the Green correspondence, an indecomposable summand of Y is a direct summand of a module of the form $R[N_G(P)/H]$ for H a subgroup of some proper conjugate gP of P . We have

$$R[N_G(P)/H]^{[P]} \downarrow_P = \left(R \uparrow_H^{N_G(P)} \downarrow_P \right)^{[P]} \cong \bigoplus_{x \in P \backslash N_G(P)/H} (R \downarrow_{xH \cap P} \uparrow^P)^{[P]} = 0,$$

because each ${}^xH \cap P \subseteq {}^gP \cap P$ is a proper subgroup of P .

2. The map f extends by Frobenius reciprocity to a unique RG -module homomorphism $\tilde{f} : E \uparrow^G \rightarrow M$ given on pure tensors by $\tilde{f}(g \otimes e) = gf(e)$. Note that $\tilde{f}s = f$. Define $\hat{f} = \tilde{f}j$. We have

$$\hat{f}^{[P]} = \tilde{f}^{[P]} j^{[P]} = \tilde{f}^{[P]} s^{[P]} ((s^{[P]})^{-1} j^{[P]}) = f^{[P]} \gamma,$$

where $\gamma = (s^{[P]})^{-1} j^{[P]}$. □

Let X be a pseudocompact RG -module. Given a p -subgroup P of G , let $\varphi_P : E_P \rightarrow X^{[[P]]}$ be the projective cover of $X^{[[P]]}$ as an $R[N_G(P)/P]$ -module. By projectivity of E_P , φ_P lifts to a homomorphism $E_P \rightarrow X^P$. Composing with the inclusion we obtain a homomorphism $\theta_P : E_P \rightarrow X \downarrow_{N_G(P)}$, which by Lemma 6.12 yields an RG -module homomorphism $\widehat{\theta}_P : M(P, E_P) \rightarrow X$ such that $\widehat{\theta}_P^{[P]} = \theta_P^{[P]} \gamma_P$ for some isomorphism $\gamma_P : M(P, E_P)^{[P]} \rightarrow E_P^{[P]}$. Note that $\varphi_P^{[[P]]} \gamma_P^{[[P]]} = \theta_P^{[[P]]} \gamma_P^{[[P]]} = \widehat{\theta}_P^{[[P]]}$. Denote by

$$\theta : C = \prod_P M(P, E_P) \rightarrow X$$

the homomorphism that acts as $\widehat{\theta}_P$ on the factor $M(P, E_P)$, as P ranges through a set of representatives of the G -conjugacy classes of p -subgroups of G . For each P , denote by $m_P : M(P, E_P) \rightarrow C$ the inclusion, noting that $m_P^{[[P]]}$ is an isomorphism by Lemma 6.11.

Theorem 6.13. *The map θ is a permutation cover of X .*

Proof. By Lemmas 6.4 and 6.10, to check that θ is a precover we must check that $\theta^{[[P]]} : C^{[[P]]} \rightarrow X^{[[P]]}$ is surjective for any p -subgroup P of G . But

$$\theta^{[[P]]} m_P^{[[P]]} = \widehat{\theta}_P^{[[P]]} = \varphi_P^{[[P]]} \gamma_P^{[[P]]},$$

and hence $\theta^{[[P]]}$ is surjective.

It remains to check that θ is a cover. If not, there exists a permutation precover $d : D \rightarrow X$ and a non-surjective homomorphism $f : D \rightarrow C$ such that $\theta f = d$. By Lemma 6.10 there is a p -subgroup P such that $f^{[[P]]} : D^{[[P]]} \rightarrow C^{[[P]]}$

is not surjective. For this P , let $\rho : S \rightarrow D^{\llbracket P \rrbracket}$ be a projective cover of $D^{\llbracket P \rrbracket}$ as an $R[\mathrm{N}_G(P)/P]$ -module. By the projectivity of S we obtain a map δ yielding the following commutative diagram:

$$\begin{array}{ccccccc}
& & & & S & & \\
& & & & \downarrow \rho & & \\
& & & & D^{\llbracket P \rrbracket} & & \\
& & \delta & & \downarrow f^{\llbracket P \rrbracket} & \searrow d^{\llbracket P \rrbracket} & \\
& & & & C^{\llbracket P \rrbracket} & \xrightarrow{\theta^{\llbracket P \rrbracket}} & X^{\llbracket P \rrbracket} \\
& \swarrow \alpha & & & \uparrow m_P^{\llbracket P \rrbracket} & & \\
E_P & \xrightarrow{(\gamma_P^{-1})^{\llbracket P \rrbracket}} & M(P, E_P)^{\llbracket P \rrbracket} & \xrightarrow{m_P^{\llbracket P \rrbracket}} & C^{\llbracket P \rrbracket} & \xrightarrow{\theta^{\llbracket P \rrbracket}} & X^{\llbracket P \rrbracket} \\
& \searrow \varphi_P & & & & & \\
& & & & & &
\end{array}$$

