THE COHOMOLOGY OF PRO-p GROUPS WITH A POWERFULLY EMBEDDED SUBGROUP

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Abstract. We calculate the cohomology of a pro-p group with an extendable and almost powerfully embedded subgroup.

We consider the mod-p cohomology of p-groups and pro-p groups with an almost powerfully embedded subgroup satisfying an extendibility condition. The result has a strikingly simple form.

If G is a pro-p group for some prime p we say that a closed, normal, finitely generated subgroup, N, is almost powerfully embedded if:

- $[G,N] \subset N^p$ for $p > 2$;
- $[G,N] \subset N^2$ and $[N,N] \subset (N^2)^2$ for $p = 2$.

Where N^p denotes the closure of the subgroup of N generated by p-th powers of elements of N.

Let $\Omega_1 N$ denote the subgroup of N generated by elements of order p. We say that N is Ω-extendable in G if $\Omega_1 N$ is central in G, so in particular is an elementary abelian subgroup, and if also there is a central extension $E \to \tilde{G} \to G$, where $E \cong \Omega_1 N$ and every non-trivial element of $\Omega_1 N$ is the image of an element of $\tilde{G}$ of order p^2. (So every torsion free group is extendable.)

For any closed normal subgroup $M \subset G$ we define $\Phi_G(M) = [G,M]M^p$.

Note: It follows from [4, 3.1 and 3.8] that the subgroups defined above are closed, indeed all but the last are open. Also every element of N^p is a p-th power and $\Omega_1 N$ is finite.

Denote by $H^*(G)$ the cohomology of G with coefficients in $\mathbb{Z}/p$ (the *Galois* or *continuous* cohomology if G is infinite, see [11]), and by $\beta : H^*(G) \to H^{*+1}(G)$ the Bockstein homomorphism. Our main theorem is the following consequence of Theorem 3.13.

Theorem. *Let G be a pro-p group and N an almost powerfully embedded subgroup, Ω-extendable in G. Then there exist elements $z_i^{(1)}, \dots, z_d^{(1)}$ of $H^2(G/\Phi_G(N))$, $z_1, \dots, z_k$ of $H^2(G)$ such that*

(i) $H^*(G) \cong H^*(G/\Phi_G(N))/(z_1^{(1)}, \dots, z_d^{(1)}) \otimes \mathbb{Z}/p[z_1, \dots, z_k]$;

(ii) $z_1^{(1)}, \dots, z_d^{(1)}$ *classify the extension*

$$(\mathbb{Z}/p)^d \cong \Phi_G(N)/\Phi_G\Phi_G(N) \to G/\Phi_G\Phi_G(N) \to G/\Phi_G(N);$$

(iii) $z_1, \dots, z_k$ *restrict to a basis of* $\beta H^1(\Omega_1(N) \cap N^p)$.

We also give several partial converses, which give group theoretic information when the cohomology has the form given above. Our basic computational tool is the spectral sequence argument of Proposition 2.4. Finally, in section 4, we calculate the Bocksteins up to an error term which vanishes if the extension is itself extendable.

This problem was investigated by Weigel [13], who considered the case when p is odd and $N = G$ and also by Browder and Pakianathan [2], who considered the case when $N = G$ is uniform and also required

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$p \geq 5$. The latter authors also considered the Bocksteins. In all cases the key spectral sequence argument is similar.

1. Regular Sequences

A regular sequence in ring R is a sequence of elements $a_1, \dots, a_n$ such that for each $i = 1, \dots, n$, a_i does not annihilate any non-trivial element of $R/(a_1, \dots, a_{i-1})$.

Recall the definition of the Koszul complex over R. Given a sequence of elements $a_1, \dots, a_n$ in the centre of R, $K = K_R(a_i, \dots, a_n)$ is the the free R-module on certain symbols, which we can take to be monomials $x_{i_1} \dots x_{i_r}$ in the symbols $x_1, \dots, x_n$ such that $i_1 < i_2 < \dots < i_r$ (so no squares occur). It is graded by the degree of the monomial. The differential $d_r : K_r \to K_{r-1}$ is the R-linear map defined by

$$d_r(x_{i_1} \dots x_{i_r}) = \sum_{j=1}^{r} (-1)^{j+1} a_{i_j} x_{i_1} \dots \hat{x}_{i_j} \dots x_{i_r}.$$

If $a_1, \dots, a_n$ is a regular sequence of central elements of R then it is well known that the homology of the associated complex is just $R/(a_1, \dots, a_n)$ in degree 0 and 0 elsewhere.

It is tempting to identify K with the exterior algebra over R on $x_1, \dots, x_n$, and indeed this makes K into a differential graded algebra, but the algebra structure is not part of the definition.

If $2R = 0$ there is a slight variation on this theme, which we will need. Let $J = J_R(a_1, \dots, a_n)$ be the free R-module on all monomials in $x_1, \dots, x_n$, again graded by the degree, and define $d_r : J_r \to J_{r-1}$ by the same formula as before.

We identify J with the polynomial ring $R[x_1, \dots, x_n]$ and it becomes a differential graded algebra. Now we consider J as a module over $R[x_1^2, \dots, x_n^2]$. We do this because $d_2(x_i^2) = 0$ and hence d is linear over $R[x_1^2, \dots, x_n^2]$.

But as an $R[x_1^2, \dots, x_n^2]$-module, $J \cong R[x_1^2, \dots, x_n^2] \otimes K$ (even though this is not compatible with any multiplicative structure) and so has homology $R[x_1^2, \dots, x_n^2] \otimes R/(a_1, \dots, a_n) \cong R/(a_1, \dots, a_n)[x_1^2, \dots, x_n^2]$. We record this as:

Lemma 1.1. *Let $a_1, \dots, a_n$ is a regular sequence of central elements in a ring R. Then:*

1. $H_*(K_R(a_1, \dots, a_n)) \cong R/(a_1, \dots, a_n)$,
2. *If $2R = 0$ then $H_*(J_R(a_1, \dots, a_n)) \cong R/(a_1, \dots, a_n)[x_1^2, \dots, x_n^2]$.* □

We now provide some regular sequences in the mod-p cohomology of a finite p-group. We shall use the following notation. Let $R = C_{p^{r_1}}(e_1) \times C_{p^{r_2}}(e_2) \times \dots \times C_{p^{r_k}}(e_k)$ be an abelian p-group, with $C_{p^{r_i}}(e_i)$ the cyclic group of order p^{r_i} generated by e_i.

For each $1 \leq i \leq k$, let $x_i \in H^1(C_{p^{r_i}}(e_i)) \cong \mathrm{Hom}(C_{p^{r_i}}(e_i), \mathbb{Z}/p)$ correspond to the homomorphism that takes e_i to 1 and let $y_i \in H^2(C_{p^{r_i}}(e_i))$ be a generator. It is well known that

$$H^*(C_{p^{r_i}}) = \begin{cases} \mathbb{Z}/2[x_i] & \text{for } p = 2 \text{ and } r_i = 1, \\ \Lambda[x_i] \otimes \mathbb{Z}/p[y_i] & \text{otherwise.} \end{cases}$$

We identify $H^*(R)$ with $H^*(C_{p^{r_1}}(e_1)) \otimes H^*(C_{p^{r_2}}(e_2)) \otimes \dots \otimes H^*(C_{p^{r_k}}(e_k))$ under the Künneth isomorphism and identify x_i and y_i with their images in $H^*(R)$.

Let $\lambda^*(R)$ denote the subalgebra of $H^*(R)$ generated by the x_i (this need *not* be an exterior algebra if $p = 2$) and set $\lambda(R) = \lambda^2(R)$ for brevity. Let $B(R) \subset H^*(R)$ be the subspace generated by the y_i. Note that $B(R)$ can be characterized as the image of $H^2(R; \mathbb{Z})$ in $H^2(R)$. Also $B(R)$ contains the image of the Bockstein map from $H^1(R)$, but is not equal to it unless R is elementary abelian. We record the following elementary consequences:

Lemma 1.2. (i) $H^2(R) = \begin{cases} \lambda(R) + B(R) & \textit{always} \\ \lambda(R) \oplus B(R) & \textit{if } p > 2 \textit{ or } p = 2 \textit{ and each } r_i \geq 2 \\ \lambda(R) & \textit{for } p = 2 \textit{ and } r_1 = 1. \end{cases}$

(ii) *$y_1, \dots, y_k$ is a regular sequence in $H^*(R)$. Furthermore, given elements $\gamma_1, \dots, \gamma_k$ of $\lambda(R)$, the sequence $y_1 + \gamma_1, \dots, y_k + \gamma_k$ is regular in $H^*(R)$, provided that $p > 2$, or $p = 2$ and $\min(r_1, \dots, r_k) > 1$.* □

Define $H^+(G)$ to be the ideal of $H^*(G)$ consisting of elements of positive degrees.

Proposition 1.3. *Given a central extension of p-groups*

$$0 \xrightarrow{\theta} A \to G \to K \to 1$$

with $A \cong (\mathbb{Z}/p)^m$, we have:

(i) *If A is contained in the Frattini subgroup of G, then* Im $(H^2(G) \xrightarrow{\text{Res}} H^2(A)) \subset B(A)$.
(ii) *If there exist elements $z_1, \dots, z_m$ of $H^2(G)$ and $\gamma_1, \dots, \gamma_m \in \lambda(A)$ such that either $z_1|_A, \dots, z_m|_A$ is linearly independent in $B(A)$, or $p > 2$ and $z_1|_A - \gamma_1, \dots, z_m|_A - \gamma_m$ is linearly independent in $B(A)$, then $z_1, \dots, z_m$ is a regular sequence in $H^*(G)$.*

Proof. (i) was given in [10, Proposition 1.5]. We now prove (ii). Let $i_A : A \to A \times G, i_G : G \to A \times G, f : A \times G \to G$ be defined by $i_A(a) = (a, 1), i_G(g) = (0, g), f(a, g) = \theta(a)g, a \in A, g \in G$. Since A is central, f is a homomorphism of groups. Note that $(f \circ i_A)^*$ (resp. $(f \circ i_G)^*$) is just the restriction (resp. identity) map on cohomology. So, for $1 \leq i \leq m$,

$$f^*(z_i) = z_i|_A \otimes 1 + 1 \otimes z_i \text{ mod } H^+(A) \otimes H^+(G).$$

We now use the argument similar to that of [1, Proof of Theorem 1.1]. The special form of $f^*(z_i)$ induces for $1 \leq i \leq n$ a homomorphism of algebras

$$f_i^* : H^*(G)/(z_1, \dots, z_{i-1}) \to H^*(A)/(z_1|_A, \dots, z_{i-1}|_A) \otimes H^*(G)/(z_1, \dots, z_{i-1}).$$

Let y be a non-trivial element of $H^*(G)/(z_1, \dots, z_{i-1})$. We now show that $z_i y$ is non-trivial, by claiming that $f_i^*(z_i y)$ is non-trivial.

Write $f_i^*(y) = \sum_{s \geq 0} v_s$ and $f_i^*(z_i y) = \sum_{s \geq 0} w_s$ with v_s, w_s in

$$(H^*(A)/(z_1|_A, \dots, z_{i-1}|_A))^s \otimes H^*(G)/(z_1, \dots, z_{i-1}).$$

Then $v_0 = 1 \otimes y$ is non-trivial. Let s_1 be maximal such that v_{s_1} is non-trivial. It follows that, for $s_2 = s_1 + 2$, $w_{s_2} = (z_i|_A \otimes 1) \cdot v_{s_1}$. By Lemma 1.2 (ii), $z_1|_A, \dots, z_i|_A$ is a regular sequence in $H^*(A)$. hence $w_{s_2} \neq 0$. Thus $z_1, \dots, z_m$ is then a regular sequence in $H^*(G)$. □

2. Powerfully embedded subgroups and a cohomological characterization

Let P be a pro-p group and let N be a closed normal subgroup of P . Set $Q = P/N$ and denote by $\pi : P \to Q$ the projection map. Let A be an elementary abelian p-group of rank m and fix a basis $a_1, \dots, a_m$ of A. Regard A as a trivial Q-module, so

$$H^2(Q, A) = \oplus_{i=1}^m H^2(Q, \mathbb{Z}/p \langle a_i \rangle) = H^2(Q)^{\oplus m}.$$

Pick an element $z \in H^2(Q, A)$. For every closed subgroup R of Q, denote by

$$0 \to A \to R_z \xrightarrow{\pi_{R,z}} R \to 1$$

the central extension of groups corresponding to the cohomology class $z|_R = \text{Res}_R^Q(z) \in H^2(R, A)$.

