STRUCTURE THEOREMS OVER POLYNOMIAL RINGS

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ABSTRACT. DRAFT 21 December 2005. Given a polynomial ring R over a field k and a finite group G, we consider a finitely generated graded RG-module S. We regard S as a kG-module and show that various conditions on S are equivalent, such as only containing finitely many isomorphism classes of indecomposable summands or satisfying a structure theorem in the sense of [8].

1. INTRODUCTION

Consider a polynomial ring $R = k[d_1, \ldots, d_n]$, finitely generated over a commutative ring k and graded in such a way that the d_i are homogeneous of positive degree. We are most interested in the case when k is a field of finite characteristic, but we allow k to be any complete local noetherian commutative ring, for example the p-adic integers. Let G be a finite group and let $S = \oplus_{i=N}^{\infty} S^i$ be a finitely generated graded left RG-module, where G preserves the grading.

By a *structure theorem* for S over RG we mean a set of finitely generated graded kG-submodules $\bar{X}_I \subseteq S$, one for each $I \subseteq \{1, \ldots, n\}$, such that $S \cong \bigoplus_{I \subseteq \{1,\ldots,n\}} k[d_i \mid i \in I] \otimes_k \bar{X}_I$ as a graded kG-module, where the map from right to left is induced by the action of R on S.

This concept originates in work of Karagueuzian and the author [7, 8], where it is shown that if $S = k[x_1, \ldots, x_n]$, k is finite, U_n is the group of $n \times n$ upper triangular matrices acting in the natural way on S and R is the ring of invariants then S has a structure theorem over RU_n.

In this paper we investigate this property further and prove the following result.

Theorem 1.1. *For R and S as above the following are equivalent.*

(1) *Only finitely many isomorphism classes of indecomposable kG-modules occur as summands of S.*
(2) *S has a relatively projective resolution over RG relative to kG of finite length and finitely generated in each degree.*
(3) *There is a number N such that the group $\mathrm{Ext}^*_{RG/kG}(S, T)^0$ of relative homological algebra vanishes for any graded RG-module T that is 0 in degrees less than N.*
(4) *S has a structure theorem over RG.*
(5) *There are integers $a_1, \ldots, a_t$ such that $S^r = \sum_{i=1}^{t} a_i S^{r-i}$ for r sufficiently large (in the Green ring of graded kG-modules).*

Of course, there are many other possible equivalent properties, but in some heuristic sense (1) is the weakest and (2) is the strongest. The significance of (4) is that it is what is usually proved in specific examples, (e.g. [7, 8] or the proof of 7.1).

Partially supported by a grant from the Leverhulme Trust.

Corollary 1.2. *If S is a polynomial ring in n variables of degree 1 over a finite field k and R denotes the ring of invariants under $\mathrm{GL}_n(k)$ then S has a structure theorem over $R\,\mathrm{GL}_n(k)$. More generally, let k be the algebraic closure of a finite field or let k be any field but $n \leq 3$. Let $G \leq \mathrm{GL}_n(k)$ be finite and let R be a polynomial subring of the ring of invariants under G that has n generators. Then S has a structure theorem over RG.*

Proof. In [8] it was shown that condition (1) of the theorem holds when k is finite, and this generalizes easily to the algebraic closure (see [9], discussion after 4.2). If $n \leq 3$ then [9] 6.1 applies without any restriction on the field. □

Note that the proof in [8] only yields a structure theorem for p-groups, where p is the characteristic of k.

An example of a module S for which the conditions of the theorem do not hold is given in [9] 4.4.

Condition (1) of the theorem is independent of the ring R, so if, for given S, k and G, one of the other conditions is satisfied for some ring R then it is also satisfied by any other ring R satisfying the hypotheses of the theorem.

We thank the referee for the elegant proofs of 3.3 and 8.1.

2. Change of Category

We want to move to a category in which all of the indecomposable kG-modules occurring become projective. This could be done using functor categories, but we choose to present a module-theoretic approach.

As is customary, given any ring A (perhaps without an identity element) we let ${}_A\mathrm{Mod}$ denote the category of left A-modules and let ${}_A\mathrm{mod}$ denote the full subcategory of finitely generated modules. We also let ${}_A\mathrm{Proj}$ denote the full subcategory of projective modules and ${}_A\mathrm{proj}$ the full subcategory of finitely generated projective modules. The corresponding categories of right modules are denoted by Mod_A, etc.. Given a left A-module M we let ${}_A\mathrm{Add}$ denote the full subcategory of ${}_A\mathrm{Mod}$ consisting of the modules that are summands of some sum (possibly infinite) of copies of M and let ${}_A\mathrm{add}$ the full subcategory of this consisting of the finitely generated modules.

Our conditions on k imply that, for any finitely generated indecomposable kG-module Y, the endomorphism ring $\mathrm{End}_{kG}(Y)$ is a local ring and the quotient ring $\ell = \mathrm{End}_{kG}(Y)/\operatorname{rad}\mathrm{End}_{kG}(Y)$ is a finite dimensional division algebra over $\bar{k}$, the residue class field of k. For proofs see [4] 6.10 or [1] 1.9.3. In fact this is the only reason for which we need completeness before section 6. It would also be possible for k to be a discrete valuation ring in a splitting field for G, by [4] ex.36.1.

In particular, it follows that finitely generated k-modules satisfy the unique decomposition (or Krull-Schmidt) property, which is that any other decomposition into indecomposables involves the same indecomposables up to isomorphism with the same multiplicities.

Let M_m be the sum of the indecomposable kG-modules that occur as summands of $\oplus_{i\leq m} S^i$, with each isomorphism type appearing once only. M_m is finitely generated over k; let $E_m = \mathrm{End}_{kG}(M_m)$, which we consider to act on M_m on the left.