The map α is the canonical projection from $E_P = E_P^P$ to $E_P^{\llbracket P \rrbracket}$. The lower shape commutes because $\varphi_P = \varphi_P^{\llbracket P \rrbracket} \alpha$ and $\varphi_P^{\llbracket P \rrbracket} \gamma_P^{\llbracket P \rrbracket} = \theta^{\llbracket P \rrbracket} m_P^{\llbracket P \rrbracket}$. Note that δ is surjective because $\varphi_P \delta = d^{\llbracket P \rrbracket} \rho$ is a surjective map from the projective $R[\mathrm{N}_G(P)/P]$ -module S to $X^{\llbracket P \rrbracket}$ and φ_P is a projective cover of $X^{\llbracket P \rrbracket}$. This yields a contradiction, because the path from S to $C^{\llbracket P \rrbracket}$ going via δ is surjective, being a composition of surjective maps, while $f^{\llbracket P \rrbracket} \rho$ is not surjective. \square

The proof of Theorem 6.13 does not suppose the existence of covers and thus provides a proof of the existence of permutation covers independent of Theorem 6.2.

We now use the explicit description of the permutation cover to obtain a characterization of R -permutation modules. Here and in future, if U is an RG -module then \overline{U} denotes the module $U/\pi U$, which may be treated as an RG -module or as an $(R/\pi)G$ -module, depending on context.

Proposition 6.14. *Let R be a complete discrete valuation ring whose residue class field $k = R/\pi$ has characteristic p . If X is a coflasque pseudocompact RG -lattice and $\overline{X} = X/\pi X$ is a k -permutation kG -module, then X itself is an R -permutation RG -module.*

Proof. One may check using the explicit construction of the permutation cover $\theta_X : P_X \rightarrow X$ that its reduction $\overline{\theta}_X : \overline{P}_X \rightarrow \overline{X}$ modulo π is the permutation cover of \overline{X} using the following facts (cf. [7]):

- For any finite group H , reduction modulo π induces a natural bijection between projective RH -modules and projective kH -modules;
- Since πX (being isomorphic to X) is coflasque, applying $(-)^P$ to the short exact sequence $0 \rightarrow \pi X \rightarrow X \rightarrow \overline{X} \rightarrow 0$ yields the isomorphism

$$\overline{X}^P \cong X^P / (\pi X)^P = X^P / \pi X^P = \overline{X}^P.$$

Hence $X^{\llbracket P \rrbracket} = \overline{X}^{\llbracket P \rrbracket}$.

- Reduction modulo π induces a natural bijection between Green correspondents of indecomposable R -permutation modules and those of the corresponding indecomposable k -permutation modules.

But $\overline{\theta}_X$ is an isomorphism by hypothesis. So the kernel of θ_X is a pure submodule of P_X contained in πP_X , and hence is 0. That is, $P_X \cong X$ and X is a permutation module. \square

7 A theorem of Cliff and Weiss

From now on, R will be a complete discrete valuation ring whose field of fractions has characteristic 0 and whose residue class field $k = R/\pi$ has prime characteristic p . Given an RG -lattice L we want to find a small power π^e such that when $L/\pi^e L$ is a direct summand of a permutation module over R/π^e , then L is coflasque. We adapt the argument in [11] to see that if some given e has this property for every lattice for a cyclic group of order p , then this same e will work for the whole group G . Indeed, we may restrict to the case where G is a p -group because $L \downarrow_{P \uparrow}^G$ for any Sylow p -subgroup of G so that, by the Eckmann-Shapiro Lemma and Mackey decomposition, we need only check that $L \downarrow_P$ is coflasque. Suppose that G is a p -group and proceed by induction on its order, the case of order p being by hypothesis. Let Q be a central subgroup of order p in G . Since $H^1(Q, L) = 0$ the sequence

$$0 \rightarrow L^Q \xrightarrow{\pi^e} L^Q \rightarrow (L/\pi^e L)^Q \rightarrow 0$$

is exact, so that $L^Q/\pi^e L^Q \cong (L/\pi^e L)^Q$ is a permutation $(R/\pi^e)[L/Q]$ -lattice. By induction, L^Q is coflasque as a module for G/Q . Now, the inflation-restriction exact sequence (see [35, 6.8.3])

$$0 \rightarrow H^1(G/Q, L^Q) \rightarrow H^1(G, L) \rightarrow H^1(Q, L)^{G/Q}$$

shows that $H^1(G, L) = 0$, as required.