Let $u_1, \dots, u_m$ be the basis of $H^1(A) = \text{Hom}(A, \mathbb{Z}/p)$, dual to that of A. We then have the projection map

$$(u_i)_* : H^2(Q, A) \to H^2(Q), 1 \leq i \leq m,$$

and z can be expressed as $z = (z_1, \dots, z_m)$ with $z_i = (u_i)_*(z)$.

Lemma 2.1. (i) *All the* $\mathrm{Inf}_P^Q(z_i)=0$ *if and only if there exists a surjective homomorphism* $\tau : P \to Q_z$ *satisfying* $\pi_{Q,z}\circ\tau=\pi$.

(ii) *Let* $R = C_{p^{r_1}}(e_1)\times\cdots\times C_{p^{r_k}}(e_k)$ *be an abelian subgroup of* Q. *Then:*

(iia) R_z *is abelian if and only if* $z_\ell|_R \in B(R), 1\le \ell\le m$;

(iib) *let* S *be the maximal elementary abelian subgroup of* R^2. *Then* $(S_z)^p=1$ *if and only if* $z_\ell|_R\in\lambda(R), 1\le\ell\le m$.

Proof. (i) Let $0\to A\to P_z\to P\to 1$ be a group extension classified by the element $z = \mathrm{Inf}(^Q_P(z_1),\dots,\mathrm{Inf}_P^Q(z_m))$. There exists then a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P_z & \xrightarrow{\rho} & P & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\pi} & & \\ 0 & \longrightarrow & A & \longrightarrow & Q_z & \xrightarrow{\pi_{Q,z}} & Q & \longrightarrow & 1. \end{array}$$

If all the $\mathrm{Inf}_P^Q(z)=0$ then ρ is split, by σ say. Hence, by setting $\tau\circ\sigma$, we have $\pi_{Q,z}\circ\tau=\pi$.

Conversely, given τ, we get a splitting $P_z\to A$ by sending x to $\tau(\rho x)\mu(x)^{-1}$.

(iia) This is well known: see [3, IV 3 ex.8] or use the calculations in §4.

(iic) We have

$$\begin{aligned}(S_z)^p=1 &\Leftrightarrow |\langle x\rangle|=p, && \text{for all } x\in S_z\\ &\Leftrightarrow z_\ell|_{\langle y\rangle}=0, && \text{for all } y\in S, 1\le\ell\le m\\ &\Leftrightarrow z_\ell|_R\in\lambda(R), && 1\le\ell\le m.\end{aligned}$$

□

We also have

Lemma 2.2. *Suppose that* R *is an elementary abelian subgroup of* Q. *Then* $(R_z)^p=A$ *if and only if there exists a basis* $b_1,b_2,\dots,b_m,\dots$ *of* R *such that* $\mathrm{Res}^Q_{\langle b_i\rangle}(z_i)\neq 0$, $\mathrm{Res}^Q_{\langle b_j\rangle}(z_i)=0, 1\le i\le m, 1\le j\neq i$.

Proof. If such a basis exists let $\tilde b_i$ be an element of the inverse image of b_i. Then $\tilde b_i$ has order p^2 and $\langle \tilde b_i^p\rangle=\langle e_i\rangle$.

Conversely, if $(R_z)^p=A$ then every element of A is a p-th power of an element of R_z (R_z is powerful). Let $\tilde b_i$ be such that $\tilde b_i^p=e_i, \tilde b_j=1, j>m$, and let b_i be the image of $\tilde b_i$ in R. □

Lemma 2.3. *Consider the extension*

$$0\to A\to Q_z\to Q\to 1$$

discussed above and suppose that there is no non-trivial relation $q_1z_1+\cdots+q_mz_m=0$ *with* $q_i\in H^1(Q)$. *Then the sequence*

$$H^2(Q)\overset{\mathrm{Inf}}{\to}H^2(Q_z)\overset{\mathrm{Res}}{\to}H^2(A)$$

is exact at the middle term.

Proof. The Lyndon-Hochschild-Serre spectral sequence for the extension has E_2 term $H^*(Q)\otimes H^*(A)$. Note that $d_2(u_i)=z_i, 1\le i\le m$ (see the remark after Proposition 2.4). Let $q=q_1u_1+\cdots+q_mu_m$ be an element of $\ker d_2^{1,1}$, with $q_1,\dots,q_m\in H^1(Q)$. Then $0=d_2(q)=-(q_1z_1+\cdots+q_mz_m)$ which implies $q_1=\cdots=q_m=0$, by the hypothesis. Therefore $d_2^{1,1}:E_2^{1,1}\to E_2^{3,0}$ is injective and so $E_3^{1,1}=0$ and thus $E_\infty^{1,1}=0$.

This yields an exact sequence

$$0\to E_\infty^{2,0}\to H^2(Q_z)\to E_\infty^{0,2}\to 0$$

and identifying the edge maps gives the result claimed. □

Proposition 2.4. *Suppose that A is contained in the Frattini subgroup of Q_z and that $z_1, \dots, z_m$ is a regular sequence in $H^*(Q)$. Then the inflation and restriction maps induce an isomorphism of rings*

$$H^*(Q_z) \cong H^*(Q)/(z_1, \dots, z_m) \otimes \mathbb{Z}/p[B(A)]$$

provided that one of the following conditions is satisfied:

(a) $H^2(Q_z) \stackrel{\text{Res}}{\to} B(A)$ *is surjective;*
(b) $\beta(z_i) \in (z_1, \dots, z_m), 1 \leq i \leq m$;
(c) $\ker \text{Inf}^Q_{Q_z} = (z_1, \dots, z_m)$.

Proof. Let $\{E_r, d_r\}$ be the Lyndon-Hochschild-Serre spectral sequence corresponding to the central extension

$$0 \to A \to Q_z \to Q \to 1.$$

Now $E_2 = H^*(Q) \otimes H^*(A)$.

Assume that $p > 2$ and that condition (a) holds. By considering the maps $B(A) = \text{Im}(\text{Res} : H^2(Q_z) \to H^2(A)) \cong E_\infty^{0,2} \subset E_2^{0,2}$ we can find elements $\tilde{y}_1, \dots, \tilde{y}_m \in E_2^{0,2}$ corresponding to $y_1, \dots, y_m \in B(A)$ such that $d_2(\tilde{y}_i) = 0$. Under the isomorphism $E_2^{0,2} \cong H^2(A)$ we find that $\tilde{y}_i = y_i (\text{mod}\, \lambda(A))$. Let $S = \mathbb{Z}/p[\tilde{y}_1, \dots, \tilde{y}_m]$. Then $E_2^{0,*} = S \otimes \Lambda^*(E_2^{0,1})$ and so $E_2 = S \otimes \Lambda^*(E_2^{0,1}) \otimes H^*(Q)$.

But d_2 vanishes on S, by construction, and thus d_2 is $S \otimes H*(Q)$-linear. So $\{E_2, d_2\}$ is just the Koszul complex $K_{S \otimes H^*(Q)}(z_1, \dots, z_m)$ of Lemma 1.1. Thus $E_3 = S \otimes H^*(Q)/(z_1, \dots, z_m) = E_3^{0,*} \otimes E_3^{*,0}$.

Now E_3 is generated as a ring by the $\tilde{y}_i$ and $E_3^{*,0}$ so, by the product structure, $d_3 = 0$ and $E_4 = E_3$. Subsequently all the differentials are 0 for degree reasons and so $E_\infty = E_3$.

Let $\hat{y}_i \in H^2(Q_z)$ have image $\tilde{y}_i \in E_\infty^{0,2}$ and let $\hat{S} = \mathbb{Z}/p[\hat{y}_1 \dots, \hat{y}_m]$. Let $I^* = \text{Im}(\text{Inf} : H^*(Q) \to H^*(Q_z) \cong H^*(Q)/(z_1 \dots, z_m)$. Then we have a natural ring homomorphism $\phi : \hat{S} \otimes I \to H^*(Q_z)$. If we filter $\hat{S} \otimes I$ by $F^p(\hat{S} \otimes I) = \hat{S} \otimes I^{\geq p}$ and $H^*(Q_z)$ in the way that yields E_∞ then ϕ is a homomorphism of filtered rings and induces and isomorphism of the associated graded modules. The filtration is finite in each degree, so ϕ must be an isomorphism. This concludes the proof in this case.

If condition (a) is not available then note that as u_i is transgressive, βu_i survives to E_3 and $d_3(\beta u_i) = -\beta z_i$ (see e.g. [9] and the remark below). Since $E_3^{3,0} = H^3(Q)/(H^3(Q) \cap (z_1, \dots, z_m))$, either of the conditions (b) or (c) implies that $d_3(\beta u_i) = 0$. We set $\tilde{y}_i = \beta(u_i)$ and proceed as before.

If $p = 2$ then E_2 is just the complex $J_{H^*(Q)}(z_1, \dots, z_m)$ of Lemma 2.1(2), so

$$E_3 = H^*(Q)/(z_1, \dots, z_m)[u_1^2, \dots, u_m^2].$$

Set $\tilde{y}_i = u_i^2$ $(= \beta u_i)$. Then condition (a) implies that $d_3(\tilde{y}_i) = 0$ for dimension reasons, and the rest of the proof is just as for p odd. □

Remark. By projecting A onto its cyclic factors and comparing spectral sequences, it is easy to see that $d_2(u_i) = \lambda_i z_i$ for some non-zero $\lambda_i \in \mathbb{Z}/p$. In fact $\lambda_i = \pm 1$, see [6] or [5]. The sign appears to be sensitive to the sign convention used in constructing the double complex for the spectral sequence. Similarly with the formula $d_3(\beta u_i) = \pm \beta d_2(u_i)$.

From now on, assume that N is a closed normal and finitely generated subgroup of a pro-p group G. Recall (see *e.g.* [4, 4]) that N is *powerfully embedded in* G if $[G, N] \subset N^{2p}$, that G is *powerful* if it is powerfully embedded in itself. When $p = 2$, N is *almost powerfully embedded in* G if $[G, N] \subset N^2$ and $[N, N] \subset (N^2)^2$ (which also implies $[N, N] \subset N^4$ (see [7])). For convenience, we define *almost powerfully embedded* as *powerfully embedded* for $p > 2$. As noted in [7], we have the following implications:

N powerfully embedded in $G \Rightarrow N$ almost powerfully embedded in $G \Rightarrow N$ powerful.

We quote two characterizations of powerfully embedded and almost powerfully embedded.

Proposition 2.5.

(i) ([4, Lemma 2.2(iv)]) *If N is not powerfully embedded in G, then there exists a normal subgroup J of G such that*

$$N^{2p}[N,G,G] \subset J \subset N^{2p}[N,G]$$
$$|N^{2p}[N,G] : J| = p;$$

in other words, G/J is given by the central extension

$$1 \to \mathbb{Z}/p \to G/J \to G/N^{2p}[N,G] \to 1$$

such that N/J is abelian of exponent $\leq 2p$, but $N/J \not\subset Z(G/J)$.

(ii) ([7, Proposition 2]) *Set $M = N^p[N,G], K = M^p[M,G], A = N/M, B = M/K$. N is almost powerfully embedded in G if and only if the map induced from the p-power $A \xrightarrow{p} B$ is linear and surjective.* □

We now give some cohomological criteria for N to be (almost) powerfully embedded in G. Set $\widehat{G} = G/N^{2p}[N,G], \widehat{N} = N/N^{2p}[N,G]$. We have

Theorem 2.6. *The following are equivalent:*

(i) *N is powerfully embedded in G;*

(ii) $\mathrm{Res}^{\widehat{G}}_{\widehat{N}}(\ker \mathrm{Inf}^{\widehat{G}}_{G}) \cap \lambda(\widehat{N}) = \{0\}$.

