THE BREDON COHOMOLOGY OF SUBGROUP COMPLEXES

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Abstract. We develop the homological algebra of coefficient systems on a group, in particular from the point of view of calculating higher limits. We show how various sequences of modules associated to a class of subgroups of a given group can be analysed by methods from homological algebra. We are particularly interested in when these sequences are exact, or if not, when their homology is equal to the higher limits of the coefficient system.

1. Introduction

This paper is concerned with the homological algebra of coefficient systems on a class of subgroups of a group G . It is partly structured around the investigation of three sequences associated to some class $\mathcal{X}$ of subgroups of G, in particular their cohomology.

The three sequences of particular interest are:

$$L(G) \to \prod_{\sigma \in \mathrm{ch}_0(\mathcal{X})/G} L(N_G(\sigma)) \to \prod_{\sigma \in \mathrm{ch}_1(\mathcal{X})/G} L(N_G(\sigma)) \to \dots$$

$$L(G) \to \prod_{\sigma \in \mathrm{ch}_0(\mathcal{X})/G} L(\sigma_b)^{N_G(\sigma)} \to \prod_{\sigma \in \mathrm{ch}_1(\mathcal{X})/G} L(\sigma_b)^{N_G(\sigma)} \to \dots$$

$$L(G) \to \prod_{\sigma \in \mathrm{ch}_0(\mathcal{X})/G} L(C_G(\sigma_t))^{N_G(\sigma)} \to \prod_{\sigma \in \mathrm{ch}_1(\mathcal{X})/G} L(C_G(\sigma_t))^{N_G(\sigma)} \to \dots$$

Here $\mathrm{ch}_n(\mathcal{X})$ denotes the set of chains in $\mathcal{X}$ (without repetition) of length $n+1$. The smallest element of a chain σ is denoted σ_b and the largest by σ_t.

The first of these sequences was investigated by Webb [24], when L is a Mackey functor, and is by now well known. The second sequence first appeared in work of Bouc [5], again for Mackey functors. The preprint dates from 1991, but remained unpublished until 1998. An infinite version of the third sequence is implied by results of Jackowski and McClure [13]. Later Dwyer [10] had infinite versions of the second and third sequences, which arose from topology, and he investigated the properties of all three. Following him, we will sometimes refer to these as the normaliser, subgroup and centraliser sequences. Finally the version of the third sequence given above (and also the second) appeared in work of Grodal [12] and of Villarroel-Flores and Webb [23].

The original interest was in conditions on L and $\mathcal{X}$ which forced these sequences to be exact. But when they are not exact, it turns out that their cohomology can often be described as the higher limits of L, and these are often of interest in their own right.

Our aim is to present a unified treatment of all the results entirely within the homological algebra of coefficient systems. The only geometric results used are some standard ones about the fixed point sets of a group acting on some subgroup complex, and then only for examples.

A non-geometric but more category-theoretic treatment of some of these results by Jackowski and Słomińska has recently appeared [15].

Our strategy is to show that each these sequences is representable as the complex of homomorphisms from some complex of coefficient systems $\tilde{C}_\bullet$ to L, in other words that the sequence is of the form $\mathrm{Hom}(\tilde{C}_\bullet, L)$. It then turns out that $\tilde{C}_\bullet$ is homotopy equivalent to a projective resolution of the trivial coefficient system $\bar{R}$. Thus the cohomology of our sequence is $\mathrm{Ext}^*(\bar{R}, L)$, which is the definition of the higher limits of L.

When these sequences are not exact then their cohomology is usually equal to the higher limits of L. This insight was developed by Grodal in [12], and the later sections of the present paper were inspired by his work, being essentially an attempt to formulate the geometric proofs given there in algebraic terms.

Many statements in group theory can be phrased succinctly in terms of higher limits. For example Robinson's reformulation of Alperin's Weight Conjecture (§5) and Quillen's conjecture on the contractibility of subgroup complexes (§11).

We use a lot of basic results about coefficient systems, in particular we make great use of the adjoint functors of several forgetful functors and their properties, and these are collected together in §2.

2. Coefficient Systems

Here we collect together some constructions on coefficient systems and Mackey functors and record their properties.

We will always work over a fixed unital ring R and refer to a fixed prime p.

We have been careful wherever possible to allow infinite groups in the basic definitions, although this is not a direction that we pursue here, and it is abandoned later, when we need a Sylow p-subgroup.

For a given group G, we will consider various classes of subgroups, assumed to be closed under conjugation. For example $\mathcal{S}(G)$, the class of all subgroups; $\mathcal{S}_p(G)$, the class of finite p-subgroups; or $\mathcal{A}_p(G)$, the class of finite elementary abelian p subgroups. We will often omit G from the notation. The superscript 1 will be used to denote the given class with the trivial subgroup removed.

Many of our results are phrased in terms of adjoint properties. Recall that if A and B are two categories then two functors $L : B \to A$ and $R : A \to B$ are adjoint if and only if there exist two natural transformations, the unit $\eta : I_B \to RL$ and the counit $\epsilon : LR \to I_A$, such that the compositions $(R\epsilon)(\eta R) : R \to R$ and $(\epsilon L)(L\eta) : L \to L$ are both the identity. Our proofs will usually consist of giving explicit formulas for η and ϵ and leaving to the reader the straightforward task of checking these identities. We will also omit many sub- and superscripts where this simplifies the formulas.

Note that these adjoint functors are known to exist for abstract reasons and can be defined in much greater generality, but we want explicit formulas so that we can investigate their properties.

When we refer to results or proofs in the literature the authors of these results usually assume that $\mathcal{W}$ and any other class of groups are either $\mathcal{S}$ or $\mathcal{S}_p$, but the change to general $\mathcal{W}$ does not present any difficulties. They also tend to assume that G is finite and here again the generalisation is straightforward in the cases mentioned, except that we have to be careful to distinguish $\oplus$ and Π.

For a given class $\mathcal{W}$ of subgroups of a group G we construct two categories $S_\mathcal{W}$ and $T_\mathcal{W}$. They both have the elements of $\mathcal{W}$ as objects. The morphisms are given as follows:

$$S_\mathcal{W}(H,K) = \{b_{g,H,K} | g \in G, H^g \leq K\},$$
$$T_\mathcal{W}(H,K) = H\backslash S_\mathcal{W}(H,K).$$

The composition law is $b_{h,K,L}b_{g,H,K} = b_{gh,H,L}$.

A weak coefficient system is an object of the category $\mathrm{WCS}_{\mathcal{W}}(G)$ of contravariant functors from $S_{\mathcal{W}}$ to $R-\mathrm{Mod}$, and a coefficient system is an object of the category $\mathrm{CS}_{\mathcal{W}}(G)$ of contravariant functors from $T_{\mathcal{W}}$ to $R-\mathrm{Mod}$. The morphisms are the natural transformations of functors and are denoted by $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}$. This is equivalent to defining $\mathrm{CS}_{\mathcal{W}}(G)$ to be the category of contravariant functors from G-sets with stabilisers in $\mathcal{W}$ to $R-\mathrm{Mod}$, by setting the value of $C \in \mathrm{CS}_{\mathcal{W}}(G)$ on the G-set G/H to be $C(H)$ and in general taking the value of C on an arbitrary G-set to be the direct sum of its values on the orbits. (See [7], and also [25] for the similar case of Mackey functors.)

We write c_{g,H,H^g}, or just c_g, for the map of R-modules $C(H^g) \to C(H)$ induced by b_{g,H,H^g}, and call these the conjugations. Then $c_g c_h = c_{gh}$, and for any weak coefficient system C, this makes $C(H)$ into a left $RN_G(H)$-module, and if C is a coefficient system then H acts trivially, so $C(H)$ is naturally an $RN_G(H)/H$-module. We also write res^K_H for the map induced by $b_{e,K,H}$, and call these restrictions. Since $b_{g,H,K} = b_{e,H^g,K}b_{g,H,H^g}$, we see that it is enough to check identities for conjugations and restrictions only.

The forgetful functor from coefficient systems to weak coefficient systems has a right adjoint given by taking invariants under H at each evaluation $C(H)$, and a left adjoint given by taking coinvariants.

A coefficient system $C \in \mathrm{CS}_{\mathcal{W}}(G)$ is called *geometric* if, for all $H \in \mathcal{W}$, $C_G(H)$ acts trivially on $C(H)$. This occurs, for instance, if C is the restriction to G of a global coefficient system or Mackey functor. The full subcategory of geometric objects in $\mathrm{CS}_{\mathcal{W}}(G)$ will be denoted by $\mathrm{GCS}_{\mathcal{W}}(G)$. The inclusion $I : \mathrm{GCS}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{W}}(G)$ has both left and right adjoints, G_0 and G^0 respectively. They are formed by taking the largest quotient (respectively largest sub coefficient system) that is geometric. G_0 has the explicit description $(G_0L)(P) = H_0(C_G(P)/Z(P), L(P))$.

We will also occasionally mention primitive coefficients systems (PCS), which have no conjugations at all, only restrictions.

For a finite group, each of these categories of coefficient systems is equivalent to the category of modules for some finite rank R-algebra. In this way we can import various results from the theory of representations of algebras: for example if R is a complete local ring then we have the Krull-Schmidt property.

If $H \leq G$ there is a forgetful map $\mathrm{Res}^G_H : \mathrm{CS}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{W}}(H)$ (we should really write $\mathrm{CS}_{\mathcal{W}\cap\mathcal{S}(H)}(H)$).

There is an obvious concept of tensor product $\otimes : \mathrm{CS}_{\mathcal{W}}(G) \times \mathrm{CS}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{W}}(G)$, defined groupwise by $(M \otimes N)(J) = M(J) \otimes N(J)$ and the obvious restriction and conjugation maps.

Given $L, M \in \mathrm{CS}_{\mathcal{W}}(G)$, an element of $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}$ can be considered as a collection of maps of R-modules $L(H) \to M(H)$ for each $H \in \mathcal{W}$ that commute with the restriction and conjugation maps. Notice that $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(L, M)$ is naturally an R-module.

We can extend $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}$ to a pairing $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)} : \mathrm{CS}_{\mathcal{W}}(G) \times \mathrm{CS}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{T}}(G)$ for any class $\mathcal{T}$, given by $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(M, N)(J) = \hom_{\mathrm{CS}_{\mathcal{W}}(J)}(\mathrm{Res}^G_J M, \mathrm{Res}^G_J M)$. The restrictions are just the forgetful maps, and the conjugations are the usual ones, $c_g(f) = c_g f c_g^{-1}$.

Thus Hom takes its values in coefficient systems. We will assume that $\mathcal{T} = \mathcal{W}$ unless otherwise indicated. Notice that if we take $\mathcal{T} = \{G\}$ then we recover hom.

We will often abbreviate $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}$ to $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}$.

There is the usual adjunction:

Lemma 2.1. *For $L, M, N \in \mathrm{CS}_{\mathcal{W}}(G)$ we have*
$\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(L \otimes M, N) \cong \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(L, \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(M, N))$.

Proof. The isomorphism Φ is given by $((\Phi_H f)_J(l))_I(m) = f_I(\mathrm{res}_I^J\, l \otimes m)$, for $H \geq J \geq I,\ f \in \mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(L \otimes M, N)(H),\ m \in M(I)$. □

When $\mathcal{X} \subseteq \mathcal{W}$ then there is the forgetful functor $\mathrm{Res}_\mathcal{X}^\mathcal{W} : \mathrm{CS}_\mathcal{W}(G) \to \mathrm{CS}_\mathcal{X}(G)$. There are functors $\varprojlim_\mathcal{X}^\mathcal{W}$ and $\varinjlim_\mathcal{X}^\mathcal{W}$ in the opposite direction. Now $\varprojlim_\mathcal{X}^\mathcal{W} L$ is obtained on $H \in \mathcal{W}$ by taking the inverse limit of $L(I)$ for $I \in \mathcal{X} \cap \mathcal{S}(H)$: the limit is over all inclusions and conjugations in H. We define $\varinjlim_\mathcal{X}^\mathcal{W} L(H)$ to be the direct limit of the $L(J)$ for all $J \in \mathcal{X}, J \geq H$ where the limit is taken over inclusions only.

Proposition 2.2. *$\varinjlim_\mathcal{X}^\mathcal{W}$ is the left adjoint and $\varprojlim_\mathcal{X}^\mathcal{W}$ is the right adjoint of* $\mathrm{Res}_\mathcal{X}^\mathcal{W} : \mathrm{CS}_\mathcal{W}(G) \to \mathrm{CS}_\mathcal{X}(G)$.