Lemma 7.1. *Let G be a finite group and let L be an RG -lattice in $RG\text{-Mod}^C$. Suppose that F is a free kG -submodule of $L/\pi L$. Then there is a free RG -lattice $\tilde{F} \leq L$ that is a summand of L and such that $\tilde{F}/\pi\tilde{F} \cong (\tilde{F} + \pi L)/\pi L = F$.*

Proof. This is well known. Let $\hat{F} \rightarrow F$ be the projective cover of F as an RG -module. The composition $\hat{F} \rightarrow F \rightarrow L/\pi L$ lifts to $\gamma: \hat{F} \rightarrow L$ since \hat{F} is projective. Let $\tilde{F} = \gamma(\hat{F})$. Then \tilde{F} is pure over R in L and so is an R -summand of L . The induction and coinduction functors being isomorphic for finite groups, and γ being injective since it is injective modulo π , mean that \tilde{F} is injective relative to the the trivial subgroup of G (see the discussion after Lemma 6.11). Thus, being an RG -submodule and an R -summand of L , it is an RG -summand. \square

Remark: Let G be a p -group. Replacing F with a permutation module $V_H \uparrow_H^G \leq L/\pi L$, let \hat{V}_H be the projective cover of V_H as an R -module, treated as a trivial RH -module. Then \hat{V}_H lifts to $\tilde{V}_H \leq L$ if $H^1(H, L) = 0$, and thus we get a submodule $\tilde{V}_H \uparrow_H^G \leq L$. This is another way to prove Proposition 6.14.

Let C be a group of order p and let L be an RC -lattice such that $L/\pi^e L$ is a permutation $(R/\pi^e)C$ -lattice. The module $L/\pi^e L$ decomposes as the sum of a free $(R/\pi^e)C$ -lattice and a trivial $(R/\pi^e)C$ -lattice. By Lemma 7.1 the free part lifts to a summand of L , whose complement we denote by M . Note that $M/\pi^e M$ is trivial.

Observe that a short exact sequence of lattices $J \rightarrow M \rightarrow X$ is necessarily split over R , so that the reduction $J/\pi^e J \rightarrow M/\pi^e M \rightarrow X/\pi^e X$ is exact, hence if $M/\pi^e M$ is trivial then so are $J/\pi^e J$ and $X/\pi^e X$. The RC -lattice M is an inverse limit of lattices of finite rank by Lemma 2.1 and these quotients are necessarily trivial modulo π^e by the above observation. A lattice of finite rank is an iterated extension of irreducible lattices [15, Cor. 23.15 and §25], where by irreducible lattice we mean a lattice S such that $K \otimes_R S$ is irreducible as a KC -module, (K the field of fractions of R). Thus if we express each finite rank lattice as an iterated extension of irreducible lattices, then each of these must also be trivial modulo π^e .

The irreducible KC -modules are either trivial or of the form $K(\omega)$ for ω a primitive p th root of unity, where a given generator c of C acts by multiplication by ω^r with r not divisible by p . A lattice in such a module corresponds to a fractional ideal of $K(\omega)$ and such ideals are principal. Thus, such a lattice is isomorphic to $R[\omega]$.

We have the following two cases:

- $\omega \notin K$: Then the extension $K(\omega)/K$ is ramified and so $\omega - 1 \notin \pi R[\omega]$. In this case we can take $e = 1$ so that the irreducible lattices are trivial. But extensions and inverse limits of trivial lattices are trivial, so that our infinite lattice is also trivial.
- $\omega \in K$, hence $\omega \in R$: Then $\omega - 1 \notin \langle (\omega - 1)\pi \rangle$, so the action of C on $R[\omega]$ is not trivial modulo $(\omega - 1)\pi$. If we choose e so that $\langle \pi^e \rangle = \langle (\omega - 1)\pi \rangle$ then the infinite lattice must have been trivial.