Proof. Let N be powerfully embedded in G (so $N^{2p}[N,G] = N^{2p}$, as $[N,G] \subset N^{2p}$) and let $0 \neq z \in \ker^{\widehat{G}}_{G}$. Consider the central extension

$$0 \to \mathbb{Z}/p \xrightarrow{i} \widehat{G}_z \to \widehat{G} \to 1$$

corresponding to z. By Lemma 2.1 (i), $\widehat{G}_z$ is a homomorphic image of G, via τ. This implies $i(\mathbb{Z}/p) \subset \tau(N^{2p}) = (\widehat{N}_z)^{2p}$. So $(\widehat{N}_z)^{2p} \neq 1$. Therefore, by Lemma 2.1 (iic), $z|_{\widehat{N}} \notin \lambda(\widehat{N})$.

Suppose now that N is not powerfully embedded in G. By Proposition 2.5, there exists a central extension

$$0 \to \mathbb{Z}/p \to \widehat{G}_w \to \widehat{G} \to 1$$

corresponding to an element $0 \neq w \in H^2(\widehat{G})$ such that $\widehat{G}_w$ is a quotient of G, $(\widehat{N}_w)^{2p} = 1, [\widehat{N}_w, \widehat{G}_w] \neq 1$. It follows from Lemma 2.1 that $w \in \mathrm{Ker}\ \mathrm{Inf}^{\tilde{G}}_{G}$ and $w|_{\widehat{N}} \in \lambda(\widehat{N})$. □

We then have the following corollary, of which the case $p > 2$ was given in [12, Theorem 5.1.6].

Corollary 2.7. *The following are equivalent:*

(i) *G is powerful.*

(ii) *p is odd and the map induced from the inflation*

$$H^1(\widehat{G}) \wedge H^1(\widehat{G}) \to H^2(G)$$

is injective, or $p = 2$ and

$$\ker(\mathrm{Inf}^{\widehat{G}}_{G}) \cap \lambda(\widehat{G}) = \{0\}.$$

(iii) *the map induced from the inflation*

$$\lambda(\widehat{G}) \to H^2(G)$$

is injective. □

Proof. Recall that G is powerful if and only if it is powerfully embedded in itself. Thus, by Theorem 2.6, G is powerful if and only if $(\ker \mathrm{Inf}^{\hat{G}}_{G}) \cap \lambda(\hat{N}) = \{0\}$, so (i) $\Leftrightarrow$ (iii). Note that, for $p \neq 2$, $\lambda(\widehat{G}) = H^1(\widehat{G}) \wedge H^1(\widehat{G})$ so (ii) is just a restatement of (iii). □

If $p = 2$, set $\tilde{G} = G/N^2[N,G], \tilde{N} = N/N^2[N,G]$. A characterization of powerfully embedded normal subgroup N of a 2-group G via the inflation $\mathrm{Inf}^{\tilde{G}}_{G}$ can also be obtained, as follows. First we prepare

Lemma 2.8. *Let L be a central subgroup of a 2-group H, and let w be a cohomology class of H/L^4 satisfying $w|_{L^2/L^4} = 0$. Then $w \in \text{Im Inf}_{H/L^4}^{H/L^2}$.*

Proof. We may suppose that $L^2 \neq 1$. Set $A_1 = L/L^2, A_2 = L^2/L^4, H_1 = H/L^2, H_2 = H/L^4, K = H/L$. A_1, A_2 are then vector spaces over $\mathbb{Z}/2$ and we have the central extensions

$$\begin{array}{ccccc} 0 & \to A_1 & \to H_1 & \to K & \to 1, \\ 0 & \to A_2 & \to H_2 & \to H_1 & \to 1 \end{array} .$$

Let $z = (z_1, \dots, z_i)$ be the cohomology class classifying the extension $0 \to A_2 \to H_2 \to H_1 \to 1$. Since L is abelian, by Proposition 1.3 (ii), $z_j|_{A_1} \in B(A_1), 1 \le j \le i$. Then $((A_1)_z)^2 = A_2$ and $z_1|_{A_1}, \dots, z_i|_{A_1}$ are linearly independent in $B(A_1)$, by Lemma 2.2. It follows from Proposition 1.3 (ii) that $z_1, \dots, z_i$ is a regular sequence in $H^*(H_1)$. Hence, by Lemma 2.3, $w|_{A_2} = 0$ implies $w \in$ Im $\text{Inf}_{H_2}^{H_1}$. □

For $a \in \tilde{G}$ (resp. $b \in \widehat{G}$), define $N_a = \langle \tilde{N}, a \rangle$ (resp. $\widehat{N}_b = \langle \widehat{N}, b \rangle$).

Corollary 2.9. *For $p = 2$, the following are equivalent:*

(i) *N is powerfully (resp. almost powerfully) embedded in G;*
(ii) *for every non-zero element μ of* ker (Inf:$H^2(\tilde{G}) \to H^2(G)$), *$\mu|_{N_a} \in B(N_a)$ for every $a \in \tilde{G}$ (resp. $\mu|_{\tilde{N}} \in B(\tilde{N})$).*

Proof. Let N be powerfully (resp. almost powerfully) embedded in G and let $0 \neq z \in \ker_G^{\tilde{G}}$. Consider the central extension

$$0 \to \mathbb{Z}/p \to \tilde{G}_z \to \tilde{G} \to 1$$

corresponding to z. By Lemma 2.1 (i), $\tilde{G}_z$ is a homomorphic image of G. Since $\tilde{N}$ is elementary abelian, $(\tilde{N}_z)^4 = 1$. As $[\tilde{G}_z, \tilde{N}_z] \subset (\tilde{N}_z)^4 = 1$ (resp. $[\tilde{N}_z, \tilde{N}_z] \subset (\tilde{N}_z)^4 = 1$), $\tilde{N}_z$ is central (resp. abelian). Note that $\tilde{N}_z$ is central iff N_a is abelian for every $a \in \tilde{G}$. By Lemma 2.1 (iia), it follows that $z|_{N_a} \in B(N_a)$ for every $a \in \tilde{G}$ (resp. $z|_{\tilde{N}} \in B(\tilde{N})$).

Suppose now that N is not powerfully (resp. almost powerfully) embedded in G. From the proof of Theorem 2.6, there exists an element $w \in H^2(\widehat{G})$ such that $\widehat{G}_w$ is a quotient of G, $(\widehat{N}_w)^4 = 1$ and $[\widehat{N}_w, \widehat{G}_w] \neq 1$ (resp. $[\widehat{N}_w, \widehat{N}_w] \neq 1$). By Lemma 2.1, $[\widehat{N}_w, \widehat{G}_w] \neq 1$ implies that there exists $b \in \widehat{G}$ such that $w|_{\widehat{N}_b} \notin B(\widehat{N}_b)$. So $w|_{\widehat{N}^2} = 0$, and $w|_{\widehat{N}_b} \notin B(\widehat{N}_b)$ for some $b \in \widehat{G}$ (resp. $w|_{\tilde{N}} \notin B(\tilde{N})$). Applying Lemma 2.8 with $H = \widehat{G}, L = \widehat{N}$ yields $w =$ $\text{Inf}_{\widehat{G}}^{\tilde{G}}(t)$ for some $t \in \tilde{H}^*(\tilde{G})$. Let a be the projection of b on $\tilde{G}$, then $w|_{\widehat{N}_b} \notin B(\widehat{N}_b)$ (resp. $w|_{\tilde{N}} \notin B(\tilde{N})$) implies $t|_{N_a} \notin B(N_a)$ (resp. $t|_{\tilde{N}} \notin B(\tilde{N})$).

As $\tilde{G}$ is a quotient of G, it follows from Lemma 2.1 (i) that ker $\text{Inf}_G^{\tilde{G}}$ contains t. □

We then have

Corollary 2.10. *For $p = 2$, the following are equivalent:*

(i) *G is powerful;*
(ii) ker (Inf:$H^2(\tilde{G}) \to H^2(G)$) $\subset B(\tilde{G})$. □

Remark 2.10. Another proof of Theorem 2.6 (for p odd) and Corollary 2.9 (for N almost powerfully embedded in G) also follows from Proposition 2.5 (ii). Indeed, one can deduce that the statement (ii) in Theorem 2.6, together with $\mu|_{\tilde{N}} \in B(\tilde{N})$ for $p = 2$, is equivalent to the condition that the map $A \xrightarrow{p} B$ be linear and surjective.

3. Cohomology of pro-p groups with powerfully embedded subgroups

Let N be a closed normal and finitely generated subgroup of a pro-p group G. Set $\Phi_G(N) = N^p[N, G]$. For $i \ge 1$, define recursively a sequence of closed normal subgroups $\Phi_G^i(N)$ of G as follows:

$$\Phi_G^1(N) = N, \Phi_G^{i+1}(N) = \Phi_G(\Phi_G^i(N)).$$

It is clear that $[\Phi^i_G(N), \Phi^j_G(N)] \subset \Phi^{i+j}_G(N)$ and the p-power map induces a map $\Phi^i_G(N) \xrightarrow{p} \Phi^{i+1}_G(N), i, j \geq 1$; also, $[G, N] \subset \Phi^2_G(N)$.

For $i \geq 0$, set $G_i = G/\Phi^{i+1}_G(N), A_i = A_i(N) = \Phi^i_G(N)/\Phi^{i+1}_G(N)$ (with the convention that $\Phi^0_G(N) = N$). By [4, Proposition 1.16], $G = \varprojlim G_i$, hence $H^*(G) = \varinjlim H^*(G_i)$. Each A_i is a vector space over $\mathbb{Z}/p$ and is central in G_i. We also have successive central extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & 1 \\
0 & \longrightarrow & A_2 & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & 1 \\
 & & & & \cdots & & & & \\
0 & \longrightarrow & A_s & \longrightarrow & G_s & \longrightarrow & G_{s-1} & \longrightarrow & 1 \\
 & & & & \cdots & & & &
\end{array}.$$

Define $\ell = \ell(\Phi^*_G(N))$ to be the largest integer s, if any, satisfying $\Phi^s_G(N) \neq 1$, or $\ell(\Phi^*_G(N)) = \infty$ otherwise (so $G = G_\ell$ if $\ell < \infty$). Denote by $\phi_{i,j} : G_j \to G_i$ and $\phi_i : G \to G_i$ the projection maps, $j \geq i$. For $1 \leq i$, let n_i be the dimension of A_i and let $\kappa_{i-1} = (z^{(i-1)}_1, \ldots, z^{(i-1)}_{n_i}) \in H^2(G_{i-1})^{\oplus n_i}$ be the cohomology class corresponding to the central extension

$$0 \to A_i \to G_i \to G_{i-1} \to 1.$$

We first have

Lemma 3.1. *For every $1 \leq i \leq \ell - 1$, no non-zero linear combination of the $z^{(i)}_j$'s belongs to* Im $\phi^*_{i-1,i}$.

Proof. Given $1 \leq i \leq \ell - 1$, assume that z is a non-zero linear combination of the $z^{(i)}_j$'s with $z \in$ Im $\phi^*_{i-1,i}$. Without loss of generality, we may assume that $z = \sum_{j=1}^{n_{i+1}} \lambda_j z^{(i)}_j$ with $\lambda_1 \neq 0$. There exists then a basis $(e_1, \ldots, e_{n_{i+1}})$ of A_{i+1} such that $\kappa_i = (z, z^{(i)}_2, \ldots, z^{i)}_{n_{i+1}}) \in H^2(G_i)^{\oplus n_{i+1}}$.