There are functors $U_m = \mathrm{Hom}_{kG}(M_m, -) : {}_{kG}\mathrm{Mod} \to \mathrm{Mod}_{E_m}$ and $V_m = - \otimes_{E_m} M_m : \mathrm{Mod}_{E_m} \to {}_{kG}\mathrm{Mod}$, which restrict to functors between the finitely generated subcategories.

Proposition 2.1. *The functors U_m and V_m induce inverse equivalences of categories between ${}_{M_m}\operatorname{Add}$ and $\operatorname{Proj}_{E_m}$ and these restrict to an equivalence between ${}_{M_m}\operatorname{add}$ and $\operatorname{proj}_{E_m}$.*

This is well known, but for the convenience of the reader we sketch a proof.

Proof. Let $(E_m)_{E_m}$ denote the right regular representation for E_m, so $\operatorname{Proj}_{E_m} = \operatorname{Add}_{(E_m)_{E_m}}$. Then M_m and $(E_m)_{E_m}$ correspond under these functors. The functors preserve direct sums and, since they must preserve idempotents, they preserve direct summands. □

If we define modules N_m by writing $M_{m+1} = M_m \oplus N_{m+1}$ then we can extend an element of E_m to E_{m+1}, by defining it to be zero on N_{m+1}. Thus it makes sense to define $E = \varinjlim E_m$ and E acts on $M = \varinjlim M_m$.

Now define $U = \varinjlim U_m : {}_{kG}\operatorname{Mod} \to \operatorname{Mod}_E$ and $V = - \otimes_E M : \operatorname{Mod}_E \to {}_{kG}\operatorname{Mod}$.

Proposition 2.2. *U and V induce inverse equivalences between ${}_M\operatorname{Add}$ and Proj_E and these restrict to equivalences between ${}_M\operatorname{add}$ and proj_E, where modules X for E are required to satisfy the property $XE = X$. (This extra condition is required because E does not, in general, contain an identity element.)*

For full details see [13] ch.10, but we sketch a proof.

Proof. Let $e_m \in E$ be the idempotent that is the identity on M_m and 0 on its complement $\oplus_{i>m} N_i$. The way that U was constructed as a direct limit means that $U(X)$ is the submodule of $\operatorname{Hom}_{kG}(M, X)$ consisting of $f \in \operatorname{Hom}_{kG}(M, X)$ satisfying $fe_m = f$ for some m. Thus each f has a finite dimensional image, and it follows that U commutes with direct sums; clearly V commutes with direct sums.

It also follows that $(E_E)E = E$, and the condition $XE = X$ shows that $\operatorname{Proj}_E = \operatorname{Add}_{E_E}$. Clearly M corresponds to E_E, so U and V induce inverse equivalences between ${}_M\operatorname{Add}$ and Proj_E, as before.

In order to see that U takes ${}_M\operatorname{add}$ into proj_E it is sufficient to show that $U(L)$ is finitely generated when L is finitely generated and indecomposable. This L must be a summand of some M_m; let $f : M \to L$ be e_m followed by projection onto the summand of M_m isomorphic to L and then an isomorphism with L. Then any homomorphism $M \to L$ factors through f and it follows that $U(L) \cong fE$.

Conversely, if X is finitely generated over E then $X = Xe_m$ for some m so $V(X) = X \otimes_E M = Xe_m \otimes_E M = X \otimes_E e_m M$ is also finitely generated. □

Notice that if X is a graded RG-module then we can regard it as a kG-module and apply U. The result is naturally a graded $R \otimes_k E$-module. Similarly, if Y is a graded $R \otimes_k E$-module then $V(Y)$ is naturally a graded RG-module.

It is shown in [4] §19 and [13] proof of 54.1 that if we write $M_m = Y_1 \oplus \cdots \oplus Y_t$ as a sum of indecomposables and we express an element of E_m as a matrix with entries in $\operatorname{End}_{kG}(Y_i, Y_j)$ then $\operatorname{rad} E_m$ consists of those matrices with no entry an isomorphism. In particular $E_m / \operatorname{rad} E_m \cong \oplus_{i=1}^t \operatorname{End}_{kG}(Y_i) / \operatorname{rad} \operatorname{End}_{kG}(Y_i) \cong \oplus_{i=1}^t \ell_i$, where each ℓ_i is a finite dimensional division algebra over k.

By taking the limit it follows that the same is true for E.

Lemma 2.3. *$E / \operatorname{rad} E \cong \oplus_I \ell_i$, where I indexes the indecomposable modules occurring in M.*

Observe that J must annihilate any simple right E-module V satisfying $VE = V$. It follows that there is a version of Nakayama's Lemma for E-modules, by the usual proof ([13], 49.7).

Lemma 2.4. *If $M \in \mathrm{mod}_E$ (so in particular $ME = M$) and $L \leq M$ is such that $MJ + L = M$ then $L = M$.*

3. Polynomial Rings

Let $R = \ell[x_1, \dots, x_n]$ be a graded polynomial ring over ℓ, with the x_i having positive grading. For future use we only require ℓ to be a division ring, rather than a field, but we do not consider anything more general.

First we prove 1.1 (4) in the case of the trivial group.

Proposition 3.1. *Any finitely generated graded module over R has a structure theorem over R.*

For the proof we need two lemmas.

Lemma 3.2. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules and A and C have structure theorems then B has a structure theorem.*

Proof. The $\bar{X}_I$ in the structure theorem for C can be lifted arbitrarily to graded subspaces of B and added to those for A. □

Lemma 3.3. *Any finitely generated R-module has a finite filtration in which each composition factor is isomorphic to a module of the form R/I for some left graded ideal I with the property that if $c, d \in R$ are homogenous, c is central and $cd \in I$ then $c \in I$ or $d \in I$.*

When ℓ is a field this is well known (e.g. [1] 2.2.2).