Proof. For $\varprojlim$, we have $\eta_M : M \to \varprojlim \mathrm{Res}\, M$ is the identity on subgroups in $\mathcal{X}$, and this extends uniquely by the definition of $\varprojlim$. Then $\epsilon_N : \mathrm{Res} \varprojlim N \to N$ is the identity.

For $\varinjlim$, we have $\eta_M : M \to \mathrm{Res} \varinjlim M$ is the identity, and $\epsilon_N : \varinjlim \mathrm{Res} \to N$ follows from the definition of $\varinjlim$. □

Clearly $\varprojlim$ and $\varinjlim$ are transitive.

Notice that from the definition of hom we find that $\mathrm{hom}_{\mathrm{CS}_\mathcal{W}(G)}(L, M) = (\varprojlim_\mathcal{W}^\mathcal{S} \mathrm{Hom}_{\mathrm{CS}_\mathcal{W}(G)}(L, M))(G)$, where we regard $\mathrm{Hom}_{\mathrm{CS}_\mathcal{W}(G)}(L, M)) \in \mathrm{CS}_\mathcal{W}(G)$. This observation and the previous proposition now yield:

Lemma 2.3. *If $\mathcal{W} \subseteq \mathcal{V}$ and $L, M \in \mathrm{CS}_\mathcal{W}(G)$ then $\mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(L, M) \cong \varprojlim_\mathcal{W}^\mathcal{V} \mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(L, M)$ in $\mathrm{CS}_\mathcal{V}(G)$ (but we regard $\mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(L, M) \in \mathrm{CS}_\mathcal{W}(G)$) .*

Also, for $N \in \mathrm{CS}_\mathcal{V}(G)$ we have $\mathrm{Hom}_{\mathrm{CS}_\mathcal{V}}(N, \varprojlim_\mathcal{W}^\mathcal{V} M) \cong \mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(\mathrm{Res}_\mathcal{W}^\mathcal{V} N, M)$ and $\mathrm{Hom}_{\mathrm{CS}_\mathcal{V}}(\varinjlim_\mathcal{W}^\mathcal{V} M, N) \cong \mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(M, \mathrm{Res}_\mathcal{W}^\mathcal{V} N)$, both in $\mathrm{CS}_\mathcal{S}(G)$.

If we denote the constant coefficient system by $\bar{R}$, then the following is a consequence of the definition of $\varprojlim$.

Lemma 2.4. *$\mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(\bar{R}, L) \cong \varprojlim_\mathcal{W}^\mathcal{S} L$ in $\mathrm{CS}_\mathcal{S}(G)$.*

Proof. $\mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(\bar{R}, L) \cong \mathrm{Hom}_{\mathrm{CS}_\mathcal{S}}(\bar{R}, \varprojlim_\mathcal{W}^\mathcal{S} L) \cong \varprojlim_\mathcal{W}^\mathcal{S} L$, by taking the image of $1 \in \bar{R}(H)$, where H is the group that we are evaluating on. □

For convenience we denote $\varprojlim_\mathcal{W}^G L = (\varprojlim_\mathcal{W}^\mathcal{S} L)(G)$, i.e. the usual inverse limit of L. By a *component* of $\mathcal{W}$ we mean an equivalence class under the equivalence relation generated by inclusion.

Lemma 2.5. *In $\mathrm{CS}_\mathcal{W}(G)$, $\bar{R}$ is projective if and only if each component of $\mathcal{W}$ has a unique maximal element M, say, and $|N_G(M) : M|$ is finite and invertible in R.*

When the conditions of this lemma are satisfied we say that G is *tight* with respect to $\mathcal{W}$.

Proof. It follows from 2.4 that $\bar{R}$ is projective if and only if the functor $L \mapsto \varprojlim_\mathcal{W}^G L$ is exact on $\mathrm{CS}_\mathcal{W}(G)$.

If the conditions involving M are satisfied then we claim that $\varprojlim_\mathcal{W}^G L$ is the sum over the conjugacy classes of maximal elements M in $\mathcal{W}$ of $L(M)^{N_G(M)/M}$. From this it will follow that $\varprojlim_\mathcal{W}^G$ is exact, since for any group A the functor $X \mapsto X^A$ is exact on RA-modules if and only if $|A|$ is finite and invertible in R.

This claim is equivalent to the one that $\varprojlim_\mathcal{W}^G L \cong (\prod_M L(M))^G$, where M runs over all maximal elements of $\mathcal{W}$. We will denote the right hand side by X and indicate the components of $x \in X$ by $x = (x_M)$. There is a family of maps $\phi_H : X \to L(H)$ for $H \in \mathcal{W}$ defined by $\phi_H(x) = \mathrm{res}_H^M\, x_M$, where M is the unique

maximal element of $\mathcal{W}$ containing H (the uniqueness is one of the conditions). These maps are compatible with restriction and conjugation, and it is easy to see that any other such compatible family of maps must factor through X, proving the claim.

Conversely, if $\bar{R}$ is projective then, by considering coefficient systems that are non-zero only on a maximal element M and its conjugates, we see that taking fixed points in $R[N_G(M)/M]$-modules must be exact, so the condition on the index must hold. If the other condition is not met then it easy to see that we can find two different maximal elements of $\mathcal{W}$, M and N say, such that $M \cap N$ contains an element U of $\mathcal{W}$.

Let $\tilde{R} \in \mathrm{CS}_{\mathcal{W}}(G)$ take the value R on the conjugacy classes of M and N (which we denote by $\langle M\rangle$ and $\langle N\rangle$), and 0 elsewhere. There is a surjection $\varinjlim^{\mathcal{W}}_{\{\langle M\rangle,\langle N\rangle\}} \bar{R} \to \tilde{R}$. Now if we apply $\varprojlim^{G}_{\mathcal{W}}$ then, by the observation at the beginning of this proof, we obtain another surjection $\varprojlim^{G}_{\mathcal{W}} \varinjlim^{\mathcal{W}}_{\{\langle M\rangle,\langle N\rangle\}} \bar{R} \to \varprojlim^{G}_{\mathcal{W}} \tilde{R}$.

The domain of this surjection is 0, because the images of the restrictions from M and N to $(\varinjlim^{\mathcal{W}}_{\{\langle M\rangle,\langle N\rangle\}} \bar{R})(U)$ are linearly independent.

Now if M and N are not conjugate then $\varprojlim^{G}_{\mathcal{W}} \tilde{R} \cong R^2$. If M and N are conjugate then $\varprojlim^{G}_{\mathcal{W}} \tilde{R} \cong R$. In either case we have a contradiction. □

Next, if $H \leq G$, consider the restriction functor $\mathrm{Res}^G_H : \mathrm{CS}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{W}}(H)$.

One functor in the other direction is Ind^G_H. This is defined on subgroups by

$$(\mathrm{Ind}^G_H L)(J) = \bigoplus_{g\in {}^J(G/H)} L(J^g).$$

The restrictions are the obvious compositions of restriction in L and inclusion, but the conjugation maps are less clear. For this purpose it is better to use a representative-free description.

First set

$$\hat{L}_{G,H}(J) = \bigoplus_{g\in G,\, J^g\leq H} L(J^g).$$

This is a PCS on $\mathcal{W}$. In fact it is a WCS as follows. Write $(g,l)^J$ for $l \in L(J^g) \subseteq \hat{L}_{G,H}(J)$ and define $\mathrm{res}^J_K(g,l)^J = (g, \mathrm{res}^J_K l)^K$. In this way $\hat{L}_{G,H}$ becomes a PCS. Now define conjugation by $f \in G$ by $c_f(g,l)^{J^f} = (fg,l)^J$. This makes $\hat{L}_{G,H}$ in to a WCS.

There is also an action of H on the right by $(g,l)^J h = (gh, c_h^{-1} l)^J$. These two actions commute and we set

$$(\mathrm{Ind}^G_H L)(J) = H_0(H, \hat{L}_{G,H}(J)).$$

The conjugation and restriction maps are the induced ones, and it is routine to check that if $f \in J$ then c_f acts trivially on the evaluation at J, so we have a coefficient system.

In fact Ind is the left adjoint of Res. The unit and counit of the adjunction are:
$\eta_M : M \to \mathrm{Res}\,\mathrm{Ind}\, M$ is the inclusion of $M(J)$ as $M(J^e)$, and
$\epsilon_N : \mathrm{Ind}\,\mathrm{Res}\, N \to N$ is $\oplus c_g$.

For the right adjoint we have coinduction, which, at least if $\mathrm{Stab}_W(G/H) \in \mathcal{W}$ for each $W \in \mathcal{W}$, can be defined in terms of G-sets just as induction is for Mackey functors, by $(\mathrm{Coind}^G_H L)(G/J) = L(\mathrm{Res}^G_H(G/J))$.

In explicit form, this is

$$(\mathrm{Coind}^G_H L)(J) = \prod_{g\in J\backslash G/H} L(H \cap J^g).$$

For a representative-free form, set

$$\check{L}_{G,H}(J) = \prod_{g \in G} L(H \cap J^g).$$

Just as before, there is a left action of G and a right action of H. There are restriction maps given for $I \leq J$ by specifying that $\mathrm{res}^J_I(g,l) = (g, \mathrm{res}^{H\cap {}^gI}_{H\cap {}^gJ} l)$. We can now set $(\mathrm{Coind}^G_H L)(J) = H^0(J \times H, \check{L}_{G,H}(J))$ (where J acts on the left).

The adjunction is given as follows: $\eta_M : M \to \mathrm{Coind}\,\mathrm{Res}\, M$ is $(\mathrm{res}\, c_g^{-1})$, and $\epsilon_N : \mathrm{Res}\,\mathrm{Coind}\, N \to N$ is projection on to $N(H \cap {}^eJ) = N(J)$. The identities can be checked just as in [21].

If $\mathcal{W}$ is not closed under intersections with H then it is not clear what to use for $L(H \cap {}^gJ)$ in the formulas above. In fact we fill in these gaps using $\varprojlim$, i.e. we define $\mathrm{Coind}^G_{H,\mathcal{W}} L = \mathrm{Res}^{\mathcal{V}}_{\mathcal{W}} \mathrm{Coind}^G_{H,\mathcal{V}} \varprojlim^{\mathcal{V}}_{\mathcal{W}} L$, for some $\mathcal{V} \supseteq \mathcal{W}$ that is closed under intersections with H, e.g. $\mathcal{V} = \mathcal{S}$.

The unit and counit extend in the obvious way, but in case this seems too much like sleight of hand, and since the matter is important for this paper, we will give a proof of the adjunction using only the properties of the functors already defined.

Working always within CS, we have:

$$\begin{aligned}
\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(L, \mathrm{Coind}^G_H M) &= \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(L, \mathrm{Res}^{\mathcal{V}}_{\mathcal{W}} \mathrm{Coind}^G_H \varprojlim^{\mathcal{V}}_{\mathcal{W}} M) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{V}}(G)}(\varinjlim^{\mathcal{V}}_{\mathcal{W}} L, \mathrm{Coind}^G_H \varprojlim^{\mathcal{V}}_{\mathcal{W}} M) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{V}}(H)}(\mathrm{Res}^G_H \varinjlim^{\mathcal{V}}_{\mathcal{W}} L, \varprojlim^{\mathcal{V}}_{\mathcal{W}} M) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{V}}(H)}(\varinjlim^{\mathcal{V}}_{\mathcal{W}} \mathrm{Res}^G_H L, \varprojlim^{\mathcal{V}}_{\mathcal{W}} M) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}^G_H L, \mathrm{Res}^{\mathcal{V}}_{\mathcal{W}} \varprojlim^{\mathcal{V}}_{\mathcal{W}} M) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}^G_H L, M)
\end{aligned}$$

Summing up we have shown:

Proposition 2.6. Ind^G_H *is the left adjoint and* Coind^G_H *is the right adjoint of* $\mathrm{Res}^G_H : \mathrm{CS}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{W}}(H)$.

If X is a left G-set we define $R[X^?] \in \mathrm{CS}_{\mathcal{W}}(G)$ by letting its value on $H \in \mathcal{W}$ be the free R-module on the points of X fixed under H, that is on ${}^HX$. We will usually denote this by $R[X^H]$. Writing ? and H on the right is confusing but traditional.

The restrictions are induced by the inclusions of subsets and conjugation c_g is induced by left multiplication by g.

Notice that it follows from the definitions that:

Lemma 2.7. $R[G/H^?] \cong \mathrm{Ind}^G_H \bar{R}$ *in* $\mathrm{CS}_{\mathcal{W}}(G)$ *for any* $\mathcal{W}$.