Since $z \in$ Im $\phi^*_{i-1,i}$, its restriction to A_i is 0, by Lemma 2.1. Thus $\langle e_1 \rangle$ is a factor of $\Phi^i_G(N)/\Phi^{i+1}_G(N)$, a contradiction. □

Proposition 3.2. *The following are equivalent:*

(i) *N is almost powerfully embedded in G;*
(ii) *$((A_1)_{\kappa_1})^p = A_2$; also, $0 \neq z^{(1)}_j|_{A_1} \in B(A_1), 1 \leq j \leq n_2$, for $p = 2$.*

Proof. Suppose that N is almost powerfully embedded in G. By Lemma 3.1, $z^{(1)}_j \neq 0, 1 \leq j \leq n_2$. It follows from Theorem 2.6 (for $p > 2$) and Corollary 2.9 (for $p = 2$) that, for any non-zero linear combination z of the $z^{(1)}_j$'s, $z|_{A_1} \notin \lambda(A_1)$ for $p > 2$ and $z|_{A_1} \in B(A_1)$ for $p = 2$. There exists then a basis $e_1, \ldots, e_{n_1}$ of A_1 such that $z^{(1)}_j|_{\langle e_j \rangle} \neq 0, z^{(1)}_j|_{\langle e_i \rangle} = 0, 1 \leq i \neq j \leq n_2$. Hence, by Lemma 2.2, $((A_1)_{\kappa_1})^p = A_2$.

Conversely, suppose that (ii) holds. If $p = 2$, (i) also holds, by Corollary 2.9. Assume that $p > 2$ and N is not powerfully embedded in G. From the proof of Theorem 2.6, there exists a non-zero linear combination z of the $z^{(1)}_j$'s such that $z|_{A_1} \in \lambda(A_1)$. Arguing as in the proof of Lemma 3.1, there exists an element of A_2 not belonging to $((A_1)_{\kappa_1})^p$, a contradiction. □

For $i < j \leq \ell$ and for every element $\xi \in H^*(G_i)$, consider ξ as an element of $H^*(G_j)$ via the inflation map $\phi^*_{i,j}$. We then have

Theorem 3.3. *Let N be almost powerfully embedded in a p-group G. We have:*

(i) *for $1 \leq i \leq \ell-1$, $((A_i)_{\kappa_i})^p = A_{i+1}$, $n_1 \geq n_2 \cdots \geq n_\ell$, and the sequence $H^*(G_i) \overset{\phi^*_{i,i+1}}{\to} H^*(G_{i+1}) \overset{\text{Res}}{\to} B(A_{i+1})$ is exact;*

(ii) *for $1 \leq i \leq \ell-1$, $z_1^{(i)}, \dots, z_{n_{i+1}}^{(i)}$ is a regular sequence in $H^*(G_i)$;*

(iii) *if $n_1 = \cdots = n_r = d, 2 \leq r \leq \ell$, then*

(iiia) *the map $H^2(G_i) \overset{\text{Res}}{\to} B(A_i)$ is surjective, for $1 \leq i \leq r-1$;*

(iiib) *for $2 \leq i \leq r-1$,*

$$\begin{aligned} H^*(G_i) &= H^*(G_{i-1})/(z_1^{(i-1)}, \dots, z_d^{(i-1)}) \otimes \mathbb{Z}/p[z_1^{(i)}, \dots, z_d^{(i)}] \\ &= H^*(G_1)/(z_1^{(1)}, \dots, z_d^{(1)}) \otimes \mathbb{Z}/p[z_1^{(i)}, \dots, z_d^{(i)}]. \end{aligned}$$

Proof. It follows from Proposition 3.2 that $((A_1)_{\kappa_1})^p = A_2$. As N is powerful, we also have $((A_i)_{\kappa_i})^p = A_{i+1}, i \geq 2$, and $n_1 \geq n_2 \cdots \geq n_\ell$ (see *e.g.* [4, Theorem 2.7]). By Lemma 2.2 (and Proposition 3.2 (ii) for $p = 2$), $z_1^{(i)}|_{A_i}, \dots, z_{n_{i+1}}^{(i)}|_{A_i}$ satisfies the assumption of Proposition 1.3 (ii). So $z_1^{(i)}, \dots, z_{n_{i+1}}^{(i)}$ is a regular sequence in $H^*(G_i)$. Therefore $H^*(G_i) \overset{\phi^*_{i,i+1}}{\to} H^*(G_{i+1}) \overset{\text{Res}}{\to} B(A_{i+1})$ is exact by Lemma 2.3. (i), (ii) are then proved. (iii) is straightforward from (i), (ii), Lemma 2.3 and Proposition 2.4. □

Remark 3.4. Theorem 3.3 also holds if G is a pro-p group.

Definition. Let N be almost powerfully embedded in G and let $d = d(N)$ be the minimal number of generators of N. Set $\ell = \ell(\Phi_G^*(N)), \Omega_1(N) = \overline{\langle x \in N | x^p = 1\rangle}$.

- N is said to be *uniform* if $n_i = \dim_{\mathbb{Z}/p} A_i(N) = d$ for every $1 \leq i \leq \ell$. In such a case, N is also said to be *almost uniformly embedded in* G.
- N is called Ω-*extendable in* G if $\Omega_1(N)$ is central in G and there exists a central extension $0 \to Z \to \Gamma \to G \to 1$ classified by $z \in H^2(G, Z)$, with $Z = (\mathbb{Z}/p)^r, r =$ rank $(\Omega_1(N))$ and $Z = (\Omega_1(N)_z)^p$ and Γ is called an $\Omega_1(N)$-*extension* of G. G is Ω-*extendable* if it is Ω-extendable in itself.

Remark 3.5. 1. If N is almost uniformly embedded in G and if $\Omega = \Omega_1(N) \subset Z(G) \cap \Phi(N)$ with $d(\Omega) = d$, the Ω-extendable property of N in G is equivalent to the existence of elements $u_1, \dots, u_d$ of $H^2(G)$ satisfying $B(\Omega) = \langle u_1|_\Omega, \dots, u_d|_\Omega\rangle$, by Lemma 2.2. Let $(w_1, \dots, w_d) \in H^2(G/\Omega)^{\oplus d}$ be any cohomology class classified the extension $0 \to (\mathbb{Z}/p)^d \to G \to G/\Omega \to 1$, and let $\{E_r, d_r\}$ be the Lyndon-Hochschild-Serre spectral sequence corresponding to this extension. It follows from the proof of Proposition 2.4 that $B(\Omega) \subset \text{Im Res}_\Omega^G$ if and only if $d_3 = 0$, or, equivalently, $\beta(w_i) \in (w_1, \dots, w_d), 1 \leq i \leq d$. So the Ω-extendibility of N in G is equivalent to the condition that $\beta(w_i) \in (w_1, \dots, w_d), 1 \leq i \leq d$.

2. If N is almost powerfully embedded in the pro-p group G and $T = T(N)$ is the set of elements of finite order of N, then T is a normal subgroup of G (see [4], Theorem 4.20) and N/T is almost uniformly embedded in G/T. In such a case, $\Omega_1(N) = \Omega_1(T)$. Furthermore, as $N^{p^k} = \{g^{p^k} | g \in N\}$ ([4], Corollary 3.5), it follows that $N^{2p} \cap T = T^{2p}$ ($N^2 \cap T = T^2, (N^2)^2 \cap T = (T^2)^2$ for $p = 2$). So T is also almost powerfully embedded in G.

Examples. 1. If G is an pro-p group of finite rank then, by [4, Corollary 4.3], G contains a characteristic, open, uniform subgroup. If G is powerful then, by [4, Theorem 2.7], there exists a term $P_i(G)$ in the lower p-series of G which is uniform and powerfully embedded in G.

2. Let $D_{2^n} = \langle a_n, b_n | a_n^{2^{n-1}} = b^2 = 1, (a_n b_n)^2 = 1\rangle$ be the dihedral group of order 2^n. We then have a projection map $\varphi_{ij} : D_{2^j} \to D_{2^i}$ which maps a_j (resp. b_j) to a_i (resp. b_i), for $j \geq i$. Define $\mathbb{D} = \varprojlim D_{2^n}$. Then $\mathbb{D} = \langle a, b | b^2 = 1, (ab)^2 = 1\rangle$ so that the projection map $\varphi_i : \mathbb{D} \to D_{2^i}$ maps a (resp. b) to a_i (resp. b_i), $i \geq 1$. It is clear that $\langle a_n\rangle$ is almost powerfully embedded, but not powerfully embedded, and Ω-extendable in D_{2^n}. Also, $\langle a\rangle$ is almost uniformly embedded in $\mathbb{D}$.

However, any cyclic subgroup of index 2 of a quaternion group Q of order 8 is powerfully embedded in Q, but not Ω-extendable in Q. Also, if M is the extraspecial p-group of order p^3 and exponent p^2 with $p > 2$, then M is powerful, the center of M is powerfully embedded but not Ω-extendable in M.

3. Let n be an integer. For every $k \geq 1$, define $\Gamma_n(k) = \{A \in GL_n(\mathbb{Z}/p^{k+1}) | A = I_n \mod (p^k)\}$. Then $\Gamma_n(k)$ with $k \geq 1$ for $p > 2$, $k \geq 2$ for $p = 2$ is uniform and Ω-extendable.

Corollary 3.6. *If N is almost uniformly embedded in a p-group G with $d = d(N)$ and $\ell = \ell(\Phi_G^*(N)) \geq 3$, then, for $2 \leq j \leq \ell - 1$,*

$$\begin{aligned} H^*(G_j) &= H^*(G_{j-1})/(z_1^{(j-1)}, \dots, z_d^{(j-1)}) \otimes \mathbb{Z}/p[z_1^{(j)}, \dots, z_d^{(j)}] \\ &= H^*(G_1)/(z_1^{(1)}, \dots, z_d^{(1)}) \otimes \mathbb{Z}/p[z_1^{(j)}, \dots, z_d^{(j)}]. \end{aligned}$$

Furthermore, if N is also Ω-extendable in G, there exist $z_1, \dots, z_d$ in $H^2(G)$ such that

$$H^*(G) = H^*(G_1)/(z_1^{(1)}, \dots, z_d^{(1)}) \otimes \mathbb{Z}/p[z_1, \dots, z_d]. \qquad \square$$

Proof. By the first assumption, $n_1 = \dots = n_\ell = d$. The first equalities follows then from Theorem 3.3.

Suppose that N is also Ω-extendable in G. By Remark 3.5.1, there exist $z_1, \dots, z_d$ in $H^2(G)$ such that $z_1|_{\Omega_1(N)}, \dots, z_d|_{\Omega_1(N)}$ is a basis of $B(\Omega_1(N))$. Consider the central extension $1 \to (\mathbb{Z}/p)^d \to \Gamma \to G \to 1$ corresponding to $z = (z_1, \dots, z_d)$. It follows that N_z is, in turn, almost uniformly embedded in Γ. The last equality follows then from the first part of the Corollary. $\square$

The above corollary can be generalized to the case where G is almost powerfully embedded in G (see Theorem 3.13 below).

Lemma 3.7. *Suppose that N is almost powerfully embedded in a p-group G and $n_i > n_{i+1}$ with a given i.*

(i) *Let $a_1, \dots, a_j, \dots, a_{n_i}$ be elements of $\Phi_G^i(N) \backslash \Phi_G^{i+1}(N)$ satisfying:*
  (ia) *$\phi_i^*(a_1), \dots, \phi_i^*(a_{n_i})$ is a basis of A_i;*
  (ib) *$\mathrm{ord}(a_1) = \dots = \mathrm{ord}(a_j) = p <$ $\mathrm{ord}(a_k)$, for $k > j$.*
  Then there exist elements $b_{j+1}, \dots, b_{n_i}$ of $\Phi_G^i(N)$ such that, for $k > j$, $\mathrm{ord}(b_k) = p, \phi_i^*(b_k) = a_k$. Precisely, for every $1 \neq x \in \Phi_G^i(N)$ satisfying $1 \neq x^p \in \Phi_G^j(N)$ with $j > i + 1$, there exists $y \in \Phi_G^{j-1}(N)$ such that $(xy)^p = 1$;

(ii) *For every element a of order p of $\Phi_G^i(N) \backslash \Phi_G^{i+1}(N)$, there exists $\xi \in H^+(G_i)$ satisfying $\xi|_{\langle \phi_i(a) \rangle} \neq 0$; hence, as an element of $H^+(G)$, $\xi|_{\langle a \rangle} \neq 0$, ξ is nilpotent in $H^+(G)$ and $\xi \notin$ Im ϕ_{i-1}^*.*

Proof. (i) follows from [4, Proof of Theorem 4.5]. Set $b = \phi_i(a)$. Then $b \neq 1$. Pick an element w of $B(A_i)$ satisfying $w|_{\langle b \rangle} \neq 0$ and set $\xi = \mathcal{N}_{A_i \to G_i}(w)$ with $\mathcal{N}_{A_i \to G_i}$ the Evens norm map. So $\xi|_{\langle b \rangle}$ is not nilpotent. From the commutative diagram

$$\begin{array}{ccc} H^*(G_i) & \xrightarrow{\phi_i^*} & H^*(G) \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ H^*(\langle b \rangle) & \xrightarrow{\mathrm{Inf}} & H^*(\langle a \rangle) \end{array},$$

since $H^*(\langle b \rangle) \overset{\mathrm{Inf}}{\to} H^*(\langle a \rangle)$ is an isomorphism, it follows that $\xi|_{\langle a \rangle} \neq 0$. So ξ is not nilpotent in $H^*(G)$ and $\xi \notin$ Im ϕ_{i-1}^*. $\square$

We now have a sufficient and necessary condition for an (almost) powerfully embedded subgroup in G to be also (almost) uniformly embedded, as follows.