Putting these observations together we obtain a form of a result of [11]:

Theorem 7.2. *Let G be a finite group and let L be a lattice in $RG\text{-Mod}^C$.*

1. *If R does not contain a primitive p th root of unity then L is an R -permutation lattice over R if, and only if, $L/\pi L$ is a k -permutation module.*
2. *If R contains a primitive p th root of unity ω then L is an R -permutation lattice over R if, and only if, $L/(\omega - 1)\pi L$ is an $R/(\omega - 1)\pi$ -permutation lattice.*

Observe that, by considering lattices of rank 1, we already see that the power of π in part 2 is the best possible.

8 Weiss' Theorem

Our goal for this section is to prove Theorem 1.2. A special case of this result was proved in [23] with $R = \mathbb{Z}_p$. The missing link needed to obtain the result in generality is Theorem 8.6, a generalization of [36, Thm 3] for infinitely generated lattices. Throughout this section, G is a finite p -group and R is a complete discrete valuation ring whose

residue class field has characteristic p and whose field of fractions has characteristic 0. Denote by π a generator of the maximal ideal of R and let $k = R/\pi$.

If U is an RG -module, recall that by \widetilde{U} we denote the kG -module $U/\pi U$. When R contains a primitive p th root of unity ω , let $\widetilde{R} = R/(\omega - 1)$ and denote by \widetilde{U} the $\widetilde{R}G$ -module $U/(\omega - 1)U$. For S any quotient ring of R (including R), a pseudocompact SG -lattice is said to be *diagonal* if it is isomorphic to a direct product of indecomposable lattices of rank 1 over S (diagonal is thus a special case of monomial).

Lemma 8.1. *Suppose that R contains a primitive p th root of unity ω . If a short exact sequence of pseudocompact $\widetilde{R}G$ -modules $0 \rightarrow X' \xrightarrow{s} X \xrightarrow{h} X'' \rightarrow 0$ splits as \widetilde{R} -modules and if the middle term is diagonal, then the sequence splits as $\widetilde{R}G$ -modules.*

Proof. We mimic the proof of [38, Lem. 52.2]. Write $X = \prod_I X_i$ with each X_i an $\widetilde{R}G$ -sublattice of X of \widetilde{R} -rank 1. The sequence remains split over \widetilde{R} and the \widetilde{X}_i are indecomposable \widetilde{R} -modules, so by Proposition 3.5 applied to these \widetilde{R} -modules there is a subset J of I such that $\widetilde{X} = \overline{s}(\widetilde{X}') \oplus \prod_J \widetilde{X}_i$, and this remains true over $\widetilde{R}G$. Thus the $\widetilde{R}G$ -module homomorphism $\widetilde{h}' = \widetilde{h}|_{\prod_J \widetilde{X}_i}$ is an isomorphism, where \widetilde{h} is the map induced by h . By Nakayama's Lemma, $h|_{\prod_J X_i} = h' : \prod_J X_i \rightarrow X''$ is an isomorphism. Now the composition of h'^{-1} with the inclusion $\prod_J X_i \rightarrow X$ is a splitting of h . \square

In the above proof we work modulo π because there need not be finitely many isomorphism classes of rank 1 $\widetilde{R}G$ -lattice, so that Proposition 3.5 need not apply directly to X .

Proposition 8.2. *Suppose that R contains a primitive p th root of unity ω . Let U be a pseudocompact RG -lattice and V a diagonal pseudocompact RG -lattice. If $\widetilde{U} \cong \widetilde{V}$, then U is diagonal.*

Proof. Write $U = \varprojlim_i U/X_i$ with each U/X_i an RG -lattice of finite rank over R , as we may by Lemma 2.1. As there are only finitely many isomorphism classes of indecomposable diagonal RG -lattices, by Theorem 3.8 applied with M the direct sum of a representative of each isomorphism class of indecomposable diagonal RG -lattice, it suffices to check that each U/X_i is diagonal. The sequence $0 \rightarrow X_i \rightarrow U \rightarrow U/X_i \rightarrow 0$ is split over R because U/X_i is a lattice, so applying $\widetilde{(-)}$ it follows that $\widetilde{U/X_i} \cong \widetilde{U}/\widetilde{X_i}$. Denoting by $\rho : \widetilde{U} \rightarrow \widetilde{V}$ the given isomorphism, we have

$$\widetilde{U/X_i} \cong \widetilde{U}/\widetilde{X_i} \cong \widetilde{V}/\rho(\widetilde{X_i}).$$