Proof. Since the R-module, M say, is finitely generated and R is noetherian it is sufficient to show that M has a submodule of the given form, because then the lemma will follow by a standard induction argument. For each $m \in M$ let $\mathrm{Ann}_R(m) = \{r \in R \mid rm = 0\}$ be its annihilator. Consider the set of all the annihilators of the non-zero homogeneous elements of M and let $I = \mathrm{Ann}_R(m)$ be a maximal element, so $R/I \cong Rm \subseteq M$. We claim that Rm has the right properties.

If $c, d \in R$ are homogeneous, c is central and $cd \in I$ then $I \leq \mathrm{Ann}_R(cm)$ and $d \in \mathrm{Ann}_R(cm)$. Since I is maximal amongst the annihilators of the non-zero homogeneous elements we must have either $d \in I$ or otherwise $cm = 0$, implying that $c \in I$. □

We now prove 3.1 by induction on n. The result is clear when $n = 0$. In view of the lemmas above it is sufficient to show that an R-module of the form $M = R/I$ with I as in 3.3 has a structure theorem.

If I contains all the x_i then the result is clear; otherwise there is an x_i that is not contained in I, say x_1. By 3.3, multiplication by x_1 on R/I is an injection.

The quotient M/x_1M is finitely generated over $\ell[x_2, \dots, x_n]$, so has a structure theorem over $\ell[x_2, \dots, x_n]$ by the induction hypothesis, say $M/x_1M \cong \bigoplus_{I \subseteq \{2,\dots,n\}} \ell[d_i \mid i \in I] \otimes_\ell \bar{X}_I$. Each $\bar{X}_I$ can be lifted over ℓ to $\bar{Y}_I \subseteq M$, say. The multiplication map $\bigoplus_{I \subseteq \{2,\dots,n\}} \ell[d_i \mid i \in I] \otimes_\ell \bar{Y}_I \to M$ is injective and we denote its image by N. We know that N maps isomorphically to M/x_1M under the quotient map $M \to M/x_1M$, so $M = N \oplus x_1M$.

Because $M \cong x_1 M$, repeated substitution for M yields $M = \oplus_{i=0}^{\infty} x_1^i N$, since the sum is finite in each degree. The natural map $\ell[x_1] \otimes_\ell N \to M$ is thus an isomorphism and this yields a structure theorem $M \cong \bigoplus_{I \subseteq \{2,\dots,n\}} \ell[d_i \mid i \in \{1\} \cup I] \otimes_\ell \bar{Y}_I$.

4. Lifting Resolutions

A relatively projective resolution of S over RG relative to kG is a chain complex of graded RG-modules $\cdots \to C_r \to \cdots \to C_0 \to S \to 0$. Each C_r must be of the form $R \otimes_k X$ (or, equivalently, $RG \otimes_{kG} X$), where X is a graded kG-module, and the complex must be split exact over kG.

Such a resolution is unique up to homotopy.

One might expect to see "a summand of a sum of terms of the form $R \otimes_k X$" in the definition of a relatively projective resolution. But the "sum of terms" part is unnecessary since $(R \otimes_k X) \oplus (R \otimes_k Y) \cong R \otimes_k (X \oplus Y)$. In fact, we will only ever consider modules that are a summand of a sum of terms of the form $R \otimes_k X$ where X is finite dimensional. In this case the "summand" part makes no difference to the class of resolutions either, in view of the following lemma.

Lemma 4.1. *If X is a homogeneous graded kG-module with local endomorphism ring (e.g. if X is finite dimensional and indecomposable) then the grading-preserving endomorphism ring of the RG-module $R \otimes_k X$ is local and, in particular, $R \otimes_k X$ is indecomposable.*

Hence any (possibly infinite) sum of such modules $M = \oplus_{i \in I} R \otimes_k X_i$ has the unique decomposition property, i.e. any other decomposition into homogeneous indecomposables involves the same indecomposables up to isomorphism with the same multiplicities. In addition any graded summand of M is isomorphic to $\oplus_{i \in J} R \otimes_k X_i$ for some $J \subseteq I$.

Proof. The grading-preserving endomorphism ring of $R \otimes_k X$ is isomorphic to $\mathrm{End}_{kG}(X)$, so local.

It is well known that the second part follows for finite sums ([1] 1.4.3). For infinite sums it is a theorem of Warfield ([11] Thm. 7). □

For any graded left RG-modules S, T the relative Ext-groups $\mathrm{Ext}^*_{RG/kG}(S,T)$ are defined to be the homology of the complex $\mathrm{Hom}_{RG}(C_*, T)$ of graded homomorphisms. We usually only have need of $\mathrm{Ext}^*_{RG/kG}(S,T)^0 = H(\mathrm{Hom}_{RG}(C_*, T)^0)$, where the superscript 0 indicates the maps that preserve grading.

Similarly, if T is a graded right RG-module we have $\mathrm{Tor}_*^{RG/kG}(T, S)$, which is the homology of the graded tensor product $T \otimes_{RG} C_*$.

For more on relative homological algebra see [5, 6, 12].

In the next proposition we only require R to be a graded ring $\oplus_{i=0}^{\infty} R^i$ with $R^0 = k$ such that R is finitely generated as a k-algebra. S is as in the introduction.

Proposition 4.2. *S has a minimal relatively projective resolution, C_*, over RG relative to kG in which, for each indecomposable kG-module Y, $R \otimes_k Y$ occurs as a summand in each C_i only finitely many times and only in (grading) degrees greater than or equal to those in which Y appears in S.*

If R is polynomial then this resolution is of finite length bounded by the number of generators of R.