Corollary 3.8. *Let N be (almost) powerfully embedded in a p-group G with $d(N) = d$ and $\ell(\Phi_G^*(N)) = \ell \geq 2$. The following are equivalent:*

(i) *N is (almost) uniformly embedded in G;*
(ii) *$n_2 = d$ and* Im ϕ_1^* = Im $\phi_{\ell-1}^*$.

Proof. (i) $\Rightarrow$ (ii) follows from Corollary 3.6. Suppose that (ii) holds. If N is not uniform, then, by Lemma 3.7, there exist $j \geq 2, \xi \in H^*(G_j)$ such that $\xi|_{A_j} \neq 0$ and ξ is not nilpotent in $H^*(G)$. This contradicts the fact that $\text{Im } \phi_1^* = \text{ Im } \phi_{\ell-1}^*$ implies $0 = \xi \in H^*(G)$. So N is uniform. □

The next result gives a necessary and sufficient condition for an (almost) powerfully embedded subgroup to be also (almost) uniformly embedded and Ω-extendable in G.

Corollary 3.9. *Let N be (almost) powerfully embedded in a p-group G with $d(N) = d$ and $\ell(\Phi_G^*(N)) = \ell \geq 2$. The following are equivalent:*

(i) *N is (almost) uniformly embedded and Ω-extendable in G;*

(ii) *$n_\ell = d$ and there exist $z_1, \ldots, z_d$ in $H^2(G)$ such that*

$$H^*(G) = H^*(G_1)/(z_1^{(1)}, \ldots, z_d^{(1)}) \otimes \mathbb{Z}/p[z_1, \ldots, z_d]$$

and $z_1|_{A_\ell}, \ldots, z_d|_{A_\ell}$ is a basis of $B(A_\ell)$;

(iii) *$n_2 = d$, $\text{Im } \phi_1^* = \text{Im } \phi_{\ell-1}^*$ and there exist $z_1, \ldots, z_d \in H^2(G)$ such that $z_1|_{A_\ell}, \ldots, z_d|_{A_\ell}$ is a basis of $B(A_\ell)$; .*

Proof. For convenience, write $H^*(G_1)/(z_1^{(1)}, \ldots, z_d^{(1)}) = H$. The implication (i) $\Rightarrow$ (ii) follows from Corollary 3.6. Also, by Corollary 3.6, (i) implies $n_2 = d$ (as N is uniform), $\ker \phi_1^* = (z_1^{(1)}, \ldots, z_d^{(1)})$, $\ker \phi_{\ell-1}^* = (z_1^{(\ell-1)}, \ldots, z_d^{(\ell-1)})$, hence

$$\text{Im } \phi_1^* = \text{ Im } \phi_{\ell-1}^* = H^*(G_1)/(z_1^{(1)}, \ldots, z_d^{(1)}).$$

So (i)$\Rightarrow$ (iii).

Suppose that (ii) holds. By Theorem 3.3 (i), $n_1 \geq n_2 \cdots \geq n_\ell$, so $n_1 = \cdots = n_\ell = d$, hence N is (almost) uniformly embedded in G. As $B(A_\ell) \subset \text{ Im Res}_{A_\ell}^G$, it follows from Remark 3.5.1 that N is Ω-extendable in G. So (i) $\Leftrightarrow$ (ii).

Suppose that (iii) holds. By Corollary 3.8, N is (almost) uniformly embedded in G. The existence of the z_i's shows that N is Ω-extendable in G. □

The following is straightforward from Corollaries 3.6, 3.8 and 3.9.

Theorem 3.10. *Let N be a normal subgroup of a p-group G with $d(N) = d$. The following are equivalent:*

(i) *N is uniformly (resp. almost uniformly, for $p = 2$) embedded and Ω-extendable in G with $\ell(\Phi_G^*(N)) \geq 2$;*

(ii) *$n_\ell = d$ and there exists a system of linearly independent elements $\psi_1, \ldots, \psi_d$ of $H^2(G_1)$ satisfying:*

 (a) *the inflation and restriction maps induces an isomorphism*

$$H^*(G) \cong H^*(G_1)/(\psi_1, \ldots, \psi_d) \otimes \mathbb{Z}/p[B(A_\ell)],$$

 and

 (b) *for every non-zero linear combination ψ of the ψ_i's, either $p > 2$ and $\psi|_{A_1} \notin \lambda(A_1)$, or $p = 2$ and $\psi|_{\langle A_1, a\rangle} \in B(\langle A_1, a\rangle)$ for every $a \in G_1$ (resp. $\psi|_{A_1} \in B(A_1)$).* □

We now have the following corollary, of which the last assertion is a celebrated theorem of Lazard ([8]).

Corollary 3.11. *Let G be an infinite pro-p group and let N be a closed normal subgroup of G with $d(N) = d < \infty$. The following are equivalent:*

(i) *N is uniformly (resp. almost uniformly, for $p = 2$) embedded in G;*

(ii) *There exists a system of linearly independent elements $\psi_1, \ldots, \psi_d$ of $H^2(G_1)$ satisfying:*

 (a) *the inflation map induces an isomorphism*

$$H^*(G) \cong H^*(G_1)/(\psi_1, \ldots, \psi_d),$$

 and

 (b) *for every non-zero linear combination ψ of the ψ_i's, either $p > 2$ and $\psi|_{A_1} \notin \lambda(A_1)$, or $p = 2$ and $\psi|_{\langle A_1, a\rangle} \in B(\langle A_1, a\rangle)$ for every $a \in G_1$ (resp. $\psi|_{A_1} \in B(A_1)$).*

In particular, if G is any finitely generated pro-p group, the inflation map $\mathrm{Inf}_G^{G/G^p[G,G]}$ *induces an isomorphism*

$$\Lambda_*((G/G^p[G,G])^*) \cong H^*(G)$$

if and only if G is uniform.

Proof. Suppose that (i) holds. It follows that, for every $i \geq 2$, $A_i(N)$ is uniformly (resp. almost uniformly, for $p = 2$) embedded and Ω-extendable in G_i. By Theorem 3.10, for $i \geq 2$,

$$H^*(G_i) \cong H^*(G_1)/(\psi_1, \ldots, \psi_d) \otimes \mathbb{Z}/p[B(A_i)],$$

and the condition (ii b) of Theorem 3.10 is satisfied. Since $H^*(G) = \varinjlim H^*(G_i)$, (ii) also holds.

Conversely, assume that (ii) holds. It follows that $\ker \phi_1^* = (\psi_1, \ldots, \psi_d)$. By Theorem 2.6 and Corollary 2.9, N is powerfully (resp. almost powerfully) embedded in G. If $n_1 > n_2$, Lemma 3.7 shows the existence of a non-nilpotent element of $H^+(G)$ not belonging to $\mathrm{Im}\ \phi_1^*$, a contradiction. Hence $n_2 = n_1$. For $i \geq 2$, as $\mathrm{Im}\ \phi_1^* = \mathrm{Im}\ \phi_i^*$, $A_i(N)$ is uniformly (resp. almost uniformly, for $p = 2$) embedded and Ω-extendable in G_i, by Corollary 3.9. So N is uniformly (resp. almost uniformly, for $p = 2$) embedded in G. □

Lemma 3.12. *Let N be almost powerfully embedded in a p-group G. For $k \geq 1, a \in G, b \in \Phi_G^i(N)$,*

(i) $[a, b^{p^k}] = [a, b]^{p^k} \bmod \Phi_G^{i+k+2}(N)$;
(ii) *if* $[a, b] = xy$ *with* $x \in Z(G), y \in \Phi_G^j(N)$, *then* $[a, b^p] = [a, b]^p \bmod \Phi_G^{j+2}(N)$;
(iii) *if* $\Omega = \Omega_1(N)$ *is central and* $\mathrm{ord}(b) = p^2$, then $[a, b] \in \Omega$.

Proof. (i) We prove by induction on k. Suppose that $k = 1$. Applying [4, Chapter 4, Exercise 6] yields $[a, b^p] = [a, b]^p \bmod \Phi_G^j(N)$ with $j \geq i + 3$ if p is odd or $i > 1$. If $p = 2$ and $i = 1$, as $[a, b] \in \Phi_G^2(N)$ and N is powerful, $[[a, b], b] \in \Phi_G^4(N)$ (see [4, Chapter 2, Exercise 4]). So $[a, b^2] = [a, b]^2[[a, b], b] = [a, b]^2 \bmod \Phi_G^4(N)$.

Assume that the equality holds for $k \geq 1$. So

$$\begin{aligned}
[a, b]^{p^{k+1}} &= ([a, b]^{p^k})^p \\
&= ([a, b^{p^k}] \bmod \Phi_G^{i+k+2})^p \\
&= [a, b^{p^k}]^p \bmod \Phi_G^{i+k+3} \\
&= [a, b^{p^{k+1}}] \bmod \Phi_G^{i+k+3}.
\end{aligned}$$

(ii) Without loss of generality, we may assume that $\Phi_G^{j+2}(N) = 1$. Therefore, for every $g \in G$, $[[a, b], g] \in \Phi_G^{j+1}(N) \subset Z(G)$; in particular, $[[a, b], g] = 1$ if $p = 2$ and $g \in N$ (see [4, Chapter 2, Exercise 4]). We then have

$$\begin{aligned}
[a, b^p] &= [a, b][a, b]^b \ldots [a, b]^{b^{p-1}} \\
&= \prod_{m=0}^{p-1} [a, b][[a, b], b^m] \\
&= [a, b]^p \prod_{m=0}^{p-1} [[a, b], b^m] \qquad \text{since } [[a, b], b^m] \text{ is central} \\
&= [a, b]^p [[a, b], b]^{p(p-1)/2} \\
&= [a, b]^p \bmod \Phi_G^{j+2}(N).
\end{aligned}$$

(iii) Let j be the smallest integer with $[a, b] \in \Phi_G^j(N)$. As $(b^p)^p = 1$, b^p is central. By (ii), $1 = [a, b^p] = [a, b]^p \bmod \Phi_G^{j+2}(N)$. By Lemma 3.7 (i), $[a, b]$ is of form

$$[a, b] = cx$$

with $c \in \Phi_G^j(N) \cap \Omega, x \in \Phi_G^{j+1}(N)$. Applying (ii) yields

$$\begin{aligned} x^p = (cx)^p = [a,b]^p &= [a,b^p] \bmod \Phi_G^{j+3}(N) && \text{by (ii)} \\ &= 1 \bmod \Phi_G^{j+3}(N) && \text{since } b^p \text{ is central.} \end{aligned}$$

Therefore $x^p \in \Phi_G^{j+3}(N)$. By Lemma 3.7 (i), x is of form $x = dz$ with $d \in \Phi_G^{j+1}(N) \cap \Omega, z \in \Phi_G^{j+2}(N)$, so $[a,b] = cdz$. Hence, by induction, it follows that $[a,b] \in \Omega$. □

Suppose that N is almost powerfully embedded in a pro-p group G and $\Omega = \Omega_1(N)$ is central with $\Omega \not\subset \Phi(G)$. Write $\Omega = M \times L$ with $M = \Omega \cap \Phi(G)$. L is then a direct factor of G. Therefore, there exist subgroups K, N of G such that

(i) $G = K \times L, N = N_1 \times L, \Omega_1(N_1) = M$;
(ii) N_1 is almost powerfully embedded in K;
(iii) N is Ω-extendable in G implies that N_1 is Ω-extendable in K.