By Lemma 8.1, $\widetilde{V}/\rho(\widetilde{X_i})$ is a direct summand of \widetilde{V} , hence diagonal by the Krull-Schmidt property. It follows that the finitely generated RG -lattice U/X_i is diagonal modulo $\omega - 1$, and hence is diagonal by the finitely generated version of this result [38, Thm 50.2] (in fact, we only require Case B in the proof). Note that the cited proof is for compact discrete valuation rings, but the same proof holds for complete discrete valuation rings because we still have Nakayama's Lemma. \square

Lemma 8.3. *Suppose that $f : L \rightarrow M$ is a (continuous) homomorphism of pseudocompact R -lattices such that $\overline{f} : \overline{L} \rightarrow \overline{M}$ is injective. Then f is a split injection in $R\text{-Mod}^c$.*

Proof. If f is not injective, then there exists $x \neq 0$ in the kernel. Write $x = \pi^n y$ for some $y \notin \pi L$. Now

$$0 = f(x) = f(\pi^n y) = \pi^n f(y),$$

so that $f(y) = 0$. Hence there exists $y \notin \pi L$ in the kernel of f . But now

$$\overline{f}(\overline{y}) = \overline{f(y)} = 0,$$

contradicting the injectivity of \overline{f} .

We claim that $f(L)$ is pure in M . If not, then there is $y \in L \setminus \pi L$ such that $f(y) = \pi m$ for some $m \in M$. Now $\overline{y} \neq 0$, while

$$\overline{f}(\overline{y}) = \overline{f(y)} = 0,$$

contradicting the injectivity of \overline{f} . Thus $M/f(L)$ is torsion-free, hence free, and so f splits. \square

We next generalize [38, Prop. 53.2] to infinite rank lattices:

Lemma 8.4. *Suppose we have a short exact sequence of RG -lattices*

$$0 \rightarrow L \rightarrow T \xrightarrow{\gamma} M \rightarrow 0$$

with L, M monomial and such that the induced sequence

$$0 \rightarrow \widetilde{L} \rightarrow \widetilde{T} \rightarrow \widetilde{M} \rightarrow 0$$

is split exact. Then the sequence splits.

Proof. We first show that γ is monomial supersurjective. Writing $(\varphi \uparrow_H^G, -)$ instead of $\text{Hom}_{RG}(\varphi \uparrow_H^G, -)$ and applying it to the given sequence, we obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\varphi \uparrow_H^G, L) & \longrightarrow & (\varphi \uparrow_H^G, T) & \longrightarrow & (\varphi \uparrow_H^G, M) \xrightarrow{\partial} \text{Ext}_{RG}^1(\varphi \uparrow_H^G, L) \\ & & & & \downarrow & & \downarrow \\ & & & & (\varphi \uparrow_H^G, \widetilde{M}) & \xrightarrow{0} & \text{Ext}_{RG}^1(\varphi \uparrow_H^G, \widetilde{L}) \end{array}$$

Write $L = \coprod L_i$ as a product of indecomposable monomial modules. We have

$$\text{Ext}_{RG}^1(\varphi \uparrow_H^G, L) = \prod \text{Ext}_{RG}^1(\varphi \uparrow_H^G, L_i).$$

By [38, Prop. 53.2], on each factor of this product the vertical right map is injective, and hence the vertical right map is itself injective. But this forces ∂ to be 0, showing that $(\varphi \uparrow_H^G, T) \rightarrow (\varphi \uparrow_H^G, M)$ is surjective, so that γ is monomial supersurjective.

By Lemma 6.5, the monomial supersurjectivity of γ implies that $(M, \gamma) : (M, T) \rightarrow (M, M)$ is surjective. An element of the preimage of id_M is the required splitting of γ . \square

We need the following standard lemma (see after Lemma 6.11 for definitions).

Lemma 8.5. *Let U be a finite direct sum of modules U_i each projective relative to a proper subgroup of G . Suppose that the epimorphism $\gamma : Y \rightarrow U$ splits over every proper subgroup of G . Then γ splits.*

Proof. Suppose that each U_i is relatively H_i -projective for $H_i < G$. Then $\text{id}_{U_i} = \text{Tr}_{H_i}^G(\alpha_i)$ for some $\alpha_i \in \text{End}_{RH_i}(U_i)$. Hence $\text{id}_U = \sum_i \text{Tr}_{H_i}^G(\alpha_i)$. For each i , let s_i be a splitting of γ as an H_i -module homomorphism. Using [3, Lem. 3.6.3] we obtain

$$\gamma \left(\sum_i \text{Tr}_{H_i}^G(s_i \alpha_i) \right) = \sum_i \text{Tr}_{H_i}^G(\gamma s_i \alpha_i) = \sum_i \text{Tr}_{H_i}^G(\alpha_i) = \text{id}_U.$$

\square

The next theorem is a generalization of [38, Thm 50.2] to pseudocompact modules.