Proof. Let $J = \operatorname{rad} E$. Consider $U(S)/U(S)J$, which we will shorten to $U(S)/J$. It is a module for $R \otimes_k E/J \cong R \otimes_k (\oplus_{i\in I}\ell_i) \cong \oplus_{i\in I}(R \otimes_k \ell_i)$, where I indexes the isomorphism classes of indecomposable summands L_i of S. Thus $U(S)/J$ is a sum of pieces $(U(S)/J)_i$, and the description of J in 2.2 shows that $(U(S)/J)_i \cong \operatorname{Hom}_{RG}(L_i, S)/\operatorname{rad}\operatorname{End}(L_i)$, so is finitely generated over R and thus over $R \otimes_k \ell_i$.

Thus each $(U(S)/J)_i$ has a minimal graded projective resolution over $R \otimes_k \ell_i$,

$$\cdots \to R \otimes_k P_1^i \to R \otimes_k P_0^i \to (U(S)/J)_i \to 0,$$

where each P_j^i is a finite dimensional graded vector space over ℓ_i.

If R is polynomial then these resolutions have length bounded by the number of variables. In any case they can be summed to give a minimal projective resolution of $U(S)/J$ over $R \otimes_k E/J$.

Let Q_j^i be the sum of $\dim_{\ell_i} P_j^i$ copies of $U(L_i)$, graded in the same way as P_j^i, so $Q_j^i/J \cong P_j^i$ as graded vector spaces over ℓ_i and we will identify these spaces. We claim that $R \otimes_k (\oplus_{i\in I}Q_j^i)$ is the jth term in a projective resolution of $U(S)$ over $R \otimes_k E$. In order to verify this we need to describe the boundary maps. To simplify the notation we write $P_j = \oplus_{i\in I}P_j^i$, $Q_j = \oplus_{i\in I}Q_j^i$ and $\otimes = \otimes_k$.

Since $R\otimes Q_0$ is projective over $R\otimes E$, the natural surjection $R\otimes Q_0 \to R\otimes P_0 \to U(S)/J$ lifts to a map $f : R \otimes Q_0 \to U(S)$. Since $U(S)$ is finitely generated in each degree we can apply Nakayama's Lemma for E-modules (2.4) to see that f is surjective. Now f is split over E because $U(S)$ is projective. Let L_0 be the kernel of f and K_0 the kernel of $R \otimes P_0 \to U(S)/J$; the splitting implies that $K_0 \cong L_0/J$.

Thus we can continue in the same way with L_0 instead of $U(S)$ and so on, lifting all the boundary maps. We obtain the minimal projective resolution of $U(S)$ over $R \otimes_k E$.

The resulting resolution is split over E, so if we apply V it is still split over kG and is the minimal relatively projective resolution that we require. $\square$

Corollary 4.3. *If R is polynomial in n variables then $\operatorname{Ext}^q_{RG/kG}(S, T)$ and $\operatorname{Tor}_q^{RG/kG}(T, S)$ vanish for $q > n$.*

Proposition 4.4. *If R is a polynomial ring and S has only finitely many isomorphism classes of indecomposable kG summands then there is a structure theorem for S over RG.*

Proof. Consider $U(S)/J$ as a module over $R \otimes E/J$. By hypothesis, the module M (from §1) is finite dimensional, say $M = \oplus_{i=1}^t M_i$ as a sum of indecomposables. Let $e_i \in E$ denote projection onto M_i.

Thus $E/J \cong \oplus_{i=1}^t \ell_i$ and $R \otimes_k (E/J) \cong \oplus_{i=1}^t (R \otimes_k \ell_i)$ is a sum of polynomial rings over ℓ_i.

Each $(U(S)/J)_i$ has a structure theorem over $R\otimes\ell_i$, by 3.1. We can sum them to obtain a structure theorem for $U(S)/J$ over $R \otimes E/J$, say $U(S)/J \cong \oplus_{I\subseteq\{1,\dots,n\}} k[d_i \mid i \in I] \otimes \bar{X}_I$. Each $\bar{X}_I$ is a sum of modules ℓ_i for various i and ℓ_i lifts to the projective E-module e_iE in such a way that $(e_iE)/J \cong \ell_i$. Thus each $\bar{X}_I$ lifts to a projective module $\bar{Y}_I$ over E such that $\bar{Y}_I/J \cong \bar{X}_I$; let $T = \oplus_{I\subseteq\{1,\dots,n\}} k[d_i \mid i \in I] \otimes \bar{Y}_I$.

Consider the map $f' : T \to T/J \cong U(S)/J$. On restriction to each $\bar{Y}_I$, it lifts to $U(S)$ over E , because $\bar{Y}_I$ is projective. Also $\operatorname{Hom}_{k[d_i|i\in I]\otimes E}(k[d_i \mid i \in I]\otimes\bar{Y}_I, -) \cong \operatorname{Hom}_E(\bar{Y}_I, -)$, so f' lifts to $f : T \to U(S)$ in such a way that $f(k[d_i \mid i \in I] \otimes \bar{Y}_I) = k[d_i \mid i \in I]f(\bar{Y}_I)$.

Thus f will give a structure theorem for $U(S)$, provided that it is an isomorphism. But f is onto, by Nakayama's Lemma over E (2.4). So f is split over E, and, since it induces an isomorphism modulo J, f must indeed be an isomorphism.

Now apply V to obtain the structure theorem for S over RG. $\square$

5. Proof of the Theorem

It is clear that (2) implies (1) and that (4) implies (1). We showed in 4.4 that (1) implies (4), and 4.2 shows that (1) implies (2).