Furthermore, L is elementary abelian and $H^*(G) = H^*(K) \otimes H^*(L)$. Hence, to consider the cohomology of pro-p groups G having an almost powerfully embedded and Ω-extendable subgroups N in G, we may suppose that $\Omega = \Omega_1(N)$ is central and is contained in the Frattini subgroup of G. We have

Theorem 3.13. *Let N be almost powerfully embedded in a pro-p group G with $d(N^p) = d$. Set $\Omega = \Omega_1(N), \Omega' = \Omega \cap N^p, k = d(\Omega), k' = d(\Omega')$. The following are equivalent:*

(i) N is Ω-extendable in G;
(ii) Ω is abelian and there exist $z_1, \dots, z_k$ in $H^2(G)$ such that $z_1|_\Omega, \dots, z_k|_\Omega$ (resp. $z_1|_{\Omega'}, \dots, z_{k'}|_{\Omega'}$) is a basis of $B(\Omega)$ (resp. $B(\Omega')$) and

$$H^*(G) = H^*(G_1)/(z_1^{(1)}, \dots, z_d^{(1)}) \otimes \mathbb{Z}/p[z_1, \dots, z_{k'}].$$

Proof.
Let T be the set of elements of finite order of N. By Remark 3.5, T is a closed normal subgroup which is almost powerfully embedded in G. So $\Omega = \Omega_1(T)$ and, by Lemma 3.7, $d(T) = k$.

We prove (i) $\Rightarrow$ (ii). Suppose that N is Ω-extendable in G. Consider the following cases:

- $N = T$ (so $d = k'$). Set $\ell = \ell(\Phi_G^*(N))$. We argue by induction on ℓ. If $\ell = 2$, then, by Lemma 2.3 and Remark 3.5,

$$H^*(G) = H^*(G_1)/(z_1^{(1)}, \dots, z_d^{(1)}) \otimes \mathbb{Z}/p[z_1, \dots, z_d]$$

with $z_1|_{\Omega'}, \dots, z_d|_{\Omega'}$ a basis of $B(\Omega')$; furthermore, by Remark 3.5, there exist $z_{d+1}, \dots, z_k$ in $H^2(G)$ such that $z_1|_\Omega, \dots, z_k|_\Omega$ is a basis of $B(\Omega)$. Assume that (i) holds for $\ell - 1$ with $\ell > 2$. As discussed above, we also assume that $\Omega \subset \Phi(G)$. Let Ω'' be a complement of Ω' in Ω. It follows that $\Omega' = \Omega/\Omega'' = \Omega_1(N/\Omega'')$. Hence N/Ω'' is Ω-extendable in $Q = G/\Omega''$ $(*)$. By Lemma 3.12 (iii) (see also [13, Proposition 5.4 (a)]), $N' = N/\Omega$ is almost powerfully embedded and Ω-extendable in $G' = G/\Omega$; furthermore, $d(N') = d(\Omega_1(N')) = d$. Set $e = d((N')^p) = d(\Omega_1(N') \cap (N')^p)$ and $R = G'/(N')^p$. Since $\ell(\Phi_{G'}^*(N')) = \ell - 1$, it follows from the inductive hypothesis that there exist $u_1, \dots, u_e$ in $H^2(R)$ and $s_1, \dots, s_e, u_{e+1}, \dots, u_d$ in $H^2(G')$ such that, via the restriction maps, the images of $s_1, \dots, s_e, u_{e+1}, \dots, u_d$ (resp. $s_1, \dots, s_e$) form a basis of $B(\Omega_1(N'))$ (resp. $B(\Omega_1((N')^p))$ and

$$H^*(G') = H^*(R)/(u_1, \dots, u_e) \otimes \mathbb{Z}/p[s_1, \dots, s_e].$$

By $(*)$, Lemma 2.3 and Remark 3.5, there exist $z_1, \dots, z_d$ in $H^2(Q)$ such that $z_1|_{\Omega'}, \dots, z_d|_{\Omega'}$ form a basis of $B(\Omega')$ and

$$\begin{aligned} H^*(Q) &= H^*(G')/(s_1, \dots, s_e, u_{e+1}, \dots, u_d) \otimes \mathbb{Z}/p[z_1, \dots, z_d] \\ &= H^*(R)/(u_1, \dots, u_d) \otimes \mathbb{Z}/p[z_1, \dots, z_d]. \end{aligned}$$

As $\Omega'' \cap N^p = 1$, ker $(H^2(Q) \overset{\text{Inf}}{\to} H^2(G)) \subset$ Im Inf_Q^R. There exist then $u_{d+1}, \dots, u_k$ in $H^2(R)$ and $z_{d+1}, \dots, z_k$ in $H^2(G)$ such that $z_{d+1}|_{\Omega''}, \dots, z_k|_{\Omega''}$ form a basis of $B(\Omega'')$ and

$$H^*(G) = H^*(R)/(u_1, \dots, u_k) \otimes \mathbb{Z}/p[z_{d+1}, \dots, z_k] \otimes \mathbb{Z}/p[z_1, \dots, z_d].$$

Since $H^*(G_1) = H^*(R)/(u_{d+1},\dots,u_k)\otimes\mathbb{Z}/p[z_{d+1},\dots,z_k]$ and $z_1^{(i)} = \operatorname{Inf}_{G_1}^R(u_i), 1\le i\le d$, it follows that

$$H^*(G) = H^*(G_1)/(z_1^{(1)},\dots,z_1^{(d)})\otimes\mathbb{Z}/p[z_1,\dots,z_d];$$

- $T \subsetneq N$. By Remark 3.5.2, T is almost powerfully embedded in G. It follows from the above case that there exist $\varphi_1,\dots,\varphi_\ell \in H^2(G/T^p), z_1,\dots,z_{k'},\dots,z_k \in H^2(G)$ such that

$$H^*(G) = H^*(G/T^p)/(\varphi_1,\dots,\varphi_\ell)\otimes\mathbb{Z}/p[z_1,\dots,z_{k'}]$$

with $z_1|_{\Omega'},\dots,z_{k'}|_{\Omega'}$ (resp. $z_1|_\Omega,\dots,z_k|_\Omega$) a basis of $B(\Omega')$ (resp. $B(\Omega)$).

Set $K = G/T^p$ and $M = N/T^p$. By [4, Theorem 4.20 and its proof], M is almost powerfully embedded in K. Denote by w the element of $H^2(K_1, A_2(M))$ classifying the extension $0\to A_2(M)\to K_2\to K_1\to 1$ and set $m = \dim_{\mathbb{Z}/p} A_2(M)$. Note that $A_1(M)$ is elementary abelian. There exists then an elementary abelian subgroup B of rank m of $A_1(M)$ such that $(B_w)^p = A_2(M)$. Let C be the preimage of B via the projection map $K\to K_1 = K/C^p$. It follows that C is almost uniformly embedded in K. Hence, by Corollary 3.9, there exist $\psi_1,\dots,\psi_m\in H^2(K/C^p)$ such that

$$H^*(K) = H^*(K/C^p)/(\psi_1,\dots,\psi_m).$$

Note that K/C^p is nothing but G_1. Each φ_i is then represented by an element ψ_{m+i} of $H^2(G_1)$. Therefore

$$H^*(G) = H^*(G_1)/(\psi_1,\dots,\psi_{m+\ell})\otimes\mathbb{Z}/p[z_1,\dots,z_{k'}].$$

As $(\psi_1,\dots,\psi_{m+\ell})$ is nothing but Ker ϕ_1^*, it coincides with $(z_1^{(d)},\dots,z_d^{(1)})$. So (i) $\Rightarrow$ (ii).

Conversely, suppose that (ii) holds. We need prove that Ω is central. Let G_z be given by the central extension

$$0\to(\mathbb{Z}/p)^k \xrightarrow{i} G_z\to G\to 1$$

classified by $z = (z_1,\dots,z_k)\in H^2(G)^{\oplus k}$. It follows that $\Omega_1(G_z) = i((\mathbb{Z}/p)^k)$ is central in G_z. Set $\Omega_2 = \langle g\in G_z | g^{p^2} = 1\rangle$. Then $\Omega = \Omega_2/\Omega_1(G_z)$. By Lemma 3.12 (iii), Ω is central. □

We have the following corollary, of which the case $p\ge 5$ was given in [2, Theorem 3.16] and the case p odd in [13, Corollary 4.2], both in the finite case, when $n = k = d$.

Corollary 3.14. *Let G be a powerful pro-p group with $d(G) = n$ and $d(\Phi(G)) = d$. Set $\Omega = \Omega_1(G)$ and $k = d(\Omega)$. The following are equivalent:*

(i) *Ω is abelian and there exist $y_1,\dots,y_{k+d-n}$ in $H^2(G)$ and a basis $x_1,\dots,x_n$ of $H^1(G)$ such that*

$$H^*(G) = \begin{cases}\Lambda[x_1,\dots,x_n]\otimes\mathbb{Z}/p[y_1,\dots,y_{k+d-n},\beta x_{d+1},\dots,\beta x_n] & \text{for } p>2,\\ \Lambda[x_1,\dots,x_d]\otimes\mathbb{Z}/2[y_1,\dots,y_{k+d-n},x_{d+1},\dots,x_n] & \text{for } p=2,\end{cases}$$

and $y_1|_\Omega,\dots,y_{k+d-n}|_\Omega,\beta x_{d+1}|_\Omega,\dots,\beta x_n|_\Omega$ is a basis of $B(\Omega)$;

(ii) *G is Ω-extendable.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorem 3.13. Suppose that G is Ω-extendable. Let N be a minimal subgroup of G satisfying $N^p = \Phi(G)$. N is then a closed normal subgroup and powerfully embedded in G. Furthermore, N is Ω-extendable in G and $d(N) = d$. It follows that $\Omega/\Omega\cap\Phi(G)$ (resp. $\Omega\cap\Phi(G)$) is of rank $n-d$ (resp. $k+d-n$). By Theorem 3.13,

$$H^*(G) = H^*(G/\Phi(G))/(z_1^{(1)},\dots,z_d^{(1)})\otimes\mathbb{Z}/p[z_1,\dots,z_{k+d-n}]$$

with $z_1|_{\Omega\cap\Phi(G)},\dots,z_{k+d-n}|_{\Omega\cap\Phi(G)}$ a basis of $B(\Omega\cap\Phi(G))$. Since $G/\Phi(G)$ is elementary abelian, (i) follows from Lemma 1.2 and Corollary 2.6. □

Theorem 3.15. *In the circumstances of Theorem 3.13, let Γ be an Ω-extension of G. Let U be a subgroup of $Aut(\Gamma)$ which preserves N, and let V be the image of U in $Aut(G)$. Then, as V-modules, $H^*(G) \cong H^*(G_1)/(z_1^{(1)}, \dots, z_d^{(1)}) \otimes \mathbb{Z}/p[\Omega^*]$.*

Proof. $H^*(G) \cong H^*(G_1)/(z_1^{(1)}, \dots, z_d^{(1)})$ is the image of inflation from G_1, so is preserved by V. □

The linear span of $z_1, \dots, z_k$ is isomorphic to Ω^*. It is the kernel of inflation to Γ, so is preserved by V, hence so is $\mathbb{Z}/p[z_1, \dots, z_k]$. □

4. The Bockstein

This section is devoted to the study of the Bockstein homomorphism on $H^*(G)$ with G a pro-p group having an almost powerfully embedded subgroup N.

It is sometimes useful to be able to calculate with an explicit construction of group extensions. Let $A = C_{p^{r_1}}(e_1) \times \cdots \times C_{p^{r_n}}(e_n)$ and let U be an elementary abelian group of rank m, considered as a trivial A-module. Let $z = (z_1, \dots, z_m) \in H^2(A, U)$ and let $\tilde{z}$ be a representative normalized cocycle. Let $A_{\tilde{z}}$ be a group defined as follows. $A_{\tilde{z}} = U \times A$ as sets and the multiplication is given by

$$(u, x) \cdot (v, y) = (u + v + \tilde{z}(x, y), xy)$$

with $u, v \in U, x, y \in A$. If $\tilde{z}$ is changed by the codifferential of a normalized cochain c then there is an isomorphism $A_{\tilde{z}} \to A_{\tilde{z}+c}$ given by $(u, x) \mapsto (u + c(x), x)$.