Theorem 8.6. *Let R be a complete discrete valuation ring containing a primitive p th root of unity ω . If U is a pseudocompact lattice such that \widetilde{U} is monomial, then U itself is monomial.*

Proof. We work by induction on the order of G . If $|G| = 1$ the result is obvious. We suppose that the result holds for any proper subgroup of G .

Applying the functor $(-)$ to the monomial cover $c : P \rightarrow U$ yields the commutative square

$$\begin{array}{ccc} P & \xrightarrow{c} & U \\ \downarrow & & \downarrow \\ \widetilde{P} & \xrightarrow{\widetilde{c}} & \widetilde{U} \end{array}$$

For any proper subgroup H of G the homomorphism $c \downarrow_H$ is a surjection from the monomial module $P \downarrow_H$ to the module $U \downarrow_H$, the latter monomial by induction. Using Frobenius reciprocity, we see that $c \downarrow_H$ is a monomial

precover. Hence, since $U \downarrow_H$ is monomial, the homomorphism $c \downarrow_H$ is a split surjection for every proper subgroup H of G . Thus \tilde{c} also splits over every proper subgroup of G .

The module \tilde{U} is monomial by hypothesis, so write it as $T \oplus Y$, where T is diagonal and Y is projective relative to proper subgroups of G . Consider the composition

$$\tilde{P} \xrightarrow{\tilde{c}} T \oplus Y \xrightarrow{\pi_Y} Y.$$

This homomorphism splits over every proper subgroup of G (because \tilde{c} does) and hence, since Y is projective relative to proper subgroups, it splits over G (to see this, apply Lemma 8.5 to Y written as a finite sum of modules each projective relative to a fixed maximal subgroup of G , for example). Let $\beta : Y \rightarrow \tilde{P}$ be an $\tilde{R}G$ -module homomorphism such that $\pi_Y \tilde{c} \beta = \text{id}_Y$. Define the following submodules of \tilde{P} :

$$\begin{aligned} Y' &= \beta(Y), \\ T' &= \tilde{c}^{-1}(T). \end{aligned}$$

Routine checks show that $\tilde{P} = Y' \oplus T'$. Observe that $\tilde{c}|_{Y'} : Y' \rightarrow Y$ is an isomorphism.

Write $P = \prod_{i \in I} P_i$ with the P_i indecomposable monomial lattices. Hence $\tilde{P} = \prod_I \tilde{P}_i$. This module has the exchange property (Proposition 3.5 can be applied because there are finitely many isomorphism classes of indecomposable monomial RG -lattice, hence finitely many \tilde{P}_i up to isomorphism), so we can write

$$\tilde{P} = \prod_{i \in J} \tilde{P}_i \oplus T'$$

for some subset $J \subseteq I$.

Let $W = \prod_J P_i$ be the corresponding summand of P . It follows that $\tilde{P} = \tilde{W} \oplus T'$, and hence that

$$\tilde{W} \cong \tilde{P}/T' \cong Y'.$$

The reader may check that $\tilde{c}|_{\tilde{W}}$ is injective. We claim that $\tilde{U} = \tilde{c}(\tilde{W}) \oplus T$: That $\tilde{U} = \tilde{c}(\tilde{W}) + T$ is obvious, so consider $u \in \tilde{c}(\tilde{W}) \cap T$, and write $u = \tilde{c}(w)$ for $w \in \tilde{W}$. Then

$$w \in \tilde{W} \cap \tilde{c}^{-1}(T) = \tilde{W} \cap T' = 0,$$

so that $u = \tilde{c}(0) = 0$.

We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{c|_W} & U & \longrightarrow & U/W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{W} & \longrightarrow & \tilde{U} = T \oplus Y & \longrightarrow & \tilde{U}/\tilde{W} & \longrightarrow & 0 \end{array}$$

The second row is exact because $\widetilde{(-)}$ is right exact and $\tilde{c}|_{\tilde{W}}$ is injective. It follows from the claim above that $\tilde{U}/\tilde{W} \cong T$. Thus, by Proposition 8.2, U/W is monomial. Meanwhile, W is monomial by construction.