Corollary 2.8. $R[G/H^?]$ *is projective in* $\mathrm{CS}_{\mathcal{W}}(G)$ *if* $H \in \mathcal{W}$. *The* $R[G/H^?]$ *for* $H \in \mathcal{W}$ *satisfy* $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(R[G/H^?], L) \cong L(H)$ *for* $L \in \mathrm{CS}_{\mathcal{W}}(G)$ *and they provide enough projectives in* $\mathrm{CS}_{\mathcal{W}}(G)$

Proof. ([7]) $R[G/H^?]$ is projective by 2.5, 2.6 and the fact that left adjoints of exact functors preserve projectives.

Now $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(R[G/H^?], L) \cong \hom_{\mathrm{CS}_{\mathcal{W}}(H)}(\bar{R}, L) \cong L(H)$ for $H \in \mathcal{W}$. It is easy to see that this isomorphism is given by evaluating the homomorphism at $H \in {}^H(G/H)$.

Finally we need to show that any L is the surjective image of a projective. But, using the previous isomorphism, we can construct a map from a sum of copies of $R[G/H^?]$ to L that is surjective on evaluation at H. Now we take the sum of these over the $H \in \mathcal{W}$. □

If $\mathcal{W}$ is a class of subgroups of G and $H \leq G$ we say that H is *taut* with respect to $\mathcal{W}$ if, for each $W \in \mathcal{W}$, $H \cap W$ is tight with respect to $\mathcal{W} \cap \mathcal{S}(H \cap W)$ in the sense defined after 2.5.

Lemma 2.9. *Working in* $\mathrm{CS}_{\mathcal{W}}$*,*

(1) Ind_H^G *is always exact and preserves projectives,*
(2) Coind_H^G *always preserves injectives,*
(3) Res_H^G *is always exact and preserves injectives.*
(4) Coind_H^G *is exact and* Res_H^G *preserves projectives if and only if* H *is taut with respect to* $\mathcal{W}$ *(so in particular if* $H \in \mathcal{W}$*).*

Proof. Res_H^G and Ind_H^G are exact by construction. Therefore the left adjoint of Res_H^G, which is Ind_H^G, preserves projectives; its right adjoint, which is Coind_H^G, preserves injectives and the right adjoint of Ind_H^G, which is Res_H^G, preserves injectives.

Since the $R[G/W^?]$ with $W \in \mathcal{W}$ provide enough projectives, Res_H^G preserves projectives if and only if each $\mathrm{Res}_H^G R[G/W^?]$ is projective. But this is $R[(\mathrm{Res}_H^G G/W)^?]$, and by the double coset formula is a sum of pieces of the form $R[H/(H \cap {}^gW)^?]$, which are all projective if and only if H is taut with respect to $\mathcal{W}$, by 2.5. □

Lemma 2.10. *If* $H \leq G, \quad L \in \mathrm{CS}_{\mathcal{W}}(G), \quad M \in \mathrm{CS}_{\mathcal{W}}(H)$ *then*

$$\mathrm{Ind}_H^G(L \otimes \mathrm{Res}_H^G M) \cong (\mathrm{Ind}_H^G L) \otimes M,$$

in $\mathrm{CS}_{\mathcal{W}}(G)$*.*

Proof. For any $K \in \mathrm{CS}_{\mathcal{W}}(G)$,

$$\begin{aligned}
\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}_H^G(L \otimes \mathrm{Res}_H^G M), K) &\cong \hom_{\mathrm{CS}_{\mathcal{W}}(H)}(L \otimes \mathrm{Res}_H^G M, \mathrm{Res}_H^G K) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(H)}(L, \mathrm{Hom}(\mathrm{Res}_H^G M, \mathrm{Res}_H^G K)) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}_H^G L, \mathrm{Hom}(M, K)) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}_H^G L \otimes M, K).
\end{aligned}$$

This is natural in K and the result now follows formally. ($\hom(A, -) \cong \hom(B, -) \Rightarrow A \cong B$.) □

The adjunction can be generalised:

Proposition 2.11. *For* $H \leq G$*,* $L \in \mathrm{CS}_{\mathcal{W}}(G)$*,* $M \in \mathrm{CS}_{\mathcal{W}}(H)$ *we have*

$$\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(L, \mathrm{Coind}_H^G M) \cong \mathrm{Coind}_H^G \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}_H^G L, M),$$

$$\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}_H^G M, L) \cong \mathrm{Coind}_H^G \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(H)}(M, \mathrm{Res}_H^G L),$$

in $\mathrm{CS}_{\mathcal{W}}(G)$*.*

Proof. For any $K \in \mathrm{CS}_{\mathcal{W}}(G)$,

$$\begin{aligned}
\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(K, \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(L, \mathrm{Coind}_H^G M)) &\cong \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(K \otimes L, \mathrm{Coind}_H^G M) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}_H^G K \otimes \mathrm{Res}_H^G L, M) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}_H^G K, \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}_H^G L, M)) \\
&\cong \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(K, \mathrm{Coind}_H^G \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}_H^G L, M)).
\end{aligned}$$

Now the first formula follows formally as in 2.10. The proof of the second is similar, but needs 2.10. □

Corollary 2.12. *If* $L \in \mathrm{CS}_{\mathcal{W}}(G)$ *and* $H \leq G$*, then* $\mathrm{Ind}_H^G \mathrm{Res}_H^G L \cong R[G/H^?] \otimes L$ *and* $\mathrm{Coind}_H^G \mathrm{Res}_H^G L \cong \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(R[G/H^?], L)$ *in* $\mathrm{CS}_{\mathcal{W}}(G)$*.*

Proof. There is a map $\Theta : \mathrm{Ind}_H^G \mathrm{Res}_H^G L \to R[G/H^?] \otimes L$ given by $\Theta(g,l) = gH \otimes c_g l$. Its inverse is given by $\Phi(gH \otimes l) = (g, c_g^{-1} l)$.

For the second part note that both sides have the same left adjoint, by the first part, 2.1 and 2.6. □

We will occasionally need to deal with quotient groups, so suppose that $H \lhd G$ and let $\rho : G \to G/H$ be the quotient map. Let $\mathcal{W}$ be a class of subgroups of G/H and let $\mathcal{V}$ be a class of subgroups of G such that $\mathcal{V} \supseteq \rho^{-1}(\mathcal{W})$ and $\rho(\mathcal{V}) \subseteq \mathcal{W}$.

Given $L \in \mathrm{CS}_{\mathcal{W}}(G)$ define $Q_{G/H}^G L \in \mathrm{CS}_{\mathcal{V}}(G/H)$ by $Q_{G/H}^G L(W) = L(\rho^{-1} W)$, for $W \in \mathcal{W}$.

In the other direction we have two functors, defined on $M \in \mathrm{CS}_{\mathcal{W}}(G/H)$ by:

$(\mathrm{Inf}_{G/H}^G M)(V) = M(\rho(V))$, for $V \in \mathcal{V}$, and

$(\mathrm{Coinf}_{G/H}^G M)(V)$ is $M(\rho(V))$ if $H \leq V \in \mathcal{V}$ and 0 otherwise.

The next result is left as an easy exercise for the reader.

Proposition 2.13. $\mathrm{Inf}_{G/H}^G$ *is the left adjoint and* $\mathrm{Coinf}_{G/H}^G$ *is the right adjoint of* $Q_{G/H}^G : \mathrm{CS}_{\mathcal{V}}(G) \to \mathrm{CS}_{\mathcal{W}}(G/H)$.

Now we consider Mackey functors, so for simplicity assume that G is finite. These have been described in many other places, e.g. [25]. The only difference in our treatment is that we only evaluate the functor on a class $\mathcal{W}$ of subgroups of G and we assume that this class is closed under intersections. We require the double coset formula for $\mathrm{res}_W^V \mathrm{tr}_U^V$ whenever $U, V, W \in \mathcal{W}$. Notice that its terms are all defined because of the condition on intersections.

Let $\mathcal{V}$ and $\mathcal{W}$ be two classes of subgroups of G, such that $\mathcal{V}$ is closed under intersections and $\mathcal{W}$ is closed under intersections with $\mathcal{V}$ (that is, if $W \in \mathcal{W}$ and $V \in \mathcal{V}$, then $V \cap W \in \mathcal{W}$).

Given $C \in \mathrm{CS}_{\mathcal{W}}$, define $\hat{C} \in \mathrm{WCS}_{\mathcal{V}}$ by:

$$\hat{C}(J) = \bigoplus_{I \in \mathcal{W},\, I \leq J} C(I).$$

An element x of $C(I) \subseteq \hat{C}(J)$ will be denoted $(I,x)^J$.

For $g \in G$ define ${}^g(J,x)^H = ({}^gJ, c_g x)^{{}^gH}$. The conjugation maps in C combine to yield a map $\hat{c}_g : \hat{C}(H) \to \hat{C}({}^gH)$, where $\hat{c}_g(J,x)^H = {}^g(J,x)^H$.

Whenever $K \leq L \leq G$ there are restriction morphisms between the values of $\hat{C}$ given by $r_K^L(J,x)^L = (J \cap K, \mathrm{res}_{J \cap K}^J x)^K$. These make $\hat{C}$ into a weak coefficient system. There are also inclusion morphisms given by $i_K^L(J,x)^K = (J,x)^L$, which give $\hat{C}$ the dual structure i.e. make it into a covariant functor on the category of conjugation and inclusion morphisms.

Define:

$$TC(H) = H^0(H; \hat{C}(H))$$

and

$$SC(H) = H_0(H; \hat{C}(H)).$$

We now define restriction and transfer maps, denoted by R and I, on these groups. We denote the restriction and transfer in cohomology by res and tr.

On TC, $R_K^L = (r_K^L)_* \mathrm{res}_K^L$ and $I_K^L = \mathrm{tr}_K^L (i_K^L)_*$.

On SC,

$$R_K^L(J,x)^L = (r_K^L)_* \sum_{g \in K \backslash L / J} {}^g(J,x)^L = \sum_{g \in K \backslash L / J} ({}^gJ \cap K, \mathrm{res}_{{}^gJ \cap K^g}^{{}^gJ} c_g x)^K,$$

and $I_K^L = \mathrm{res}_K^L (i_K^L)_*$.

One can verify that SC and TC are Mackey functors on $\mathcal{V}$. The case of S goes back at least to [9] (see also [5]) and T appears in [20].

There is a forgetful functor $F : \mathrm{MF}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{W}}(G)$ and another functor $G : \mathrm{MF}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{W}}(G)$ given by

$$GM(J) = M(J)/\sum_{I \lneq J} \mathrm{Im}\,\mathrm{tr}_I^J,$$

with the zero restriction maps.

Proposition 2.14. *S is the left adjoint of F and T is the right adjoint of G.*

Proof. For T, $\eta_J : M \to TGM$ is $\prod \mathrm{res}_I^J$, and $\epsilon_J : GTN \to N$ is projection on to $L(J)$.

For S, $\eta_J : M \to FSM$ takes m to $(J, m)^J$, and $\epsilon_J : SFN \to N$ takes $(I, n)^J$ to $\mathrm{tr}_I^J\, n$. □

Remark. Some authors use a slightly different definition of T. Instead of $\hat{C}$, they use a WCS $\check{C}$, which differs from $\hat{C}$ only in that the restriction maps are given by

$$\check{r}_K^L(J, x)^L = \begin{cases} (J, x)^K & \text{if } J \leq K, \\ 0 & \text{otherwise.} \end{cases}$$

However there is a map $\phi : \hat{C} \to \check{C}$ given by

$$\phi_L(J, x)^L = \sum_{I \in \mathcal{W},\, I \leq J} (I, \mathrm{res}_I^J\, x)^L.$$

This ϕ is compatible with the maps r and $\check{r}$, and also with the i and the c_g. It is an isomorphism because it is the identity on the factors if we filter according to the order of the group.

Thus the two definitions are equivalent.

This offers a good way of constructing projective and injective coefficient systems or Mackey functors. Note that if H is normal in G then $\mathrm{CS}_{\{H\}}(G)$ is naturally equivalent to the category of $R(G/H)$-modules. Since left adjoints of exact functors preserve projectives and right adjoints of exact functors preserve injectives, we have:

Proposition 2.15. *Suppose that $H \in \mathcal{W}$, and $H \leq N \leq N_G(H)$, (and $\mathcal{W}$ is closed under intersections when we refer to $\mathrm{MF}_{\mathcal{W}}(G)$). Let P be a projective RN/H-module, I an injective RN/H-module (both regarded as elements of $\mathrm{CS}_{\{H\}}(N)$ as above) and let $\langle H \rangle$ denote the set of conjugates of H in G. Then:*

1. *$\varinjlim_{\langle H \rangle}^{\mathcal{W}} \mathrm{Ind}_N^G P \in \mathrm{CS}_{\mathcal{W}}(G)$ and $S \varinjlim_{\langle H \rangle}^{\mathcal{W}} \mathrm{Ind}_N^G P \in \mathrm{MF}_{\mathcal{W}}(G)$ are projective.*
2. *$\varprojlim_{\langle H \rangle}^{\mathcal{W}} \mathrm{Coind}_N^G I \in \mathrm{CS}_{\mathcal{W}}(G)$ is injective.*

The following result is key to many applications, including obtaining a splitting of Mackey functors in 5.3.