Condition (5) allows us to express any S^r recursively in terms of the S^i with i bounded, so (5) implies (1). We can see that (2) implies (5) as follows. For any finitely generated graded RG-module M consider the Poincaré series $\mathrm{PS}_M(t) = \sum_{i\in\mathbb{Z}} M^i t^i$, with coefficients in the Green ring of kG-modules. If C_* is the resolution of (2) then $\mathrm{PS}_S(t) = \sum(-1)^i \mathrm{PS}_{C_i}(t)$. But $C_i \cong R\otimes X_i$, where X_i is finite dimensional, so $\mathrm{PS}_{C_i}(t) = \mathrm{PS}_R(t)\,\mathrm{PS}_{X_i}(t)$ and $\mathrm{PS}_{X_i}(t)$ is a Laurent polynomial. Thus $\mathrm{PS}_S(t) = \mathrm{PS}_R(t) f(t)$ for some Laurent polynomial $f(t)$.

But $\mathrm{PS}_R(t) = \prod_{i=1}^n (1-t^{\deg d_i})^{-1}$, so $\prod_{i=1}^n(1-t^{\deg d_i})\,\mathrm{PS}_S(t) = f(t)$, which is 0 in large degrees. Thus we can take a_i to be the coefficient of t^i in $-\prod_{i=1}^n(1-t^{\deg d_i})$.

The conditions of (2) imply that the resolution C_* contains only finitely many summands $R\otimes X$ and the X are finite dimensional, hence in sufficiently large degree m, $X_m = 0$ for all the X. Since $\mathrm{Hom}_{RG}(R\otimes_k X, Y) \cong \mathrm{Hom}_{kG}(X,Y)$, we see that (2) implies (3).

To finish we need to show that (3) implies (2). In fact we will prove a stronger claim, for which we need another condition. If V is any kG-module we let V^d denote this module considered as a homogeneous graded RG-module in degree d.

(3′) There is a number N such that if V any kG-summand of any S^r then $\mathrm{Ext}^*(S, R\otimes_k V^d)^0 = 0$ for $d \geq N$.

Clearly (3) implies (3′), so we prove that (3′) implies (2). In order to achieve this we show that the minimal resolution C_* from 4.2 contains no term C_r that has a summand of the form $R\otimes V$ with $\deg V \geq N$. Since C_r is finite dimensional in each degree, this implies that it is finitely generated over R.

Our proof is by downward induction on r. Since the resolution is of finite length the induction certainly starts.

Let R^+ denote the part of R in positive grading. By the induction hypothesis, we know that the image of the boundary map $\mathrm{Im}(d_{r+1})$ is generated over R by elements in degrees less than N.

Suppose that C_r contains a summand $R\otimes_k V$ with $\deg V = d \geq N$, so $S = (R\otimes V)\oplus(R\otimes W)$; let c be the projection onto $R\otimes V$. Then $cd_{r+1} = 0$, by the remark about $\mathrm{Im}(d_{r+1})$ above, so c is a cocycle; since $\mathrm{Ext}^*_{RG/kG}(S, R\otimes V)^0 = 0$, c must be a coboundary.

Thus c factors through $d_r : C_r \to C_{r-1}$, so $d_r(R\otimes_k V)$ is a summand of C_{r-1} isomorphic to $R\otimes V$ and we obtain a smaller resolution $\cdots \to C_{r+1} \to C_r/(R\otimes_k V) \to C_{r-1}/d_r(R\otimes_k V) \to C_{r-2} \to \cdots$, a contradiction.

This completes the proof of the Theorem.

6. Further Conditions

We now consider some further equivalent conditions.

Let I denote the injective hull of $\bar{k}$ as a k-module; we consider it to have grading 0. For any graded left RG-module T, the graded dual $T^* = \mathrm{Hom}_k(T, I)$ naturally carries the structure of a graded right RG-module by $(frg)(t) = f(rgt)$, $f \in T^*$, $r \in R$, $g \in G$. Notice that our grading conventions mean that the degree r part of T^* is the dual of T^{-r}. Similarly, starting with a right module we can dualise to obtain a left module.

We remark that if $T^* = 0$ then $T = 0$ and also that if T is finitely generated over k in each degree then $T \cong T^{**}$, since k is complete.

Using this duality we can formulate a close relationship between Ext and Tor, where S is a left- and T is a right- graded RG-module

$$\mathrm{Ext}^r_{RG/kG}(S, T^*) \cong \mathrm{Tor}_r^{RG/kG}(T, S)^*.$$

The proof is quite formal and is left to the reader (cf. [1] 2.8.5).

Proposition 6.1. *The following conditions are also equivalent to those of Theorem 1.1, where V^d is as in condition $(3')$.*

$(3'')$ *There is an $N \in \mathbb{Z}$ such that $\mathrm{Ext}^*_{RG/kG}(S, V^d)^0 = 0$ for $d \geq N$ for any V that is a summand of some S^r.*

(6) *There is an $N \in \mathbb{Z}$ such that $\mathrm{Tor}_*^{RG/kG}(T, S)^0 = 0$ for any graded right RG-module T that is 0 in degrees greater than N.*

$(6')$ *There is an $N \in \mathbb{Z}$ such that $\mathrm{Tor}_*^{RG/kG}((V^*)^d, S)^0 = 0$ for $d \leq N$ and for any V that is a summand of some S^r.*

Proof. For convenience we also consider another condition.

$(6'')$ There is an $N \in \mathbb{Z}$ such that $\mathrm{Tor}_*^{RG/kG}(T, S)^0 = 0$ for any graded right RG-module T that is finitely generated over k in each degree and is 0 in degrees greater than N.

Clearly (6) implies $(6'')$, which in turn implies $(6')$. The duality formula above shows that (3) implies (6), that $(6'')$ implies $(3')$ and that $(3'')$ is equivalent to $(6')$.