We choose $x_i \in H^1(C_{p^{r_i}}(e_i))$ as in section 1 and denote by $\tilde{x}_i$ the representing normalized cocycle. We specify $y_i \in H^2(C_{p^{r_i}}(e_i))$ by the cocycle

$$\tilde{y}_i(e_i^t, e_i^s) = \begin{cases} 1 & \text{for } t + s \geq p^{r_i}, \\ 0 & \text{for } t + s < p^{r_i}. \end{cases}$$

As before we also regard these elements as cocycles on A.

Lemma 4.1. *If $z_\ell = \sum_{i=1}^n \alpha_i^{(\ell)} y_i + \sum_{1 \leq i < j \leq n} \lambda_{ij}^{(\ell)} x_i x_j$ then:*

(i) *If $r_i = 1$ then $\beta x_i = y_i$,*
(ii) $(u, e_i)^{p^{r_i}} = ((\alpha_i^{(1)}, \dots, \alpha_i^{(m)}), 1)$,
(iii) $[(u, e_i), (v, e_j)] = (u, e_i)^{-1}(v, e_j)^{-1}(u, e_i)(v, e_j) = ((\lambda_{ij}^{(1)}, \dots, \lambda_{ij}^{(m)}), 1)$, *where we define* $\lambda_{ij}^{(\ell)} = -\lambda_{ji}^{(\ell)}$ *if $i > j$ and 0 if $i = j$.*

Notice that these sets of equations are invariant under adding a coboundary to $\tilde{z}$.

Proof. Part (i) is an easy calculation with cocycles.

For (ii) one proves, by induction on s, that $(u, e_i)^s = (su + \sum_{k=1}^{s-1} \tilde{z}(e_i, e_i^s), e_i^s)$. Now set $s = p^{r_i}$ and evaluate the explicit cocycles.

For (iii), note that $(u, e_i)(v, e_j) = (v, e_j)(u, e_i)(\tilde{z}(e_i, e_j) - \tilde{z}(e_j, e_i), 1)$. Now finish by evaluating $\tilde{z}(e_i, e_j) - \tilde{z}(e_j, e_i)$ using explicit cocycles. □

Suppose now that

$$A = C_{p^r}(e_0) \times C_p(e_1) \times \cdots \times C_p(e_n)$$

be an abelian group with $r \geq 1$, and let $B = \langle e_1, \dots, e_n \rangle$

Let Γ be given by the central extension

$$0 \to (\mathbb{Z}/p)^n \xrightarrow{i} \Gamma \xrightarrow{\varphi} A \to 1$$

corresponding to a cohomology class $(z_1, \dots, z_n) \in H^2(A)^{\oplus n}$. Suppose that $i((\mathbb{Z}/p)^n) \subset \Phi(\Gamma)$, that $\varphi^{-1}(B)$ is almost powerfully embedded and Ω-extendable in Γ. By Lemma 2.2 and Theorem 3.3, the z_i can be chosen such that

$$z_i|_{\langle e_j \rangle} = \begin{cases} y_i & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varphi^{-1}(B)$ is Ω-extendable in Γ, it follows from Theorem 3.13 that each βz_i belongs to $(z_1,\dots,z_n)$, hence is of the form

$$\beta z_i = \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} x_k z_\ell \tag{1}$$

with $\alpha^{(i)}_{k\ell} \in \mathbb{Z}/p, 1 \le i \le n$. We have

Lemma 4.2. *For $1 \le i \le n, 0 \le k,\ell \le n$, we have $\alpha^{(i)}_{kk} = 0, \alpha^{(i)}_{k\ell} = -\alpha^{(i)}_{\ell k}$ and*

$$z_i = -\sum_{k<\ell} \alpha^{(i)}_{k\ell} x_k x_\ell \bmod B(A).$$

In other words, $[(0,e_k),(0,e_\ell)] = \prod_{i=1}^{n} (0,e_i)^{-p\alpha^{(i)}_{k\ell}}, 0 \le k,\ell \le n$.

Proof. Write

$$z_i = y_i + \sum_{0<\ell} \gamma^{(i)}_{0\ell} x_0 x_\ell + \sum_{1\le k<\ell} \gamma^{(i)}_{k\ell} x_k x_\ell + \gamma_i y_0, \tag{2}$$

with $\gamma^{(i)}_{st}, \gamma_i \in \mathbb{Z}/p, 1 \le i \le n$. (Note that, for $p = 2$ and $1 \le k < \ell$, $\gamma^{(i)}_{k\ell} = 0$ by Proposition 3.2, as $\varphi^{-1}(B)$ is powerful.) So

$$\beta z_i = \sum_{0<\ell} \gamma^{(i)}_{0\ell} [x_\ell \beta(x_0) - x_0 y_\ell] + \sum_{1\le k<\ell} \gamma^{(i)}_{k\ell} [x_\ell y_k - x_k y_\ell]. \tag{3}$$

Also, from (1) and (2), we have

$$\begin{aligned}
\beta z_i &= \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} x_k [y_\ell + \sum_{0<s} \gamma^{(\ell)}_{0s} x_0 x_s + \sum_{1\le t<s} \gamma^{(\ell)}_{ts} x_t x_s + \gamma_\ell y_0]\\
&= \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} x_k y_\ell + \sum_{\substack{k,\ell\\ \ell>0}} \sum_{0<s} \alpha^{(i)}_{k\ell} \gamma^{(\ell)}_{0s} x_k x_0 x_s + \sum_{\substack{k,\ell\\ \ell>0}} \sum_{1\le t<s} \alpha^{(i)}_{k\ell} \gamma^{(\ell)}_{ts} x_k x_t x_s + \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} \gamma_\ell x_k y_0\\
&= \begin{cases} \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} x_k y_\ell + \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} \gamma_\ell x_k y_0 \bmod \lambda^3(A) & \text{for } p > 2,\\ \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} x_k y_\ell + \sum_{\substack{k,\ell\\ \ell>0}} \sum_{0<s} \alpha^{(i)}_{k\ell} \gamma^{(\ell)}_{0s} x_k x_0 x_s + \sum_{\substack{k,\ell\\ \ell>0}} \alpha^{(i)}_{k\ell} \gamma_\ell x_k y_0 & \text{for } p = 2. \end{cases}
\end{aligned} \tag{4}$$

Comparing (3) and (4) yields then $\alpha^{(i)}_{kk} = 0, \alpha^{(i)}_{k\ell} = -\alpha^{(i)}_{\ell k}$ and $\alpha^{(i)}_{k\ell} = -\gamma^{(i)}_{k\ell}$ for $k < \ell$. The last equality follows from (2) and Lemma 4.1. ◻

With the notation as above, let Δ be given by the central extension

$$0 \to (\mathbb{Z}/p)^{n+1} \xrightarrow{j} \Delta \xrightarrow{\psi} A \to 1$$

corresponding to a cohomology class $(w_0, w_1, \dots, w_n) \in H^2(A)^{\oplus n+1}$. Suppose that $j((\mathbb{Z}/p)^{n+1}) \subset \Phi(\Delta)$, that $r \ge 2$ (so $\beta x_0 = 0$) and $\psi^{-1}(\Omega_1(A))$ is almost powerfully embedded and Ω-extendable in Γ. As above, the w_i can be chosen such that

$$w_i|_{\langle e_j\rangle} = \begin{cases} y_i & \text{for } i = j,\\ 0 & \text{otherwise,} \end{cases}$$

and each βw_i is of form

$$\beta w_i = \sum_{k,\ell} \sigma^{(i)}_{k\ell} x_k w_\ell \tag{5}$$

with $\sigma^{(i)}_{k\ell} \in \mathbb{Z}/p, 0 \le i \le n$. We have

Lemma 4.3. *For $0 \le i \le n, 0 \le k, \ell \le n$, we have $\sigma^{(i)}_{kk} = 0, \sigma^{(i)}_{k\ell} = -\sigma^{(i)}_{\ell k}$ and*

$$w_i = -\sum_{k<\ell} \sigma^{(i)}_{k\ell} x_k x_\ell \bmod B(A).$$

In other words, $[(0,e_k),(0,e_\ell)] = (0,e_0)^{-p^2\sigma^{(0)}_{k\ell}} \prod_{i=1}^{n} (0,e_i)^{-p\sigma^{(i)}_{k\ell}}, 0 \le k,\ell \le n.$

Proof. We use a similar argument to the one of the proof of Lemma 4.2. Write

$$w_i = y_i + \gamma^{(i)}_{0\ell} x_0 x_\ell + \sum_{1\le k<\ell} \gamma^{(i)}_{k\ell} x_k x_\ell, \tag{6}$$

with $\gamma^{(i)}_{st} \in \mathbb{Z}/p, 0 \le i \le n$. (So, for $p=2$ and $1 \le k < \ell$, $\gamma^{(i)}_{k\ell} = 0$.) Therefore

$$\beta w_i = -\sum_{0<\ell} \gamma^{(i)}_{0\ell} x_0 y_\ell + \sum_{1\le k<\ell} \gamma^{(i)}_{k\ell} [x_\ell y_k - x_k y_\ell]. \tag{7}$$

From (5) and (6), we have

$$\begin{aligned}
\beta w_i &= \sum_{k,\ell} \sigma^{(i)}_{k\ell} x_k [y_\ell + \sum_{0<s} \gamma^{(\ell)}_{0s} x_0 x_s + \sum_{1\le t<s} \gamma^{(\ell)}_{ts} x_t x_s] \\
&= \sum_{k,\ell} \sigma^{(i)}_{k\ell} x_k y_\ell + \sum_{k,\ell}\sum_{0<s} \sigma^{(i)}_{k\ell}\gamma^{(\ell)}_{0s} x_k x_0 x_s + \sum_{k,\ell}\sum_{1\le t<s} \sigma^{(i)}_{k\ell}\gamma^{(\ell)}_{ts} x_k x_t x_s \\
&= \begin{cases} \sum_{k,\ell} \sigma^{(i)}_{k\ell} x_k y_\ell \bmod \lambda^3(A) & \text{for } p>2, \\ \sum_{k,\ell} \sigma^{(i)}_{k\ell} x_k y_\ell + \sum_{k,\ell}\sum_{0<s} \sigma^{(i)}_{k\ell}\gamma^{(\ell)}_{0s} x_k x_0 x_s & \text{for } p=2. \end{cases}
\end{aligned} \tag{8}$$

Comparing (7) and (8) yields then $\sigma^{(i)}_{kk} = 0, \sigma^{(i)}_{k\ell} = -\sigma^{(i)}_{\ell k}$ and $\sigma^{(i)}_{k\ell} = -\gamma^{(i)}_{k\ell}$ for $k<\ell$. The last equality follows from (6) and Lemma 4.1. □

Suppose now that N is almost powerfully embedded in the p-group G with $d(N) = d$. Let $a_1,\dots,a_r$, $a_{r+1},\dots,a_m$ be a minimal system of generators of G with $a_1,\dots,a_r \in N$ (with the convention that $r=0$ if $N \subset \Phi(G)$). So, for every $i \ge 1$, $\phi_i(a_1),\dots,\phi_i(a_m)$ also generate G_i. Define $x_j, 1 \le j \le n$ in $H^1(G_1)$ by $x_j(\phi_1(a_k)) = \delta_{jk}$. Set $e_i = a_i, 1 \le i \le m$. There exist then $e_{r+1},\dots,e_d \in N$ such that $e_1,\dots,e_d$ generate N. As N is powerful, the e_i's can be renumbered so that $e_1^{p^{i-1}},\dots,e_{n_i}^{p^{i-1}}$ generate $\Phi^i_G(N)$. The x_j's are then considered as elements of $H^*(G_i), i \ge 2$, via the inflation map.