Proposition 2.16. *If $L \in \mathrm{CS}_{\mathcal{W}}(G)$ and $M, N \in \mathrm{MF}_{\mathcal{W}}(G)$ then $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(L, M)$ and $\mathrm{Hom}_{\mathrm{MF}_{\mathcal{W}}}(N, M)$ are naturally Mackey functors in $\mathrm{MF}_{\mathcal{W}}(G)$. These structures are consistent in the sense that they are compatible with the isomorphism $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(L, M) \cong \mathrm{Hom}_{\mathrm{MF}_{\mathcal{W}}}(SL, M)$ given on each subgroup in $\mathcal{W}$ by 2.14.*

Proof. $\mathrm{Hom}_{\mathrm{MF}_{\mathcal{W}}}(N, M)$ is defined, at least as a coefficient system, in a similar way to $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(N, M)$.

The transfer on $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(L, M)$ is defined as follows: If $K \leq H \leq G$ and $f \in \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}(K)}}(L, M)$ then $\mathrm{tr}_K^H(f)$ is defined on $J \leq H$, $J \in \mathcal{W}$ as

$$\sum_{g \in J \backslash H / K} (L(J) \xrightarrow{\mathrm{res}} L(J \cap {}^g K) \xrightarrow{c_g} L(J^g \cap K) \xrightarrow{f} M(J^g \cap K) \xrightarrow{c_{g^{-1}}} M(J \cap {}^g K) \xrightarrow{\mathrm{tr}} M(J)).$$

A similar definition works for $\mathrm{Hom}_{\mathrm{MF}_{\mathcal{W}}}(N, M)$. For full details see [6]. □

Proposition 2.17. *Suppose that $\mathcal{X} \subset \mathcal{W}$, both closed under pairwise intersection, and $M \in \mathrm{MF}_{\mathcal{X}}(G)$. Then $\varprojlim_{\mathcal{X}}^{\mathcal{W}} M$ is naturally a Mackey functor. It is the right adjoint of $\mathrm{Res}_{\mathcal{X}}^{\mathcal{W}} : \mathrm{MF}_{\mathcal{W}}(G) \to \mathrm{MF}_{\mathcal{X}}(G)$.*

Proof. To determine tr_H^K we only need to specify $\mathrm{res}_I^K \mathrm{tr}_H^K x$ for each $I \in \mathcal{X}$ in a consistent way. This can be done by setting

$$\mathrm{res}_I^K \mathrm{tr}_H^K x = \sum_{g \in I\backslash K/H} \mathrm{tr}^I_{gH\cap I} c_g \mathrm{res}^H_{H\cap I^g} x.$$

□

Corollary 2.18. *If $M \in \mathrm{MF}_{\mathcal{W}}(G)$ and $M \cong \varprojlim_{\mathcal{X}}^{\mathcal{W}} \mathrm{Res}_{\mathcal{X}}^{\mathcal{W}} M$ as a coefficient system then $M \cong \varprojlim_{\mathcal{X}}^{\mathcal{W}} \mathrm{Res}_{\mathcal{X}}^{\mathcal{W}} M$ as a Mackey functor.*

Proof. The formula used to define the transfer on $\varprojlim$ is clearly necessary, so it must agree with the transfers on L. □

In [21] there is constructed a functor $\mathrm{Ind}_H^G : \mathrm{MF}_{\mathcal{S}}(H) \to \mathrm{MF}_{\mathcal{S}}(G)$, which is both right and left adjoint to restriction. The same recipe will work if we replace $\mathcal{S}$ by $\mathcal{X}$, provided that $\mathcal{X}$ is closed under intersections with H, and, if we ignore transfers then we see that it agrees with our construction of Coind for coefficient systems. For this reason we prefer to denote it by Coind. If $\mathcal{X}$ is not closed under intersections with H then we use $\varprojlim$ as before. The following version of 2.12 is straightforward to check.

Lemma 2.19. *If $M \in \mathrm{MF}_{\mathcal{W}}(G)$ then $\mathrm{Coind}_H^G \mathrm{Res}_H^G M \cong \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}(R[G/H^?], M)$ in $\mathrm{MF}_{\mathcal{W}}(G)$.*

There is an important property of the functor S above.

Lemma 2.20. *$S\mathbb{Z}[G/H^?] \cong B^G(?, H)$, where $B^G(?, H)$ is the functor induced up to G from the Burnside ring Mackey functor B^H on H. In fact $B^G(?, H) \cong \mathrm{Hom}_{\Omega_R(G)}(G/H, -)$ in the notation of* [21].

Proof. Notice that S commutes with induction (in CS or MF, depending upon the side) because their right adjoints commute. Also $S\mathbb{Z}[G/H^?] \cong S \mathrm{Ind}_H^G \bar{\mathbb{Z}}$. Now we claim that $S\mathbb{Z} \cong B^G$, which can be checked from the definitions. For the rest, see [21]. □

3. Higher Limits

We work in the category $\mathrm{CS}_{\mathcal{W}}(G)$ of coefficient systems on a class $\mathcal{W}$ of a group G over some fixed unital ring R. This is an abelian category with enough projectives and injectives (a consequence of 2.15), so we can use homological algebra. We could just consider the derived functors of hom, but instead we look at $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}$, considered as taking values in $\mathrm{CS}_{\mathcal{T}}(G)$ for some class $\mathcal{T}$. This class $\mathcal{T}$ should, perhaps, be indicated in the notation but, instead, we will regard it as implicitly understood or mention it in the text. We take the right derived functors as a functor in the *second* variable, obtaining $\mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(G)} \in \mathrm{CS}_{\mathcal{T}}(G)$. This can be confusing, but at least some potential sources of confusion do not arise:

Lemma 3.1. (1) *$\mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(G)}(M, N)(J)$ does not depend on $\mathcal{T}$, as long as $J \in \mathcal{T}$.*

(2) *If $J \leq H \leq G$, $M, N \in \mathrm{CS}_{\mathcal{W}}(G)$ and $J \in \mathcal{T}$ then $\mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(G)}(M, N)(J) \cong \mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}_H^G M, \mathrm{Res}_H^G N)(J)$.*

Proof. Part (1) is clear from the definitions. Part (2) follows from the definition of the right derived functors and the fact that Res^G_H is exact and preserves injectives (2.9). □

A problem that does arise is that Hom is not always right balanced in the sense of, for example, [26] 2.2.7.

Lemma 3.2. *If $\mathcal{T}$ is a class of subgroups of G that are taut (defined just before 2.9) with respect to $\mathcal{W}$ then $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-,-)$ is right balanced as a functor taking values in $\mathrm{CS}_{\mathcal{W}}(G)$.*

The advantage of having a balanced functor is that its derived functors in the first and in the second variable coincide.

Proof. We need to check that if the second variable N is injective then $\mathrm{Hom}(-,N)$ is exact as a functor of the first variable. We can do this by evaluating on each $J \in \mathcal{T}$, so we are just looking at $\hom_{\mathrm{CS}_{\mathcal{W}}(J)}(-, \mathrm{Res}^G_J N)$. But $\mathrm{Res}^G_J N$ is also injective, by 2.9.

We must also check that if the first variable M is projective then $\mathrm{Hom}(M,-)$ is exact. The argument is dual to the previous one, except that for $\mathrm{Res}^G_J M$ to be projective we need J to be taut with respect to $\mathcal{W}$. □

Remark. A possible choice for $\mathcal{T}$ that satisfies the conditions of 3.2 is $\{G\}$, and this amounts to considering the derived functors of hom.

The higher limit coefficient systems are, by definition, the right derived functors of $\varprojlim^{\mathcal{W}}_{\mathcal{X}} : \mathrm{CS}_{\mathcal{X}}(G) \to \mathrm{CS}_{\mathcal{W}}(G)$ and we will write $(\varprojlim^{\mathcal{W}}_{\mathcal{X}})^i$ for $R^i \varprojlim^{\mathcal{W}}_{\mathcal{X}}$

What are normally thought of as the higher limits of $L \in \mathrm{CS}_{\mathcal{X}}(G)$ are the R-modules $((\varprojlim^{\mathcal{S}}_{\mathcal{X}})^i L)(G)$.

Strictly speaking, we only defined $\varprojlim^{\mathcal{T}}_{\mathcal{W}}$ when $\mathcal{W} \subseteq \mathcal{T}$, but the definition without this restriction is clear. It comes to the same as $\mathrm{Res}^{\mathcal{S}}_{\mathcal{T}} \varprojlim^{\mathcal{S}}_{\mathcal{W}}$.

Lemma 3.3. *For all $n \geq 0$, $\mathrm{Ext}^n_{\mathrm{CS}_{\mathcal{W}}}(\bar{R}, L) \cong (\varprojlim^{\mathcal{S}}_{\mathcal{W}})^n L$ in $\mathrm{CS}_{\mathcal{S}}(G)$.*

Proof. The case $n = 0$ is just 2.4. Both sides are, by definition, the derived functors in L of the $n = 0$ case. □

Remark. We can *not* calculate $\mathrm{Ext}^n_{\mathrm{CS}_{\mathcal{W}}}(\bar{R}, L)$ above by taking a projective resolution of $\bar{R}$ unless we are able to invoke 3.2.

Notice that if $\mathcal{W}$ consists of just the trivial group 1, then an object of $\mathrm{CS}_{\{1\}}(G)$ is just an RG-module and the higher limits are just the usual cohomology groups.

We see that the higher inverse limits are natural and unavoidable objects to consider. However if $\mathcal{W}$ is large enough they often vanish.

Lemma 3.4. *Consider $\mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(G)}$ to take values in $\mathrm{CS}_{\mathcal{S}}(G)$. For $H \leq G$ and $L \in \mathrm{CS}_{\mathcal{W}}(H)$, $M \in \mathrm{CS}_{\mathcal{W}}(G)$, we have*

$$\mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}^G_H L, M) \cong \mathrm{Coind}^G_H \mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(H)}(L, \mathrm{Res}^G_H M),$$

and if H is closed under intersections with $\mathcal{W}$ (or just H is taut with respect to $\mathcal{W}$) then

$$\mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(G)}(M, \mathrm{Coind}^G_H L) \cong \mathrm{Coind}^G_H \mathrm{Ext}^*_{\mathrm{CS}_{\mathcal{W}}(H)}(\mathrm{Res}^G_H M, L).$$

Proof. The zeroth terms are isomorphic by 2.11. We need to check that both sides calculate the right derived functors in the second variable of this common functor. Notice that the first and third occurrences of Coind^G_H are applied to $\mathrm{CS}_{\mathcal{W}}(H)$, so are exact by 2.9. The second occurrence is exact, by 2.9, because of the restrictions imposed on the intersections.

For the first formula, notice that an injective resolution of M in $\mathrm{CS}_{\mathcal{W}}(G)$ becomes an injective resolution of $\mathrm{Res}^G_H M$ in $\mathrm{CS}_{\mathcal{W}}(G)$ on applying Res^G_H since Res^G_H preserves injectives by 2.9 . For the second formula, notice that an injective resolution of L in $\mathrm{CS}_{\mathcal{W}}(H)$ becomes an injective resolution of $\mathrm{Coind}^G_H L$ after applying Coind^G_H since Coind^G_H is exact and it preserves injectives by 2.9. □

We say that a coefficient system L is injective relative to a set of subgroups $X \subseteq \mathcal{S}(G)$ if L is a direct summand of $\prod_{H\in X} \mathrm{Coind}^G_H \mathrm{Res}^G_H L$. This has many equivalent formulations along the lines of Higman's criterion (cf. [21], [1]). There is also an analogous concept for Mackey functors (where it is customarily referred to as *projective relative to* since the right and left adjoints of restriction are then isomorphic).

Remark. Since the forgetful functor F from Mackey functors to coefficient systems commutes with Coind^G_H (their left adjoints commute), a Mackey functor that is injective relative to X as a Mackey functor is also injective relative to X as a coefficient system.