The proof of 1.1 actually showed that $(3')$ implied (2), so it is sufficient to show that $(6')$ implies $(6'')$.

We assume $(6')$ and let $T = \oplus_{i=-\infty}^N T^i$, where each T^i is finitely generated over k and N is as in condition $(6')$. Thus each $\mathrm{Tor}_*^{RG/kG}(T^i, S)^0 = 0$.

Let $T^{\geq r} = \oplus_{i=r}^N T^i$. It is easy to show, by downward induction on r, that $\mathrm{Tor}_*^{RG/kG}(T^{\geq r}, S)^0 = 0$, using the long exact sequence for Tor corresponding to the short exact sequence $0 \to T^{\geq r+1} \to T^{\geq r} \to T^r \to 0$.

But $T = \mathrm{colim}_{r \to -\infty} T^{\geq r}$, the direct limit, so $\mathrm{Tor}_*^{RG/kG}(T, S)^0 = \mathrm{colim}_{r \to -\infty} \mathrm{Tor}_*^{RG/kG}(T^{\geq r}, S)^0 = 0$, since Tor commutes with direct limits. □

It is easy to see that condition (2) of the Theorem implies that $\mathrm{Ext}^*_{RG/kG}(S, S)$ is finitely generated over R. In certain circumstances we can prove the converse.

We say that S is repetitive of width D if there are integers N, D such that if an indecomposable kG-module V is a summand of some S^r for $r \geq N$ then V is a summand of $\oplus_{j=s}^{s+D-1} S^j$ for any $s \geq N$. Being repetitive of some width is a necessary condition for there to be a structure theorem that is often clearly satisfied in examples. If S is a polynomial ring then there is a $v \in S^G$ and a kG-submodule T of S such that $S \cong T \otimes k[v]$ as a kG-module, by [9] 6.3, so we can take $N = 0$ and $D = \deg v$. Alternatively, we could always adjoin an invariant element d_{n+1} and consider $S \otimes_k k[d_{n+1}]$ as a module over $R[d_{n+1}]$.

Proposition 6.2. *Suppose that k is artinian and that S is repetitive of width D. If $\mathrm{Ext}^*_{RG/kG}(S,S)^{-d}$ is finitely generated over k for $d = 0, \dots, D-1$ (so, for example, if $\mathrm{Ext}^*_{RG/kG}(S,S)$ is finitely generated over R) then S has a structure theorem over RG.*

Proof. Let $S^{\geq r} = \oplus_{i \geq r} S^i$, $\;S^{<r} = S/S^{\geq r} \cong \oplus_{i<r} S^i$ and similarly for other graded modules or inequality signs. (For the second of these the R-module structure is defined by the term in the middle.)

There is a short exact sequence of RG-modules $0 \to S^{\geq r} \to S \to S^{<r} \to 0$, which is split over kG.

As the first step we will prove that there is a number M such that $\mathrm{Ext}^*_{RG/kG}(S, S^{\geq r})^{-d} = 0$ and $\mathrm{Ext}^*_{RG/kG}(S, S^r)^{-d} = 0$ for $r \geq M$ and $d = 1, \dots, D-1$.

In fact, we will prove the dual statement, that $\mathrm{Tor}_*^{RG/kG}(S^{*\leq -r}, S)^d = 0$ and $\mathrm{Tor}_*^{RG/kG}(S^{*-r}, S)^d = 0$ for $r \geq M$ and $d = 1, \dots, D-1$.

But $\mathrm{colim}_{r\to\infty} \mathrm{Tor}_*^{RG/kG}(S^{*\leq -r}, S)^d = \mathrm{Tor}_*^{RG/kG}(\mathrm{colim}_{r\to\infty} S^{*\leq -r}, S)^d = 0$ and, since the k-module $\oplus_{d=0}^{D-1} \mathrm{Tor}_q^{RG/kG}(S^*, S)^d$ is finitely generated and hence artinian, there is a number M such that $\mathrm{Im}(\mathrm{Tor}_*^{RG/kG}(S^*, S)^d \to \mathrm{Tor}_*^{RG/kG}(S^{*\leq -r}, S)^d) = 0$ for $r \geq M$ and $d = 0, \dots, D-1$.

The long exact sequence for Tor associated to the short exact sequence above decomposes, for the same range of r and d, into short exact sequences $0 \to \mathrm{Tor}_{q+1}^{RG/kG}(S^{*\leq -r}, S)^d \to \mathrm{Tor}_q^{RG/kG}(S^{*>-r}, S)^d \to \mathrm{Tor}_q^{RG/kG}(S^*, S)^d \to 0$. We claim, by downward induction on q, that $\mathrm{Tor}_i^{RG/kG}(S^{*\leq -r}, S)^d = 0$ and $\mathrm{Tor}_i^{RG/kG}(S^{*-r}, S)^d = 0$ for $r \geq M$, $i \geq q$ and $d = 1, \dots, D-1$.

The induction starts, by 4.3, so we suppose that we know the claim for $q+1$ and we prove it for q. The short exact sequence above shows that $\mathrm{Tor}_i^{RG/kG}(S^{*>-r}, S)^d \cong \mathrm{Tor}_i^{RG/kG}(S^*, S)^d$ for $r \geq M$ and $i \geq q$. It follows that $\mathrm{Tor}_i^{RG/kG}(S^{*>-r}, S)^d \cong \mathrm{Tor}_i^{RG/kG}(S^{*>-r-1}, S)^d$ for $i \geq q$ and $r \geq M$.