Finally, the lower sequence splits ($\tilde{c}|_{\tilde{W}}$ is a split injection) so by Lemma 8.4, $U \cong W \oplus U/W$ is monomial, as required. \square

We prove Theorem 1.2. We work by induction on the order of G . If $N = 1$ the result is immediate so suppose that $N \neq 1$. There exists a central normal subgroup M of G of order p contained in N . Consider the G/M -module U^M . We have $(U^M)^N = U^N$ is a permutation $G/N = (G/M)/(N/M)$ -module, and $U^M \downarrow_{N/M} \cong (U \downarrow_N)^M$ is a free N/M -module. Hence by induction we have that U^M is a permutation G/M -lattice. But we also have that $U \downarrow_M = U \downarrow_N \downarrow_M$ is free, and hence we need only show that the theorem holds when N is central of order p . We assume this from now on. Denote by Σ_N the element $\sum_{n \in N} n \in RN$.

Lemma 8.7. *Let N be a central subgroup of G of order p and let U be a pseudocompact RG -lattice such that U^N is a permutation module and $U \downarrow_N$ is free. Let L be a pseudocompact permutation RG -lattice and let $f : L \rightarrow U/\Sigma_N U$ be a surjection having kernel $\Sigma_N L$. Then $L \cong U$.*

Proof. Note that RN/RN^N is torsion-free (as an R -module), so U/U^N is torsion-free since $U \downarrow_N$ is free. Note also that since U is free over N , we have $\Sigma_N U = U^N$. Write $L = \prod_i R[G/H_i]$ (via Proposition 2.8). If for some i we had $N \leq H_i$, then by the Mackey decomposition, $L \downarrow_N$ would have a trivial summand R . But then $R/\Sigma_N R = R/pR$ would be isomorphic to a submodule of $U/\Sigma_N U$ – impossible, since $U/\Sigma_N U$ is torsion-free. Hence $L \downarrow_N$ is free.

Because $\Sigma_N U = U^N$ is a permutation module by hypothesis, we have $H^1(H, \Sigma_N U) = 0$ for all H by Lemma 2.9. Thus the homomorphism $U \twoheadrightarrow U/\Sigma_N U$ is supersurjective and hence, by Lemma 6.4, f lifts to a homomorphism $\hat{f}: L \rightarrow U$.

Consider the triangle

$$\begin{array}{ccc} & L & \\ \hat{f} \swarrow & & \searrow f \\ U & \longrightarrow & U/\Sigma_N U \end{array}$$

Considering this triangle modulo Σ_N shows that \hat{f} is an isomorphism modulo Σ_N . But RN is local, hence \hat{f} is surjective by Nakayama's Lemma and injective by Lemma 8.3. \square

By the discussion before Lemma 8.7, Theorem 1.2 follows from the following special case:

Theorem 8.8. *Let R be a discrete valuation ring whose field of fractions has characteristic 0 and whose residue field has characteristic $p > 0$. Let G be a finite p -group and let U be a pseudocompact RG -lattice. Suppose G has a central subgroup N of order p such that*

- $U \downarrow_N$ is free,
- U^N is a permutation module.

Then U itself is a permutation module.

Proof. We prove this first in the special case that p is prime in R . Adjoin a primitive p th root of unity ω to R to obtain the ring $R[\omega] = R[x]/(1 + x + \dots + x^{p-1})$. The kernel of the surjective ring homomorphism $R[\omega] \rightarrow R/p$ sending ω to 1 is generated by $\omega - 1$, so that p prime in R implies that $\omega - 1$ is prime in $R[\omega]$. If $p = 2$ then $R[\omega] = R$, but this does not affect the argument. Fix an isomorphism of groups $\psi: N \rightarrow \langle \omega \rangle$. If p is odd, define the structure of $R[\omega]G$ -lattice on the module $U/\Sigma_N U$ via ψ by defining the action of ω as

$$\omega \cdot (u + \Sigma_N U) := \psi^{-1}(\omega)u + \Sigma_N U,$$

(if $p = 2$ there is nothing to define, but in this case note that the formula agrees with the action of $\omega = -1$ on $U/\Sigma_N U$). This structure is natural in the sense that having fixed ψ , we can consider $U \mapsto U/\Sigma_N U$ as a functor $RG\text{-Mod} \rightarrow R[\omega]G\text{-Mod}$. Multiplication by Σ_N yields an isomorphism

$$(U/\Sigma_N U)/(\omega - 1)(U/\Sigma_N U) \cong U^N/pU^N.$$

This is a permutation module (because U^N is), and hence by Theorem 8.6, $U/\Sigma_N U$ is a monomial $R[\omega]G$ -lattice. So write