Proposition 3.5. *For $L \in \mathrm{CS}_{\mathcal{W}}(G)$ and $\mathcal{X} \subseteq \mathcal{W}$ with $\mathcal{W}$ closed under intersections with $\mathcal{X}$, if L is injective relative to $\mathcal{X}$ then $\varprojlim^{\mathcal{W}}_{\mathcal{X}} \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} L \cong L$ and $(\varprojlim^{\mathcal{W}}_{\mathcal{X}})^n \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} L = 0$ for $n > 0$ in $\mathrm{CS}_{\mathcal{S}}(G)$.*

Proof. It is enough to prove this for $\mathrm{Coind}^G_H \mathrm{Res}^G_H L, \quad H \in \mathcal{X}$. But $\mathrm{Ext}^n_{\mathrm{CS}_{\mathcal{X}}(G)}(\bar{R}, \mathrm{Coind}^G_H \mathrm{Res}^G_H L) \cong \mathrm{Coind}^G_H \mathrm{Ext}^n_{\mathrm{CS}_{\mathcal{X}}(H)}(\bar{R}, \mathrm{Res}^G_H L)$ by 3.4.

But $\bar{R}$ is projective in $\mathrm{CS}_{\mathcal{X}}(H)$ by 2.8, so the higher Ext vanish, and for $n = 0$ we have the result required. □

The following vanishing result is a version of one in [14].

Proposition 3.6. *Let $\mathcal{X}$ be a class of p-subgroups of G which is closed under intersections, and such that $\mathcal{X}$ contains a Sylow p-subgroup P, and assume that all positive numbers of the form $|G/P| - np$, $n \in \mathbb{N}_0$ are invertible in R (e.g. R is p-local). Let M be a Mackey functor on $\mathcal{X}$. Then $\varprojlim^{\mathcal{S}}_{\mathcal{X}} M$ is injective relative to $\mathcal{X}$ in $\mathrm{MF}_{\mathcal{S}}(G)$. In particular M is injective relative to $\mathcal{X}$ in $\mathrm{CS}_{\mathcal{X}}(G)$, and so $(\varprojlim^{\mathcal{S}}_{\mathcal{X}})^n M = 0$ for $n \geq 1$ in $\mathrm{CS}_{\mathcal{S}}(G)$.*

Proof. Notice that $\varprojlim^{\mathcal{S}}_{\mathcal{X}} M$ is naturally a Mackey functor by 2.17.

The natural augmentation yields a map $\pi : R[G/P^?] \to \bar{R}$, which is onto in $\mathrm{CS}_{\mathcal{X}}(G)$.

Now the functor $S : \mathrm{CS}_{\mathcal{X}}(G) \to \mathrm{MF}_{\mathcal{X}}(G)$ is right exact, since its construction involves coinvariants, so $S\pi : SR[G/P^?] \to S\bar{R}$ is onto. We claim that $S\pi$ splits. To see this, use the second $(\check{C})$ model for S. The splitting is induced by sending

$$(J,1)^H \mapsto \frac{1}{|^J(G/P)|}(J, \sum_{g\in G/P,\, J\leq {}^gP} gP)^H.$$

Note that, when J acts on G/P, the orbit of every non-fixed point has size divisible by p, so the denominators are indeed invertible, by hypothesis.

Now $\varprojlim^{\mathcal{S}}_{\mathcal{X}} M \cong \mathrm{Hom}_{\mathrm{MF}_{\mathcal{X}}(G)}(S\bar{R}, M)$ is a summand of $\mathrm{Hom}_{\mathrm{MF}_{\mathcal{X}}(G)}(SR[G/P^?], M) \cong \mathrm{Hom}_{\mathrm{CS}_{\mathcal{X}}(G)}(R[G/P^?], M) \cong \mathrm{Coind}^G_P \mathrm{Res}^G_P M$ by 2.19. □

Let $\mathrm{MF}_{\mathcal{S}}(G,1)$ denote the full subcategory of Mackey functors which are projective relative to $\mathcal{S}_p$ (often denoted $\mathrm{Mack}(G,1)$ by other authors). We can deduce a result of Bouc [5].

Proposition 3.7. *If R is p-local then $\varprojlim^{\mathcal{S}}_{\mathcal{S}_p}$ and $\mathrm{Res}^{\mathcal{S}}_{\mathcal{S}_p}$ provide an equivalence of categories between $\mathrm{MF}_{\mathcal{S}_p}(G)$ and $\mathrm{MF}_{\mathcal{S}}(G,1)$.*

Proof. From 3.6 we see that $\varprojlim^{\mathcal{S}}_{\mathcal{S}_p}$ takes values in $\mathrm{MF}_{\mathcal{S}}(G,1)$.

Clearly $\mathrm{Res}^{\mathcal{S}}_{\mathcal{S}_p}\varprojlim^{\mathcal{S}}_{\mathcal{S}_p} = \mathrm{Id}$. As for $\varprojlim^{\mathcal{S}}_{\mathcal{S}_p}\mathrm{Res}^{\mathcal{S}}_{\mathcal{S}_p} = \mathrm{Id}$, it is enough to check this on a functor of the form $\mathrm{Coind}^G_P\mathrm{Res}^G_P M$ for $P \in \mathcal{S}_p$. But the left adjoint of $\varprojlim^{\mathcal{S}}_{\mathcal{S}_p}\mathrm{Res}^{\mathcal{S}}_{\mathcal{S}_p}\mathrm{Coind}^G_P\mathrm{Res}^G_P$ is $\mathrm{Ind}^G_P\mathrm{Res}^G_P\varinjlim^{\mathcal{S}}_{\mathcal{S}_p}\mathrm{Res}^{\mathcal{S}}_{\mathcal{S}_p} \cong \mathrm{Ind}^G_P\mathrm{Res}^G_P$, which in turn has right adjoint $\mathrm{Coind}^G_P\mathrm{Res}^G_P$, so $\varprojlim^{\mathcal{S}}_{\mathcal{S}_p}\mathrm{Res}^{\mathcal{S}}_{\mathcal{S}_p}\mathrm{Coind}^G_P\mathrm{Res}^G_P \cong \mathrm{Coind}^G_P\mathrm{Res}^G_P$ as required. □

For any poset $\mathcal{X}$ we define $\mathcal{X}_{\geq H} = \{K \in \mathcal{X} | K \geq H\}$ and similarly $\mathcal{X}_{>H}$. Recall that G_0 is the left adjoint of the inclusion $\mathrm{GCS} \to \mathrm{CS}$

Lemma 3.8. *If $\mathcal{X} \subseteq \mathcal{W}$ and for each $H \in \mathcal{W}$, the poset $\mathcal{X}_{\geq H}$ is connected, then $\varinjlim^{\mathcal{W}}_{\mathcal{X}}\bar{R} = \bar{R}$.*

If for each $H \in \mathcal{W}$, $C_G(H)$ acts transitively on the components of $\mathcal{X}_{\geq H}$, then $G_0\varinjlim^{\mathcal{W}}_{\mathcal{X}}\bar{R} = \bar{R}$.

Proof. From the definition, $(\varinjlim^{\mathcal{W}}_{\mathcal{X}}\bar{R})(H)$ is the free R-module on the components of $\mathcal{X}_{\geq H}$. □

Recall that we made the abbreviation $\varprojlim^G_{\mathcal{W}} L = (\varprojlim^{\mathcal{S}}_{\mathcal{W}} L)(G)$.

Proposition 3.9. *If each $\mathcal{X}_{\geq H}, H \in \mathcal{W}$ is connected then $\varprojlim^G_{\mathcal{X}}\mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} \cong \varprojlim^G_{\mathcal{W}}$ on $\mathrm{CS}_{\mathcal{W}}(G)$.*

If $C_G(H)$ acts transitively on the components of each $\mathcal{X}_{\geq H}, H \in \mathcal{W}$ then $\varprojlim^G_{\mathcal{X}}\mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} \cong \varprojlim^G_{\mathcal{W}}$ on $\mathrm{GCS}_{\mathcal{W}}(G)$.

Proof. For the first formula we need to show that $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\bar{R}, L) \cong \hom_{\mathrm{CS}_{\mathcal{X}}(G)}(\bar{R}, \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} L)$. But $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\varinjlim^{\mathcal{W}}_{\mathcal{X}}\bar{R}, L) \cong \hom_{\mathrm{CS}_{\mathcal{X}}(G)}(\bar{R}, \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} L)$, by 2.2. Now use the previous lemma. The second formula is proved similarly. □

For any poset W we denote the geometric realisation by $|W|$. This is the simplicial complex where the simplices correspong to chains in W

For any poset W we define the weakly essential elements to be

$$\mathrm{Wess}_0(W) = \{H \in W \mid |W_{>H}| \text{is empty or has more than one component}\},$$

and also

$$\mathrm{Wess}(W) = \{H \in W || W_{>H}| \text{is not contractible}\}.$$

Also, if W is a G-poset, we define the essential elements to be

$$\mathrm{Ess}_0(W) = \{H \in W || W_{>H}|/C_G(H) \text{is empty or has more than one component}\}.$$

Notice that $\mathrm{Ess}_0(W) \subseteq \mathrm{Wess}_0(W) \subseteq \mathrm{Wess}(W)$.

Remark. We allow maximal elements of W to be essential, in contrast to [20].

The next proposition will be very useful for changing classes of groups. It is based on §6.6 of [1] vol. II, attributed to Bouc.

Proposition 3.10. *Let W be a poset such that there is a bound on the length of any chain in W, and let X be a subposet.*

If X contains $\mathrm{Wess}\, W$ then the inclusions of the geometric realisations $|\mathrm{Wess}W| \subseteq |X| \subseteq |W|$ are homotopy equivalences.

If X contains $\mathrm{Wess}_0 W$ then the inclusions of the geometric realisations $|\mathrm{Wess}_0 W| \subseteq |X| \subseteq |W|$ induce a bijection on the connected components.

Proof. The first part is due to Bouc [2, 3, 4], see also II.6.6.5 of [1]. The second part is proved in the same way, replacing *homotopy equivalence* by *induces a bijection on the connected components*. The same must be done for Quillen's Lemma (II.6.6.2 in [1]), either by considering only the $E_{0,0}$-term of the spectral sequence in the proof, or just by elementary means. □

Proposition 3.11. *Suppose that there is a bound on the length of any chain in* $\mathcal{W}$. *If* $\mathrm{Wess}_0(\mathcal{W}) \subseteq \mathcal{X} \subseteq \mathcal{W}$ *then* $\varprojlim^G_{\mathcal{X}} \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} \cong \varprojlim^G_{\mathcal{W}}$ *on* $\mathrm{CS}_{\mathcal{W}}(G)$.

If, in addition, $\mathcal{X}$ *is closed under supergroups in* $\mathcal{W}$ *then* $(\varprojlim^G_{\mathcal{X}})^n \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} \cong (\varprojlim^G_{\mathcal{W}})^n$.

Proof. By the 3.9 we need to show that $\mathcal{X}_{\geq H}$ is connected for each $H \in W$. But $\mathrm{Wess}_0(\mathcal{W}_{\geq H}) \subseteq \mathcal{X}_{\geq H} \subseteq \mathcal{W}_{\geq H}$, so we can apply 3.10.

For the higher limits the result will follow if we know that $\mathrm{Res}^{\mathcal{W}}_{\mathcal{X}}$ is exact (which it clearly is) and it preserves projectives. The latter is equivalent to the right adjoint $\varprojlim^{\mathcal{W}}_{\mathcal{X}}$ being exact, which it clearly is under the condition on supergroups. □

The next result is an immediate consequence of Alperin's Fusion Theorem, as stated in, for example, [11], [20]. For the rest of this section we suppose that G is finite.

Proposition 3.12. *Suppose that* $\mathcal{W} \subseteq \mathcal{S}_p$ *is closed under supergroups in* $\mathrm{Ess}_0(\mathcal{S}_p)$, *and that* $\mathrm{Ess}_0(\mathcal{S}_p) \cap \mathcal{W} \subseteq \mathcal{X} \subseteq \mathcal{W}$. *Then* $\varprojlim^G_{\mathcal{X}} \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} \cong \varprojlim^G_{\mathcal{W}}$ *on* $\mathrm{GCS}_{\mathcal{W}}(G)$.

If, in addition, $\mathcal{X}$ *is closed under supergroups in* $\mathcal{W}$ *then* $(\varprojlim^G_{\mathcal{X}})^n \mathrm{Res}^{\mathcal{W}}_{\mathcal{X}} \cong (\varprojlim^G_{\mathcal{W}})^n$.

Proof. Pick a Sylow p-subgroup P of G and use the method of stable elements to realise $\varprojlim^G_{\mathcal{X}} L$ as the set of elements $x \in L(P)$ such that, whenever $H \in \mathcal{W}$, $H \leq P$, $g \in G$ and ${}^gH \leq P$, then x satisfies $\mathrm{res}^P_{{}^gH}\, x = c_g\, \mathrm{res}^P_H\, x$.