The long exact sequence for Tor associated to the short exact sequence $0 \to S^{*-r} \to S^{*>-r-1} \to S^{*>-r} \to 0$ now yields $\mathrm{Tor}_q^{RG/kG}(S^{*-r}, S)^d = 0$ for $r \geq M$ and $d = 0, \dots, D-1$. An easy induction argument shows that for $t \leq -r$ we have $\mathrm{Tor}_q^{RG/kG}((S^{*\leq -r})^{\geq t}, S)^d = 0$. Taking the limit as $t \to -\infty$ yields that $\mathrm{Tor}_q^{RG/kG}(S^{\leq -r}, S)^d = 0$, completing the induction and the proof of the first step.

If V is an indecomposable kG module we will write V^r for this module considered as a graded RG-module in degree r, as in the statement of condition $(3')$. If V is a summand of S^r for some $r \geq M$ then, by what we have just shown, $\mathrm{Ext}^*_{RG/kG}(S, V^r)^{-d} = 0$ for $d = 0, \dots, D-1$. But then $\mathrm{Ext}^*_{RG/kG}(S, V^{r+d})^0 = \mathrm{Ext}^*_{RG/kG}(S, V^r)^{-d} = 0$, so, since S is repetitive, if we take $L \geq M, N$ (where N comes from the definition of repetitive) then $\mathrm{Ext}^*_{RG/kG}(S, V^r)^0 = 0$ for all $r \geq L$.

But $S^{<L}$ certainly satisfies condition (1), so, by condition $(3'')$, there is a $K \geq L$ such that $\mathrm{Ext}^*_{RG/kG}(S^{<L}, V^r)^0 = 0$ for $r \geq K$. The long exact sequence for $0 \to S^{\geq L} \to S \to$

$S^{<L} \to 0$ now shows that $\mathrm{Ext}^*_{RG/kG}(S^{\geq L}, V^r)^0 = 0$ for $r \geq K+1$ and any V a summand of $S^{\geq M}$, so certainly for any V a summand of $S^{\geq L}$.

Thus $S^{\geq L}$ satisfies condition (3″), hence condition (1), and S itself must satisfy condition (1). □

7. When k is Not a Field

For more general rings k we only have results about when a ring S satisfies the conditions of the Theorem in the case when $n \leq 2$. We still require k to be complete. Recall that $\bar{k}$ denotes the residue class field of k.

Proposition 7.1. *Let S be a finitely generated graded k-algebra that is free over k and let a finite group G act on S by grading preserving k-algebra automorphisms. Suppose that G acts faithfully on $\bar{S} = \bar{k} \otimes_k S$, that $\bar{S}$ is an integral domain and that there is a bound on $\dim_{\bar{k}} \bar{S}^r$ that is linear in r.*

Then there are two algebraically independent homogeneous elements $d_1, d_2 \in S^G$ (possibly only one or zero if the bound above is constant) such that S is finitely generated over $R = k[d_1, d_2]$ and S satisfies the conditions of the theorem.

Proof. The existence of suitable elements $\bar{d}_1, \bar{d}_2 \in \bar{S}^G$ of positive degree is well known (see e.g. [2] 1.3.1, 2.2.7). If there are none of them then S is finitely generated over k, so the claim holds.

If there is only one, say d, then we lift it arbitrarily to a homogeneous element $d' \in S$ and define $d = \prod_{g\in G} gd' \in S^G$. Then S is finitely generated over $k[d]$, by Nakayama's Lemma applied in each degree.

Consider the multiplication map $S^r \xrightarrow{d} S^{r+\deg d}$. Because S is finitely generated over $k[d]$, this map must be surjective for large enough r. Also the kernels of the maps $S^r \xrightarrow{d^t} S^{r+t\deg d}$ form an ascending chain of submodules of S^r as t increases, so must eventually stabilise. These two facts together imply that the multiplication map is an isomorphism for large enough r. This implies that S contains only a finite number of isomorphism classes of indecomposable modules.

Now we assume that there are two elements $\bar{d}_1, \bar{d}_2$; we lift them arbitrarily to homogeneous elements of $d_1', d_2' \in S$ and then define $d_1, d_2 \in S^G$ by $d_i = \prod_{g\in G} gd_i'$.

According to [10], $\bar{S}$ is mostly projective over $\bar{k}G$ in the sense that it has a projective summand $\bar{P}$ over $\bar{k}G$ such that the multiplication map $\bar{S}^G \otimes_{\bar{k}} \bar{P} \to \bar{S}$ is injective, its image $\bar{S}^G\bar{P}$ is a $\bar{k}G$-summand and the complement has a uniform bound on the dimension in each degree.

Again, it is well known that, because of this bound, we can find a homogeneous element $\bar{v} \in \bar{S}^G$ such that $\bar{S}/\bar{S}^G\bar{P}$ is finitely generated over $\bar{k}[\bar{v}]$.

Let $\bar{U}$ be a complement to $\bar{v}^{|G|}\bar{S}^G$ in $\bar{S}^G$ as a $\bar{k}$-module, so $\bar{U}$ is finitely generated. Then $\bar{S}^G \cong \oplus_{r=0}^{\infty} \bar{v}^{r|G|}\bar{U} \cong \bar{k}[\bar{v}^{|G|}] \otimes_{\bar{k}} \bar{U}$, since $\bar{S}$ is an integral domain and the sum is finite in each degree. Also $\bar{U}\bar{P} \cong \bar{U} \otimes_{\bar{k}} \bar{P}$.

Let P be the projective cover of $\bar{U}\bar{P}$ over kG. The map $P \to \bar{U}\bar{P} \to \bar{S}$ must lift to a map $P \to S$. Lift $\bar{v}$ arbitrarily to a homogeneous element v' of S and set $v = \prod_{g\in G} gv' \in S^G$. Multiplication yields a map $k[v] \otimes_k P \to S$. We denote its image by $P[v]$, so the image of $P[v]$ in $\bar{S}$ is $\bar{S}^G \otimes_{\bar{k}} \bar{P}$.