$$U/\Sigma_N U = \prod_{i \in I} V_i \uparrow_{H_i}^G,$$

where V_i is an $R[\omega]H_i$ lattice of rank 1. As $R[\omega]$ contains no primitive p^2 root of unity, the action of each H_i on the corresponding V_i is given by a group homomorphism φ_i from H_i to $\langle \omega \rangle$. Denote by K_i the kernel of φ_i . The RN -module $RN/\Sigma_N RN$ has no non-zero N -fixed points and hence, since U is N -free, $U/\Sigma_N U$ has no non-zero N -fixed points. Calculating the N -fixed points of $V_i \uparrow_{H_i}^G$ differently, we obtain isomorphisms of R -modules

$$\begin{aligned} 0 = (V_i \uparrow_{H_i \downarrow_N}^G)^N &\cong \bigoplus_{g \in G/H_i N} \text{Hom}_N(R, {}^g V_i \downarrow_{H_i \cap N} \uparrow^N) \\ &\cong \bigoplus_{g \in G/H_i N} \text{Hom}_{H_i \cap N}(R, {}^g V_i \downarrow_{H_i \cap N}). \end{aligned}$$

It follows that $H_i \cap N$ is not contained in K_i and hence that N is a subgroup of H_i not intersecting K_i . In particular, φ_i is not trivial.

A surjective RH_i -module homomorphism $R[H_i/K_i] \rightarrow V_i/(\omega - 1)V_i \cong R/p$ lifts by projectivity over H_i/K_i to a surjective RH_i -module homomorphism from $R[H_i/K_i]$ to V_i . Inducing to G , we obtain a surjective RG -module homomorphism $R[G/K_i] \rightarrow V_i \uparrow_{H_i}^G$ whose kernel contains (hence is equal to by comparison of ranks) $\Sigma_N R[G/K_i]$, so we have an isomorphism of RG -modules

$$R[G/K_i]/\Sigma_N R[G/K_i] \cong V_i \uparrow_{H_i}^G.$$

Define the permutation lattice $L = \prod_{i \in I} R[G/K_i]$ together with the obvious homomorphism $L \rightarrow U/\Sigma_N U$, whose kernel is $\Sigma_N L$. Now by Lemma 8.7, $U \cong L$ is a permutation lattice. This completes the special case.

For the general case, let S be a coefficient ring for R , in the sense of Cohen's Structure Theorem [13]. Thus S is a complete discrete valuation ring contained in R with the following properties:

- The inclusion $S \rightarrow R$ realizes R as a free pseudocompact S -module of finite rank over S (cf.[32, Ch. 1 §6]).
- The number p generates the maximal ideal of S .

Denote by U' the module U considered as an SG -module. Then U' satisfies the hypotheses of the theorem, so that U' is a permutation SG -module by the special case above. As U' is free over N , it is a product of modules $S[G/H]$ with $H \cap N = 1$. Since $\text{id} : U' \rightarrow U'$ is the permutation cover of U' , we have that $R \widehat{\otimes}_S U' \rightarrow U$ given by $r \widehat{\otimes} u \mapsto ru$ is supersurjective, so is a permutation precover of U by Lemma 6.4. It follows that the permutation cover of U (a direct summand of $R \widehat{\otimes}_S U'$) is free on restriction to N . Denote by

$$0 \rightarrow K \rightarrow C \xrightarrow{f} U \rightarrow 0$$

the permutation cover of U . We will show that $K = 0$. By freeness of U over N the sequence is split over N and hence

$$0 \rightarrow K_N \rightarrow C_N \xrightarrow{f_N} U_N \rightarrow 0$$

is exact. Note that U_N is a permutation module and f_N is supersurjective, both these claims following from the natural isomorphisms $C_N \rightarrow C^N$ and $U_N \rightarrow U^N$ given by multiplication by Σ_N , again by freeness of C and U over N . Thus f_N is a permutation precover of the permutation module U_N , so it splits and hence K_N is a direct summand of C_N . Write $C = \prod_I V_i$, a product of indecomposable modules. Then each $(V_i)_N$ is indecomposable, and $C_N = \prod_I (V_i)_N$. By the extension property, there is a subset J of I such that $C_N = K_N \oplus \prod_J (V_i)_N$. By Nakayama's Lemma, $C = K + \prod_J V_i$. The restriction $f : \prod_J V_i \rightarrow U$ is thus also a precover of U , so that $I = J$ since f is a cover. It follows that $K_N = 0$, hence $K = 0$ as required. \square

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