The Fusion Theorem states that the group homomorphism $c_g : H \to {}^gH$ given by conjugation by g is equal to the composition of a sequence of conjugations $c_v : U \to {}^vU$ for $v \in G$, $U, {}^vU \leq E \leq P$ for some essential subgroup $E \in \mathrm{Ess}_0(\mathcal{S}_p)$ such that v normalises E. Since $U \in \mathcal{W}$ and $\mathcal{W}$ is closed under supergroups in $\mathrm{Ess}_0(\mathcal{S}_p)$ we see that $E \in \mathrm{Ess}_0(\mathcal{W})$ and hence is in $\mathcal{X}$.

But $(\mathrm{res}^P_{{}^vU} - c_v\, \mathrm{res}^P_U)x = \mathrm{res}^E_{{}^vU}(1 - c_v)\, \mathrm{res}^P_E\, x$. It follows that all the conditions that we want to impose on $x \in L(P)$ are already imposed when we just consider subgroups in $\mathcal{X}$.

Note that the factorisation of c_g given is only as a group homomorphism, so ignores $C_G(H)$. This is why we need to work in GCS not CS.

The claim about the higher limits follows as in the previous proof. Note that the right adjoint of $\mathrm{Res}^{\mathcal{W}}_{\mathcal{X}}$ in GCS is $G^0 \varprojlim^{\mathcal{W}}_{\mathcal{X}} I$, so is still exact. □

Let $\mathcal{B}_p$ denote the class of subgroups P of $\mathcal{S}_p$ satisfying $P = O_p N_G(P)$, often known as the radical subgroups, and let $\mathcal{C}_p$ denote the class of subgroups P in $\mathcal{S}_p$ for which the centre of P is the Sylow p-subgroup of $C_G(P)$, sometimes known as the centric or self-centralising subgroups.

It is well known that $\mathrm{Wess}_0(\mathcal{S}_p) \subseteq \mathcal{B}_p$ and $\mathrm{Ess}_0(\mathcal{S}_p) \subseteq \mathcal{C}_p \cap \mathcal{B}_p$ (see [20]).

The next lemma is well known and easy to prove.

Lemma 3.13. *If* $N \lhd G$ *and* N *is of order coprime to* p *then* $|\mathcal{S}^1_p(G/N)| \cong |\mathcal{S}^1_p(G)/N|$.

Following Grodal [12], let $\mathcal{D}_p$ be the set of centric subgroups P of G for which $N_G(P)/PC_G(P)$ has no non-trivial normal p-subgroup.

Lemma 3.14. $\mathrm{Ess}_0(\mathcal{S}_p) \subseteq \mathcal{D}_p \subseteq \mathcal{C}_p \cap \mathcal{B}_p$.

Proof. $\mathcal{D}_p \subseteq \mathcal{C}_p$ by definition. If $P \in \mathcal{C}_p \smallsetminus \mathcal{B}_p$ then there is a subgroup Q such that $P \lneqq Q \lhd N_G(P)$ and $Q \not\leq C_G(P)$. The image of Q in $N_G(P)/PC_G(P)$ is non-trivial, so $P \notin \mathcal{D}_p$.

Now if P is centric then $|\mathcal{S}_p^1(G)^{>P}|/C_G(P) \cong |\mathcal{S}_p^1(N_G(P)/P)|/(C_G(P)/Z(P)) \cong |\mathcal{S}_p^1(N_G(P)/PC_G(P))|$, by the lemma above. So if $P \notin \mathcal{D}_p$ then these spaces are contractible and $P \notin \mathrm{Ess}_0(\mathcal{S}_p)$. □

Corollary 3.15. *$L \in \mathrm{CS}_{\mathcal{S}_p}(G)$ then $\varprojlim^G_{\mathcal{B}_p} \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{B}_p} \cong \varprojlim^G_{\mathcal{S}_p}$ on $\mathrm{CS}_{\mathcal{S}_p}(G)$.*

$\varprojlim^G_{\mathcal{D}_p} \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{D}_p} \cong \varprojlim^G_{\mathcal{B}_p} \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{B}_p} \cong \varprojlim^G_{\mathcal{S}_p}$ on $\mathrm{GCS}_{\mathcal{S}_p}(G)$.

Remark. In section 9 we will see that if R is p-local then we can extend these results to higher limits. In this form the first part of the corollary appears in [14] and the second part is in [12].

4. Hyper Cohomology

Given a chain complex $C_\bullet$ in $\mathrm{CS}_{\mathcal{W}}(G)$ which is bounded below, and $L \in \mathrm{CS}_{\mathcal{W}}(G)$ we can consider the hyper-Ext groups $\mathbb{E}\mathrm{xt}^n_{\mathrm{CS}_{\mathcal{W}}}(C_\bullet, L)$. These are the hyper-derived functors of $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}}$, which takes its values in $\mathrm{CS}_{\mathcal{T}}(G)$ (where $\mathcal{T}$ is any class of subgroups), so are themselves coefficient systems. This is not consistent with our previous definition of Ext as the derived functor on the second variable, unless we are in the circumstances of 3.2. But confusion will rarely arise, and when it does we will write $R^i \mathrm{Hom}(A, -)(B)$, for example.

Lemma 4.1. *$\mathbb{E}\mathrm{xt}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L)(J)$ does not depend on $\mathcal{T}$, provided $J \in \mathcal{T}$.*

Proof. When we apply $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-, L)$ we do so groupwise. □

There are two spectral sequences (see e.g. [26] 5.7.9):

$$ {}^{II}E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{CS}_{\mathcal{W}}(G)}(H_q(C_\bullet), L) \Rightarrow \mathbb{E}\mathrm{xt}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L) \tag{1} $$

$$ {}^{I}E_2^{p,q} = H^p(\mathrm{Ext}^q_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L)) \Rightarrow \mathbb{E}\mathrm{xt}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L) \tag{2} $$

We adopt the convention that when we apply plain Ext to a chain complex, we apply it term by term to obtain another chain complex.

The following proposition is the basic result that we will use to obtain the sequences of the introduction and to identify their cohomology.

We say that a class of subgroups $\mathcal{T}$ is *taut* with respect to another class $\mathcal{W}$ if each $T \in \mathcal{T}$ is taut with respect to $\mathcal{W}$ as defined after 2.8 .

Proposition 4.2. (1) *If, in $\mathrm{CS}_{\mathcal{W}}(G)$,*

$$ H_n(C_\bullet) = \begin{cases} \bar{R} & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} $$

then $\mathbb{E}\mathrm{xt}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L) \cong \mathrm{Ext}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(\bar{R}, L)$. If $\mathcal{T}$ is taut with respect to $\mathcal{W}$ then this is also equal to $(\varprojlim^{\mathcal{T}}_{\mathcal{W}})^n L$.

(2) *If $K \leq G$ and $H^* \mathrm{Ext}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L)(K) = 0$ for $n \geq 1$ then*

$$ \mathbb{E}\mathrm{xt}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L)(K) \cong H^n \mathrm{Hom}_{\mathcal{W}}(C_\bullet, L)(K) $$

.

Remark. We can always take $\mathcal{T} = \{G\}$ and then $\mathcal{T}$ is taut with respect to $\mathcal{W}$. In this way we can always obtain $(\varprojlim^G_{\mathcal{W}})^n L$ in part (1).

Proof. For (1) we apply the ${}^{II}E$ spectral sequence and see that it collapses. The second part follows using 3.2 to see that $\operatorname{Ext}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(\bar{R}, L) \cong R^n \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(\bar{R}, -)(L)$ and then 3.3 to identify this with $(\varprojlim^{\mathcal{T}}_{\mathcal{W}})^n L$. □

Proposition 4.3. *Let $C_\bullet$ be a complex in $\mathrm{CS}_{\mathcal{W}}(G)$. Let $X \subseteq \mathcal{W}$ and suppose that $L \in \mathrm{CS}_{\mathcal{W}}(G)$ is injective relative to X and also that for each $H \in X$ we have that $\operatorname{Res}^G_H C_\bullet$ is split exact. Then $\operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, L)$ is split exact. If L is also a Mackey functor then this is split exact as a complex of Mackey functors.*

Note that L being a Mackey functor entails $\mathcal{W}$ being closed under intersections. Also L is only required to be relatively injective as a coefficient system.

Proof. When we restrict to $H \in X$, $\operatorname{Res}^G_H C_\bullet$ becomes split exact. Thus $\operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(H)}(\operatorname{Res}^G_H C_\bullet, L)$ is split exact. Now apply Coind^G_H and use

$$\operatorname{Coind}^G_H \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(H)}(\operatorname{Res}^G_H -, L) \cong \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-, \operatorname{Coind}^G_H \operatorname{Res}^G_H L)$$

(by 2.11). We see that

$$\operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet, \operatorname{Coind}^G_H \operatorname{Res}^G_H L)$$

is split exact. Thus our sequence is a summand of a product of split exact sequences, so is itself split exact by [24].

The splitting as a Mackey functor comes from 2.16. □

5. Bredon Cohomology

If Δ is a G-CW-complex on which G acts admissibly (i.e. the stabiliser of each cell fixes it pointwise), let Δ_n denote the G-set of n-cells. We can form a chain complex of coefficient systems $C_\bullet[\Delta^?]$ in $\mathrm{CS}_{\mathcal{S}}(G)$ by setting

$$C_n[\Delta^?] = R[\Delta^?_n] \cong \bigoplus_{n\text{-cells } \sigma \text{ up to } G\text{-conjugacy}} R[G/\operatorname{Stab}_G \sigma^?],$$

with the natural boundary morphisms, as described in [7].

More succinctly, we regard Δ as a simplicial G-set (in the language of [26]) and apply the functor $G/H \mapsto R[G/H^?]$ to obtain a semi-simplicial coefficient system, and then take $C_\bullet[\Delta^?]$ to be the associated chain complex of coefficient systems.

We often restrict this chain complex to some class $\mathcal{W}$ where it is better behaved.

We will also use the augmented complex, $\tilde{C}_\bullet[\Delta^?]$, where we add the term $\bar{R}$ in degree -1 and the map $R[\Delta^?_0] \to \bar{R}$ takes each 0-cell to 1.

The definition of the cohomology of Δ with coefficients in a Mackey functor M in [17] amounts to saying that it is the cohomology of the complex $\operatorname{Hom}_{\mathrm{MF}\,\mathcal{S}(G)}(SC_\bullet[\Delta^?], M)$. But this is isomorphic to $\operatorname{Hom}_{\mathrm{CS}\,\mathcal{S}(G)}(C_\bullet[\Delta^?], M)$. For a slightly different approach, see [22].

Notice that if $H \in \mathcal{W}$ then $\operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(R[G/H^?], L) \cong \operatorname{Coind}^G_H \operatorname{Res}^G_H L$ by 2.12, and so if the stabiliser of every cell in Δ is contained in $\mathcal{W}$ then $\operatorname{Hom}_{\mathcal{W}}(C_\bullet[\Delta^?], L)$ takes the form described explicitly in [7] and [24], which we sometimes refer to as $L_{\Delta_\bullet}$.

If $\mathcal{X}$ is a class of subgroups of G then we can regard $\mathcal{X}$ as a G-poset and form the geometric realisation $|\mathcal{X}|$.

Lemma 5.1. *If the normaliser of each chain in $\mathcal{X}$ is in $\mathcal{W}$ then*

$$\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\tilde{C}_\bullet(|\mathcal{X}|^?), L)$$

is the normaliser sequence of the introduction, except that we have $\varprojlim^G_{\mathcal{W}} L$ in the first place instead of $L(G)$.

We define the *Bredon cohomology* of Δ with coefficients in L to be

$$H^*_{\mathrm{CS}_\mathcal{W}}(\Delta, L) = \mathbb{E}\mathrm{xt}^*_{\mathrm{CS}_\mathcal{W}}(C_\bullet[\Delta^?], L).$$

This is again an element of $\mathrm{CS}_\mathcal{T}(G)$.

Remark. For Bredon in [7], $\mathcal{W} = \mathcal{S}$ always, so $C_\bullet[\Delta^?]$ is a complex of projectives and $H^*_{\mathrm{CS}_\mathcal{W}}(\Delta, L) = H^* \mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(C_\bullet[\Delta^?], L)$, but this is not always true for general $\mathcal{W}$.

Remark. In view of 2.20 we can see that the definition of cohomology with coefficients in a Mackey functor M given in [17] is equivalent to $H^* \mathrm{Hom}_{\mathrm{MF}_\mathcal{S}}(SC_\bullet[\Delta^?], M) \cong H^* \mathrm{Hom}_{\mathrm{CS}_\mathcal{S}}(C_\bullet[\Delta^?], M)$, so is just Bredon cohomology, with the transfer given as in 2.16.

Example. If $\mathcal{W} = \{1\}$, the trivial group, and $\mathcal{T} = \{G\}$ then $H^*_{\mathrm{CS}_{\{1\}}}(\Delta, L)(G)$ is just the usual G-equivariant cohomology of Δ as in [8], i.e. the cohomology of the Borel construction.