Let $\bar{C}$ be a complement to $\bar{S}^G \otimes_{\bar{k}} \bar{P}$ in $\bar{S}$ as a $\bar{k}$-module and denote its projective cover over k by C, so we obtain a map $C \to S$.

The combined map $P[v] \oplus C \to S$ is surjective, by Nakayama's Lemma, and S is free over k so the map splits. But after reducing to $\bar{k}$ this map is an isomorphism, by construction, so the kernel is 0 after reduction, hence 0 itself, thus $S \cong P[v] \oplus C$. The same argument now applies to the multiplication map $k[v] \otimes_k P \to P[v]$, so $P[v] \cong k[v] \otimes_k P$ and, in particular, $P[v]$ is projective over kG. But a projective kG-module is injective relative to k and $P[v]$ is certainly a summand of S over k, so $P[v]$ must be a summand of S over kG. Thus any indecomposable kG-summand of S is either projective or must be a summand of $S/P[v]$.

But $S/P[v]$ is finitely generated over $k[v]$, so the case of just one invariant element applies. □

Notice how the proof above almost yields a structure theorem for S directly, but it might not be possible to choose v to be either one of a given pair d_1, d_2.

Other results, which concern regular functions on projective varieties, particularly curves and surfaces, are given in [3].

We have already seen in section 2 that we do not always need k to be complete for the general theory to apply. Let k be a discrete valuation ring in a field in which the order of G is not 0 and let $\hat{k}$ denote its completion. In fact, if we can show that $\hat{k} \otimes S$ has a structure theorem over $\hat{k} \otimes RG$ then S has a structure theorem over RG.

This can be seen using Maranda's Theory. Condition (5) is of the form $M \cong N$ for certain modules M and N. By [4] 30.17, if this is true after completion then it was before. ([4] 30.17 is stated for lattices, but it easily extends to finitely generated modules.)

Similarly, in order to show that S has a structure theorem it is sufficient to prove it only for $(k/|G|^r k) \otimes S$ for sufficiently large r, i.e. in the artinian case. This follows from [4] 30.14. This is relevant even when k is complete.

8. Composition Factors

If k is a field and V is a simple kG-module then the multiplicity of V as a composition factor of S^r is $(\dim_k \mathrm{End}_{kG}(V))^{-1} \dim_k \mathrm{Hom}_{kG}(P_V, S^r)$, where P_V is the projective cover of V. Since $\mathrm{Hom}_{kG}(P_V, S)$ is a summand of $\mathrm{Hom}_{kG}(kG, S) \cong S$ and hence finitely generated over R, these multiplicities are described by polynomial functions (cf. [10]).

If k is not required to be a field then we can consider the Grothendieck group $G_0(kG)$, where the generators are the isomorphism classes of finitely generated kG-modules M, denoted by $[M]$, and the relations are $[B] = [A] + [C]$ for each short exact sequence $0 \to A \to B \to C \to 0$.

For any graded kG-module $M = \oplus_{i \in \mathbb{Z}} M^i$ with each M_i finitely generated over k we can consider $\sum_{i \in \mathbb{Z}} [M^i] t^i \in G_0(kG)^{\mathbb{Z}}$, which we also denote by $[M]$. Thus if $R = k[d_1, \ldots, d_n]$ then $[R] = \prod_{i=1}^n (1 - t^{\deg(d_i)})^{-1}[k]$, and we write $p_R(t) = \prod_{i=1}^n (1 - t^{\deg(d_i)})^{-1}$.

If M is a graded kG-module with $M_i = 0$ for $i \ll 0$ then it makes sense to multiply $[M]$ by a power series in t with coefficients in $\mathbb{Z}$, such as $p_R(t)$.

In [3] the question is posed of describing $[S]$ in $G_0(kG)^{\mathbb{Z}}$, and this is answered in the geometric context employed there.

Proposition 8.1. *Let k be any commutative noetherian ring, $R = k[d_1, \dots, d_n]$ with the d_i of positive degree and S a finitely generated graded RG-module.*

Then there is an $X \in G_0(kG)^{\mathbb{Z}}$ that is 0 *except in finitely many degrees and such that $[S] = p_R(t)[X]$.*

If k is a field then this is an easy consequence of the discussion at the beginning of this section.

Proof. First we claim that S has a finite filtration by RG-submodules where the composition factors M_i are such that to each M_i is associated a graded prime ideal $p_i < R$ and $\mathrm{Ann}_R(m) = p_i$ for all non-zero elements $m \in M_i$. Since R is noetherian we only need to show that any finitely generated graded RG-module L has such a submodule.

Consider all the ideals $\mathrm{Ann}_R(\ell)$ for homogeneous $0 \neq \ell \in M$ and let p be maximal among them. Then p is prime, as in 3.3, and $M = \{m \in L \mid p \leq \mathrm{Ann}_R(m)\}$ is the desired submodule.

Thus it is sufficient to prove the theorem for modules of this form and we do so by induction on the number of polynomial generators n.

If p contains all the d_i then M is finitely generated over k and we can take $[X] = p_R^{-1}(t)[M]$, since $p_R^{-1}(t)$ is a polynomial. Otherwise p does not contain d_1, say. Then d_1 is regular on M and since $[M] = [M/d_1M] + [d_1M] = [M/d_1M] + t^{\deg d_1}[M]$, we obtain $[M] = (1 - t^{\deg d_1})^{-1}[M/d_1M]$. But M/d_1M is finitely generated over $k[d_2, \dots, d_n]$ and by induction we obtain $[M/d_1M] = p_{k[d_2,\dots,d_n]}(t)[X]$, so $[M] = p_R(t)[X]$. □

Notice that the constructions in the proof do not require any knowledge of the action of G.

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