Theorem 4.4. *Suppose that N is almost uniformly embedded in G with $\ell = \ell(\Phi^*_G(N)) \ge 3$. For $2 \le i \le \ell-1$, the classes $z^{(i)}_1,\dots,z^{(i)}_d$ given in* Section 2 *can be chosen such that:*

(i) $z^{(i)}_j|_{\langle \phi_i(e_k^{p^{i-1}})\rangle} = 0$ *if $j \ne k$;*

(ii) *if*

$$\beta z^{(1)}_j = \sum_{s,t} \lambda^{(j)}_{st} x_s z^{(1)}_t \tag{9}$$

with $\lambda^{(j)}_{st} \in \mathbb{Z}/p, 1 \le j \le d$, then:

(iia) $\lambda^{(j)}_{ss} = 0$;

(iib) *for $2 \le i \le \ell-2, 1 \le j \le d$,*

$$\beta z^{(i)}_j = \sum_{st} \lambda^{(j)}_{st} x_s z^{(i)}_t,$$
$$\beta z^{(\ell-1)}_j = \sum_{st} \lambda^{(j)}_{st} x_s z^{(\ell-1)}_t + \eta_j,$$

with $\eta_j \in \text{ Im Inf}_G^{G_1}$. *If* N *is* Ω*-extendable in* G*, then* $\eta_j = 0, 1 \le j \le d$.
In particular, if $p = 2$ *or if* $[N, G] \subset N^{p^2}$ *for* $p > 2$*, we have, for* $1 \le j \le d$*,*

$$\begin{aligned} \beta z_j^{(i)} &= 0, 1 \le i \le \ell - 2, \\ \beta z_j^{(\ell-1)} &\in \text{ Im Inf}_G^{G_1}, \end{aligned}$$

and $\beta z_j^{(\ell-1)} = 0$ *if* N *is* Ω*-extendable in* G*.*

Proof. (i) is clear from Lemma 2.2 and Proposition 3.2. We now prove (ii). Set $A = \langle \phi_1(a_k), A_1 \rangle$ with a given $1 \le k \le m$. A is then an abelian subgroup of G_1. It follows from (9) that

$$\beta z_j^{(1)}|_A = \sum_{s=1}^{m} \sum_{t=1}^{d} \lambda_{st}^{(j)} x_s|_A z_t^{(1)}|_A.$$

By Lemmas 4.1, 4.2 and 4.3, this implies $\lambda_{ss}^{(j)} = 0$ and, for $s \ne t$,

$$[a_s, e_t] = \prod_{j=1}^{d} e_j^{-p\lambda_{st}^{(j)}} \mod \Phi_G^3(N).$$

Hence, by Lemma 3.12 (i), for $s \ne t$

$$\begin{aligned} [a_s, e_t^{p^{i-1}}] &= [a_s, e_t]^{p^{i-1}} \mod \Phi_G^{i+2}(N) \\ &= \prod_{j=1}^{d} e_j^{-p^i \lambda_{st}^{(j)}} \mod \Phi_G^{i+2}(N). \end{aligned} \tag{10}$$

Given $i \le \ell - 2$, for $1 \le k \le m$, define B_k to be the abelian subgroup of G_i given by $B_k = \langle e_k, A_i \rangle$. Note that $\beta z_j^{(i)}$ belongs to $(z_1^{(i)}, \dots, z_d^{(i)})$, for $i \le \ell - 2$; hence it should be a linear combination of the $x_k z_t^{(i)}$'s. By (i), (10), Lemmas 4.2, 4.3 and from the structure of $H^*(G_i)$, $\beta z_j^{(i)}|_{B_k}$ should be of form

$$\beta z_j^{(i)}|_{B_k} = \sum_{s,t} \lambda_{st}^{(j)} x_s|_{B_k} z_t^{(i)}|_{B_k}.$$

Therefore

$$\beta z_j^{(i)} = \sum_{s,t} \lambda_{st}^{(j)} x_s z_t^{(i)}.$$

Analogously, we also get the required decomposition for $z_j^{(\ell-1)}, 1 \le j \le d$, by noting that, modulo Im $\text{Inf}_G^{G_1}$, $\beta z_j^{(\ell-1)}$ also belongs to $(z_1^{(\ell-1)}, \dots, z_d^{(\ell-1)})$, and $\xi|_{B_k} = 0$ for every $\xi \in \text{Im Inf}_G^{G_1}$. Furthermore, if N is Ω-extendable in G, then $\beta z_j^{(\ell-1)}$ belongs to $(z_1^{(\ell-1)}, \dots, z_d^{(\ell-1)})$, hence $\eta_j = 0$.

Finally, if $p = 2$ or if $[N, G] \subset N^{p^2}$ for $p > 2$, then $\beta z_j^{(1)} = 0$, for $1 \le j \le d$. So the last equalities follow from what we just proved. The theorem follows. □

Suppose that N is almost powerfully embedded and Ω-extendable in a finite p-group G. As noted above, $d(\Omega_1(N)) = d(N)$ and we may consider the case where $\Omega_1(N)$ is contained in the Frattini subgroup of G. The proof of the above theorem can be applied to the subgroup $N/\Omega_1(N)$ of $G/\Omega_1(N)$ and, together with Theorem 3.13, yields the following

Theorem 4.5. *Let N be almost powerfully embedded and Ω-extendable in a finite p-group G. Assume that $\Omega = \Omega_1(N)$ is contained in N^p, $d(\Omega) = d$. Pick a basis $x_1, \ldots, x_m$ of $H^1(G_1)$ and a cohomology class $(w_1, \ldots, w_d) \in H^2(G_1)^{\oplus d}$ corresponding to the central extension $0 \to A_2 \to G_2 \to G_1 \to 1$. Then there exist $z_1, \ldots, z_d$ of $H^2(G)$, $\eta_1, \ldots, \eta_d$ of $\mathrm{Im}\,\mathrm{Inf}_G^{G_1}$ such that:*

(i) $H^*(G) = H^*(G_1)/(w_1, \ldots, w_d) \otimes \mathbb{Z}/p[z_1, \ldots, z_d]$;
(ii) $z_1|_\Omega, \ldots, z_d|_\Omega$ *is a basis of* $B(\Omega)$;
(iii) *if*

$$\beta w_j = \sum_{r,s} \lambda_{rs}^{(j)} x_r w_s, 1 \le j \le k,$$

then

$$\beta z_j = \sum_{r,s} \lambda_{rs}^{(j)} x_r z_s + \eta_j, 1 \le j \le k.$$

The extension Γ of G by $(\mathbb{Z}/p)^d$ corresponding to $(z_1, \ldots, z_d)$ is also Ω-extendable if and only if all the η_j's vanish.

In particular, if $[N, G] \subset N^{p^2}$ (for example, if N is powerfully embedded in G for $p = 2$), then $\beta w_j = 0, \beta z_j \in \mathrm{Im}\,\mathrm{Inf}_G^{G_1}, 1 \le j \le k$. □

Consider now the case where N is almost powerfully embedded and Ω-extendable in a pro-p group G with $\Omega = \Omega_1(N)$ contained in N^p. As noted in Remark 3.5, $d(\Omega) = d(T)$ with T the subgroup of N consisting of elements of finite order. It is known that N/T is almost uniformly embedded in G/T. Set $d = d(N), k = d(\Omega)$. The cohomology class $(z_1^{(1)}, \ldots, z_d^{(1)}) \in H^2(G_1)$ can be then chosen such that $(z_{k+1}^{(1)}, \ldots, z_d^{(1)})$ corresponds to the central extension $0 \to A_2(N/T) \to G/T \to G_1 \to 1$. It follows that, for $1 \le j \le k$, $\beta z_j^{(1)}$ is of form

$$\beta z_j^{(1)} = \sum_{\substack{r \\ s \le k}} \lambda_{rs}^{(j)} x_r z_s^{(1)}.$$

We then have

Corollary 4.6. *Let N be almost powerfully embedded and Ω-extendable in a pro-p group G with $d(N) = d$. Assume that $\Omega = \Omega_1(N)$ is contained in $N^p, d(\Omega) = k$. Then there exist a basis $x_1, \ldots, x_m$ of $H^1(G_1)$, a cohomology class $(w_1, \ldots, w_d) \in H^2(G_1)^{\oplus d}$ corresponding to the central extension $0 \to A_2 \to G_2 \to G_1 \to 1$, elements $z_1, \ldots, z_k$ of $H^2(G)$, $\eta_1, \ldots, \eta_k$ of $\mathrm{Im}\,\mathrm{Inf}_G^{G_1}$ and $\lambda_{rs}^{(j)} \in \mathbb{Z}/p, 1 \le r \le m, 1 \le s, j \le k$ such that:*

(i) $H^*(G) = H^*(G_1)/(w_1, \ldots, w_d) \otimes \mathbb{Z}/p[z_1, \ldots, z_k]$;
(ii) $z_1|_\Omega, \ldots, z_k|_\Omega$ *is a basis of* $B(\Omega)$;
(iii) *for* $1 \le j \le k$,

$$\beta w_j = \sum_{\substack{r \\ s \le k}} \lambda_{rs}^{(j)} x_r w_s,$$

$$\beta z_j = \sum_{r,s} \lambda_{rs}^{(j)} x_r z_s + \eta_j.$$

The extension Γ of G by $(\mathbb{Z}/p)^k$ corresponding to $(z_1, \ldots, z_d)$ is also Ω-extendable if and only if all the η_j's vanish.

In particular, if $[N, G] \subset N^{p^2}$ (for example, if N is powerfully embedded in G for $p = 2$), then $\beta w_j = 0, \beta z_j \in \mathrm{Im}\,\mathrm{Inf}_G^{G_1}, 1 \le j \le k$. □

With the notation as above, suppose that G is a powerful pro-p group and $\beta x_j = \sum_{s<t} \lambda_{st}^{(j)} x_s x_t, 1 \le j \le n_1$. Combining the above theorem with Corollary 3.14, we have the following, of which the case $p \ge 5$ was given in [2, Theorem 3.16].

Corollary 4.7. *Let G be a powerful pro-p group, Ω-extendable with $d(G) = n, \ell = \ell(\Phi^*_G(G)) \geq 2$. Set $d = d(\Phi(G)), \Omega = \Omega_1(G)$ and $k = d(\Omega)$. Then there exist a basis $x_1, \dots, x_n$ of $H^1(G)$, elements $y_1, \dots, y_{k+d-n}$ of $H^2(G)$, $\eta_1, \dots, \eta_{k-d+n}$ of $\operatorname{Im} \operatorname{Inf}_G^{G/\Phi(G)}$ and $\lambda_{rs}^{(i)}$ of $\mathbb{Z}/p, 1 \leq r, s, i \leq k+d-n$, satisfying:*

(i) $\operatorname{Res}_\Omega^G$ *maps* $\{y_1, \dots, y_{k+d-n}, \beta x_{d+1}, \dots, \beta x_n\}$ *isomorphically onto a basis of* $B(\Omega)$*;*

(ii) $H^*(G) = \begin{cases} \Lambda[x_1, \dots, x_n] \otimes \mathbb{Z}/p[y_1, \dots, y_{k-d+n}, \beta x_{d+1}, \dots, \beta x_n] & \text{for } p > 2, \\ \Lambda[x_1, \dots, x_d] \otimes \mathbb{Z}/2[y_1, \dots, y_{k-d+n}, x_{d+1}, \dots, x_n] & \text{for } p = 2; \end{cases}$

(iii) $\lambda_{rr}^{(i)} = 0, \lambda_{rs}^{(i)} = -\lambda_{sr}^{(i)}$, $\beta x_i = -\sum_{r<s} \lambda_{rs}^{(i)} x_r x_s$, $\beta y_i = \sum_{r,s} \lambda_{rs}^{(i)} x_r y_s + \eta_i, 1 \leq i \leq k-d+n$.

The extension Γ of G by $(\mathbb{Z}/p)^{k+d-n}$ corresponding to $(y_1, \dots, y_{k+d-n})$ is also Ω-extendable if and only if all the η_i's vanish.

In particular, if $p = 2$ or if $[G, G] \subset G^{p^2}$ for p odd, then all the $\lambda_{rs}^{(i)}$'s vanish. □

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