Theorem 5.2. *If $L \in \mathrm{CS}_\mathcal{W}(G)$ and $\tilde{H}_*(\Delta^H, R) = 0$ (i.e. R-acyclic) for every $H \in \mathcal{W}$ and $(\varprojlim^S_\mathcal{W})^n L = 0$ for every $n \geq 1$ and every S which is the stabiliser of a cell in Δ (e.g. $S \in \mathcal{W}$), then*

$$H^n(\mathrm{hom}_{\mathrm{CS}_\mathcal{W}(G)}(C_\bullet[\Delta^?], L)) \cong (\varprojlim^G_\mathcal{W})^n L.$$

Proof. We take $\mathcal{T} = \{G\}$ and check that the conditions of 4.2 are satisfied (with $K = G$). This is clear for the first part. For the second we calculate instead with $\mathcal{T} = \mathcal{S}$, knowing that this will not matter by 4.1. Now $\mathrm{Ext}^n_{\mathrm{CS}_\mathcal{W}(G)}(R[G/S^?], L)(G) \cong \mathrm{Coind}^G_S \mathrm{Ext}^n_{\mathrm{CS}_\mathcal{W}(S)}(\bar{R}, L)(G) \cong \mathrm{Ext}^n_{\mathrm{CS}_\mathcal{W}(S)}(\bar{R}, L)(S)$ by 3.4. Next we work in $\mathrm{CS}_\mathcal{W}(S)$ with $\mathcal{T} = \{S\}$ (invoking 4.1 again). But now Hom is balanced so, by 3.3, we have $\mathrm{Ext}^n_{\mathrm{CS}_\mathcal{W}(S)}(\bar{R}, L)(S) \cong (\varprojlim^S_\mathcal{W})^n L$, which is 0 by hypothesis. □

Theorem 5.3. *Let $L \in \mathrm{CS}_\mathcal{W}(G)$ and $X \subseteq \mathcal{S}$. Suppose that L is injective relative to X and that Δ^K is R-acyclic for every $K \in \mathcal{W}$, $K \subseteq H \in X$.*

Suppose also that for each subgroup $H \in X$ we know that H is taut with respect to $\mathcal{W}$ and also for each cell σ of Δ, $\mathrm{Stab}_H(\sigma)$ is taut with respect to $\mathcal{W}$.

Then the chain complex

$$\mathrm{Hom}_{\mathrm{CS}_\mathcal{W}}(\tilde{C}_\bullet[\Delta^?], L)$$

is split exact in $\mathrm{CS}_\mathcal{S}(G)$. If L is a Mackey functor then the complex is split as a complex of Mackey functors.

Proof. Let $H \in X$ and consider $\mathrm{Res}^G_H \tilde{C}[\Delta^?]$. It is exact, by the condition on the Δ^K, and a complex of projectives, by the tautness conditions. The conditions of 4.3 are now satisfied and the result follows . □

Remark.

(1) The statement for Mackey functors is similar to the main theorem of Webb, [24]. He has $\mathcal{W} = \mathcal{S} \setminus \mathcal{Y}$, and our X is his $\mathcal{X} \setminus \mathcal{Y}$. Notice firstly that relative injectivity is the same as relative projectivity for Mackey functors, and secondly that if a coefficient system L is injective relative to $\mathcal{X}$ and L vanishes on $\mathcal{Y}$, then L is injective relative to $\mathcal{X} \setminus \mathcal{Y}$, at least if $\mathcal{Y}$ is closed under subgroups in $\mathcal{X}$.

(2) This proof of Webb's theorem, shorn of the general notation, is in fact very short. The relative injectivity condition allows us to reduce to the the case of a group in X, and then the complex $\tilde{C}_\bullet[\Delta^?]$ is an exact complex of projectives, so splits.

(3) If R is p-local and Δ^H is R-acyclic for every $H \leq G$ of order p, then Δ^H is R-acyclic for every non-trivial p-subgroup H by Smith theory, or by using equivariant cohomology as in [8].
(4) Another proof of Webb's Theorem has been given by Bouc [5].

Usually Δ is taken to be the Quillen complex, i.e. $|\mathcal{S}_p^1(G)|$, or some variant. Webb gives many examples of 5.3, but 5.2 is also useful. It can be used to give a simpler proof of the main results in [19]: here is another application.

Example. Fix a prime p and let B denote the ring of Brauer characters, considered as a coefficient system over $\mathbb{C}$ on some finite group G. Let $\mathcal{N}_p$ be the class of subgroups which contain a non-trivial normal p-subgroup, and let Δ denote the usual Brown complex.

Since each stabiliser of a cell in Δ is in $\mathcal{N}_p$, 5.2 applies to $B \in \mathrm{CS}_{\mathcal{N}_p}(G)$ on Δ and also $\mathrm{Hom}_{\mathcal{N}_p}(C_\bullet, B) \cong B_{\Delta_\bullet}$. But by Robinson's reformulation of Alperin's Conjecture [16], $\sum(-1)^i \dim B_{\Delta_i}(G)$ should be equal to the number of non-projective simple modules for G, denoted $f_o(G)$.

It follows that (the non-blockwise version of) Alperin's weight conjecture is true for all finite groups if and only if for all finite groups G

$$f_o(G) = \sum_i (-1)^i \dim(\varprojlim{}^i_{\mathcal{N}_p} B)(G).$$

It is interesting to try and understand this by filtering B by functors which are non-zero on only one conjugacy class and then calculating the higher limits of these in the manner of [14]. The result is the original formulation of Alperin's conjecture, by essentially the same proof as in [16].

6. The structure of $C_\bullet([\Delta^?])$

First we need some lemmas.

Lemma 6.1. *For any $H \leq G$ and any class $\mathcal{W}$ of subgroups of G, let $\pi_0(|\mathcal{W} \cap \mathcal{S}(H)|)$ denote the H-set of components in $\mathcal{W} \cap \mathcal{S}(H)$. Then, as RG-modules,*

$$\mathrm{Res}^{\mathcal{W}\cup\{1\}}_{\{1\}} \varinjlim{}^{\mathcal{W}\cup\{1\}}_{\mathcal{W}} R[G/H^!] \cong R[G \times_H \pi_0(|\mathcal{W} \cap \mathcal{S}(H)|)].$$

Proof. Since $\varinjlim$ commutes with Ind (their right adjoints commute) and $R[G/H^!] \cong \mathrm{Ind}_H^G \bar{R}$ it suffices to prove the case $G = H$, observing that both sides are induced modules. But $\varinjlim^{\mathcal{W}\cup\{1\}}_{\mathcal{W}} \bar{R}$ is formed by taking one basis element for each element of $\mathcal{W} \cap \mathcal{S}(H)$ and then identifying two basis elements if there is an inclusion between the corresponding subgroups. This yields $R[\pi_0(|\mathcal{W} \cap \mathcal{S}(H)|)]$. □

It will be convenient to define a coefficient system to be *based at* $H \leq G$ if it is a summand of a sum of $R[G/H^?]$s.

Corollary 6.2. *Consider the canonical map*

$$\varinjlim{}^{\mathcal{W}\cup\{1\}}_{\mathcal{W}} R[G/H^!] \to R[G/H^!]$$

in $\mathrm{CS}_{\mathcal{W}\cup\{1\}}(G)$.

It is onto if H contains some element of $\mathcal{W}$ and an isomorphism if $\mathcal{W} \cap \mathcal{S}(H)$ is connected, in particular if $H \in \mathcal{W}$.

If H does not contain any subgroup in $\mathcal{W}$ then the left hand side is 0 and the right hand side is based at H.

Lemma 6.3. *Let $F \leq G$ be of index invertible in R, and assume all complexes to be bounded below. If $D_\bullet$ is a complex of RG-modules, then $\mathrm{Res}^G_F D_\bullet$ is homotopy equivalent to a complex of projective RF-modules if and only if $D_\bullet$ is homotopy equivalent to a complex of projective RG-modules.*

Proof. If $\mathrm{Res}^G_F D_\bullet \simeq P_\bullet$ then $\mathrm{Ind}^G_F \mathrm{Res}^G_F D_\bullet \simeq \mathrm{Ind}^G_F P_\bullet$, which is a complex of projectives. But $D_\bullet$ is a summand of $\mathrm{Ind}^G_F \mathrm{Res}^G_F D_\bullet$ by the maps

$$d \mapsto \sum_{g \in G/F} g \otimes g^{-1} d, \quad h \otimes d \mapsto |G:F|^{-1} hd, \quad d \in D_\bullet, h \in G.$$

In order to simplify the notation we write $D_\bullet \oplus X_\bullet \simeq Q_\bullet$, where $Q_\bullet$ is a complex of projectives. Let $P^D_\bullet \stackrel{\rho_D}{\to} D_\bullet$ and $P^X_\bullet \stackrel{\rho_X}{\to} X_\bullet$ be projective resolutions. The composition $P^D_\bullet \oplus P^X_\bullet \stackrel{\rho_D \oplus \rho_X}{\longrightarrow} D_\bullet \oplus X_\bullet \stackrel{\simeq}{\to} Q_\bullet$ is a quasi-isomorphism of bounded below complexes of projectives. Thus it is a homotopy equivalence and hence $\rho_D \oplus \rho_X$ is also a homotopy equivalence. It follows that ρ_D must be a homotopy equivalence. □

If G acts admissibly on a CW-complex Δ we define $\mathrm{Stab}_G(\Delta)$ to be the set of subgroups $\mathrm{Stab}_G(\delta)$, where δ is a cell of Δ.

Theorem 6.4. *Suppose that G acts admissibly on a CW-complex Δ. Suppose also that there is a subgroup $F \leq G$ with $|G:F|$ invertible in R such that Δ^F is R-acyclic and also for every $K \in \mathrm{Stab}_F(\Delta)$, $K \neq 1$, we have that Δ^K is R-acyclic.*

Then $\tilde{C}_\bullet(\Delta)$ is homotopy equivalent as a complex in $\mathrm{CS}_{\{1\}}(G) \cong RG$-Mod to a bounded complex of projectives.

If $\mathcal{W}$ is a class of subgroups of G such that for each $K \in \mathcal{W}$, $K \neq 1$ we have that Δ^K is R-acyclic, then $\tilde{C}_\bullet(\Delta^?)$ is equal in the derived category of $\mathrm{CS}_{\mathcal{W}}(G)$ to a complex of projective coefficient systems which are projective relative to 1.

Proof. Let $\mathcal{V} = \{\mathrm{Stab}_F(\Delta) \setminus \{1\}\} \cup \{F\}$, and work in $\mathrm{CS}_{\mathcal{V}}(F)$. $\mathrm{Res}_{\mathcal{V}} \mathrm{Res}^G_F \tilde{C}_\bullet(\Delta^?)$ is an exact complex of projectives, by 2.8 and the definition of $\mathcal{V}$, so must split. Thus $\varinjlim_{\mathcal{V}}^{\mathcal{V} \cup \{1\}} \mathrm{Res}_{\mathcal{V}} \mathrm{Res}^G_F \tilde{C}_\bullet(\Delta^?)$ is also a split exact complex of projectives, since $\varinjlim$ preserves projectives by 2.2 and the remark before 2.15.

Now, in $\mathrm{CS}_{\mathcal{V} \cup \{1\}}(F)$, the natural map

$$\varinjlim_{\mathcal{V}}^{\mathcal{V} \cup \{1\}} \mathrm{Res}_{\mathcal{V}} \mathrm{Res}^G_F \tilde{C}_\bullet(\Delta^?) \to \mathrm{Res}_{\mathcal{V} \cup \{1\}} \mathrm{Res}^G_F \tilde{C}_\bullet(\Delta^?)$$

is a sum of those in 6.2. Each stabiliser is either in $\mathcal{V}$ or is 1, so the map is injective and the cokernel $P_\bullet$ is a complex of projectives based at 1.

Now $\mathrm{Res}_{\mathcal{V} \cup \{1\}} \mathrm{Res}^G_F \tilde{C}_\bullet(\Delta^?) \to P_\bullet$ is a quasi-isomorphism of bounded below complexes of projectives, so must be a homotopy equivalence. Restricting to the subgroup 1 (so we are just dealing with RF-modules), we see that $\tilde{C}_\bullet(\Delta)$ is homotopy equivalent to $P_\bullet(1)$, which is a complex of projective RF-modules.

Now apply 6.3 to obtain the first claim.

For the second claim, we work in $\mathrm{CS}_{\mathcal{W}}(G)$, and notice that $\tilde{C}_\bullet(\Delta^?)$ is equal in the derived category to the complex obtained from it by changing the evaluations to 0 on every subgroup not equal to 1. Now we use the first claim. □

Lemma 6.5. *In any Abelian category, if there is a map of chain complexes $f : C_\bullet \to P_\bullet$, which is a quasi-isomorphism, and where $P_\bullet$ and $C_\bullet$ are bounded below and $P_\bullet$ is a complex of projectives, then*

$$C_\bullet \oplus S_\bullet \cong P_\bullet \oplus E_\bullet,$$

where $S_\bullet$ is a split exact complex of projectives and $E_\bullet$ is an exact complex. If f is a homotopy equivalence, then $E_\bullet$ is split exact. If $C_\bullet$ and $P_\bullet$ are bounded or of finite type (when this makes sense) then so are $S_\bullet$ and $E_\bullet$.

Proof. By adding a split exact complex of projectives bounded below to $C_\bullet$, we can assume that f is an epimorphism. Let $E_\bullet$ be the kernel, so $E_\bullet$ is exact and bounded below.

Now, since $P_\bullet$ is a complex of projectives, we can take a splitting in each degree, and in this way identify $C_\bullet$ with $E_\bullet \oplus P_\bullet$, but with boundary map $d_{E\oplus P} + \phi$, where ϕ is a collection of maps $\phi_i : P_i \to E_{i-1}$. If we set $\phi'_i = (-1)^i \phi_i$ then these combine to give a chain map $\phi' : P_\bullet \to E_\bullet$ of degree -1. But ϕ' must factor through a projective resolution of $E_\bullet$, which must be split, so ϕ' is nullhomotopic. Thus we have maps $\theta_i : P_i \to E_i$ such that $\phi' = d_E\theta + \theta d_P$

Set $\theta'_i = (-1)^i\theta_i$. An isomorphism $\Theta : C_\bullet \to E_\bullet \oplus P_\bullet$ is now given by setting $\Theta(e, p) = (e + \theta' p, p), \quad e \in E, p \in P$.

If f was a homotopy equivalence then $E_\bullet$ must be split. □

Corollary 6.6. *In the circumstances of 6.4, set* $\tilde{C}_\bullet(\Delta) = \mathrm{Res}_{\{1\}} \tilde{C}_\bullet(\Delta^!)$*, and regard it as a complex of RG-modules. Then*

$$\tilde{C}_\bullet(\Delta) \oplus S_\bullet \cong P_\bullet \oplus E_\bullet,$$

as bounded complexes of RG-modules, where $S_\bullet$ *is a split exact complex of projectives,* $P_\bullet$ *is a complex of projectives and* $E_\bullet$ *is a split exact complex.*

Proof. Apply 6.5 to the first claim of 6.4. □

Finally, we obtain Webb's original result.

Corollary 6.7. *In the circumstances of 6.4, but with* Δ *finite and* R *a complete local ring, set* $\tilde{C}_\bullet(\Delta) = \mathrm{Res}_{\{1\}} \tilde{C}_\bullet(\Delta^!)$*, and regard it as a complex of RG-modules. Then*

$$\tilde{C}_\bullet(\Delta) \cong P_\bullet \oplus E_\bullet,$$

as complexes of RG-modules, where $P_\bullet$ *is a complex of projectives and* $E_\bullet$ *is a split exact complex.*

Proof. The complexes in the isomorphism of 6.6 are of finite type so we can apply the Krull-Schmidt Theorem to cancel $S_\bullet$. □

Example. The standard examples where the hypotheses of 6.4 are satisfied are when R is p-local, F is the Sylow p-subgroup of a finite group G and either:

(i) Δ is the geometric realisation of $\mathcal{S}_p^1(G)$, or

(ii) P is a p-subgroup of some group E, $G = N_E(P)/P$ and Δ is the geometric realisation of $\mathcal{S}_p(E)_{>P}$.

In case (i) we call the homotopy class of complexes of projective modules homotopy equivalent to $\tilde{C}_\bullet(\Delta)$ the *Steinberg complex* of G, and denote it by $\mathrm{St}_\bullet(G)$. Whenever we mention this complex it will be implicit that every prime dividing $|G|$, except perhaps for p, is invertible in R. If p does not divide $|G|$ then $\mathrm{St}_\bullet(G)$ consists just of the trivial coefficient system $\bar{R}$ in degree -1.

In both cases the stabiliser of any cell contains a non-trivial normal p-subgroup, and conversely for any $K \in (\mathcal{N}_p)_{>P}$, Δ^K is R-acyclic.

In these circumstances we have a uniqueness result.

Proposition 6.8. *Suppose that* G *acts admissibly on a CW-complex* Δ *in such a way that* $\mathrm{Stab}_G(\Delta) \subseteq \mathcal{N}_p$*, and that* $\mathcal{X}$ *is a class of subgroups containing* $\mathcal{N}_p$ *such that, for each* $1 \neq K \in \mathcal{X}$*,* Δ^K *is* R*-acyclic. Then* $\tilde{C}_\bullet(\Delta^!) \simeq \tilde{C}_\bullet(|\mathcal{S}_p^1(G)|^!)$ *in* $\mathrm{CS}_\mathcal{X}(G)$ *and, in particular,* $\tilde{C}_\bullet(\Delta) \simeq \mathrm{St}_\bullet(G)$ *in RG-Mod.*

Proof. Both $\tilde{C}_\bullet(\Delta^!)$ and $\tilde{C}_\bullet(|\mathcal{S}_p^1|^!)$ are projective resolutions of $\bar{R}$ in $\mathrm{CS}_{\mathcal{X}\setminus\{1\}}(G)$, so are homotopy equivalent by maps which are the identity on $\bar{R}$ and where the homotopies take the value 0 on $\bar{R}$.

We can now apply $\varinjlim^{\mathcal{X}}_{\mathcal{X}\setminus\{1\}}$ to recover $C_\bullet(\Delta)$ and $C_\bullet(|\mathcal{S}_p^1|)$, by 6.2. We also obtain maps between them and the necessary homotopies, which we extend to $\bar{R}$ by the identity and 0 respectively. □

For example this shows that in case (ii) above the complex obtained is in fact $\mathrm{St}_\bullet(G)$. Of course in this case the defining simplicial complexes are known to be equivariantly homotopy equivalent anyway.

7. Properties of the Steinberg Complex

It is often convenient to consider $\mathrm{St}_\bullet(G)$ as a coefficient system in $\mathrm{CS}_{\mathcal{W}}(G)$ by giving it the value 0 on all non-trivial subgroups. This is in fact formally $\varinjlim_1^{\mathcal{W}} \mathrm{St}_\bullet(G)$, and is still a complex of projectives, based at 1. We will denote it by $\mathrm{St}^0_\bullet(G)$.

There is an alternative description $\mathrm{St}^0_\bullet(G) \simeq \tilde{C}_\bullet(|\mathcal{S}^1_p|^?) \otimes \bar{R}_1$, where $\bar{R}_1$ is the coefficient system which takes the value R on 1 and 0 elsewhere.

Proposition 7.1. *For any class of subgroups $\mathcal{W}$ of G containing 1 and any $M \in \mathrm{CS}_{\mathcal{W}}(G)$, the following are all homotopy equivalent as complexes in $\mathrm{CS}_{\mathcal{W}}(G)$:*

1. $\mathrm{Hom}_{R?}(\mathrm{St}_\bullet(G), M(1))$,
2. $\mathrm{Hom}_{\mathcal{W}}(\mathrm{St}^0_\bullet(G), M)$,
3. $\mathrm{Hom}_{R?}(\tilde{C}_\bullet(|\mathcal{S}^1_p|), M(1))$,
4. $\mathrm{Hom}_{\mathcal{W}}(\tilde{C}_\bullet(|\mathcal{S}^1_p|^?), M(1)^?)$,
5. *If $\mathcal{W} \subseteq \mathcal{N}_p$, $R\,\mathrm{Hom}_{\mathcal{W}}(\tilde{C}_\bullet(|\mathcal{S}^1_p|^?), M)$. (Where $R\,\mathrm{Hom}$ is the complex used to define $\mathbb{E}\mathrm{xt}$.)*

Proof. Now (1) and (2) are homotopy equivalent because $\mathrm{St}^0_\bullet(G) = \varinjlim_1^{\mathcal{W}} \mathrm{St}_\bullet(G)$, and $\varinjlim$ is the left adjoint of restriction. Also (1) and (3) are homotopy equivalent by the first part of 6.4. But (3) and (4) are homotopic because $M(1)^? \cong \varprojlim_1^{\mathcal{W}} M(1)$ and $\varprojlim$ is the right adjoint of restriction. Finally $\mathrm{St}^0_\bullet(G)$ is a projective resolution of $\tilde{C}_\bullet(|\mathcal{S}^1_p|^?)$ by the last part of 6.4, so (5) is homotopic to (2) by definition. □

Lemma 7.2. *If $H \lhd G$ and p divides $|H|$ then for any p-subgroup P of G:*

1. *p divides $|C_G(P) \cap H|$,*
2. *If σ is any chain of p-subgroups of G then p divides $|N_H(\sigma)|$.*

Proof. P permutes the Sylow p-subgroups of H by conjugation. The number of these is coprime to p, so at least one of them is fixed: call it S.

Now P permutes the non-trivial elements of S. Again, the number of these is coprime to p, so one is fixed, say s. Now $s \in C_G(P) \cap H$.

For (2), let P be the largest subgroup in the chain and apply (1). □

Now we can state a fundamental result from [14], (although our proof is based on a preliminary version of [12]).

Theorem 7.3. *Suppose that R is a p-local discrete valuation ring, and that M is an RG-module such that either M is finitely generated or M is projective over R . If the order of the kernel (i.e. the subgroup of elements of G which act trivially on M) is divisible by p, then the complex of RG-modules $\mathrm{hom}_{RG}(\mathrm{St}_\bullet(G), M)$ is split exact.*

Proof. First we assume that $pM = 0$.

Define $\widehat{\mathrm{hom}}_{RG}$ to be hom_{RG} modulo the image of $\mathrm{tr}_1^G : \mathrm{hom}_R \to \mathrm{hom}_{RG}$, (with the usual transfer on Hom). Using the complex (3) of 7.1, each term is a sum of pieces of the form $\mathrm{hom}_{RG}(R[G/S], M)$, where S is the stabiliser of a chain σ.

By 7.2 with H as the kernel of M, we see that $S \cap H$ contains a non-trivial p-subgroup Q, say, so Q is in both S and H. Under the isomorphism $\mathrm{hom}_{RG}(R[G/S], M) \cong \mathrm{hom}_{RS}(R, M)$ the image of tr_1^G on the left corresponds to the image of tr_1^S on the right. But tr_1^S factors through tr_1^Q, and this is multiplication by $|Q|$, which is

equal to 0. Thus $\hom_{RG}(\mathrm{St}_\bullet(G), M) = \widehat{\hom}_{RG}(\mathrm{St}_\bullet(G), M)$; but $\widehat{\hom}$ vanishes on projectives.

In general, since $\mathrm{St}_\bullet(G)$ is homotopy equivalent to a complex of projectives, $\hom_{RG}(\mathrm{St}_\bullet(G), M)/p \cong \hom_{RG}(\mathrm{St}_\bullet, M/p)$, and M/p also satisfies the conditions of the theorem, so is split by the proof above.

If M is finitely generated over R, then so is $\hom_{RG}(\mathrm{St}_\bullet(G), M)$, and so split modulo p implies split, (by an obvious generalisation of Nakayama's Lemma).

If M is not finitely generated then it is a direct limit of finitely generated submodules, and the homology of $\hom_{RG}(\mathrm{St}_\bullet(G), -)$ commutes with direct limits, so the latter is exact. It is also a complex of projective R-modules if M is, because its terms are summands of sums of terms $\hom_{RG}(RG, M) \cong M$. Thus the complex must split. □

We can generalise this slightly in a way that will be useful in section 9. Example (ii) in §6 suggests that we consider the complex $\mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \mathrm{St}^0_\bullet(N_G(P)/P)$.

Corollary 7.4. *For simplicity we assume that P is a p-subgroup of G, that all coefficient systems are over $\mathcal{S}_p$ and that R is p-local.*

1. $\mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \mathrm{St}^0_\bullet(N_G(P)/P)$ *is a complex of projectives, based at P.*
2. $\hom_{\mathrm{CS}_{\mathcal{S}_p}(G)}(\mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \mathrm{St}^0_\bullet(N_G(P)/P), M) \simeq 0$ *for any $M \in \mathrm{CS}_{\mathcal{S}_p}(G)$ with $M(P)$ either finitely generated or projective over R and on which p divides the order of the kernel of $M(P)$ as an $N_G(P)/P$-module.*
3. $G_0 \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \mathrm{St}^0_\bullet(N_G(P)/P) \simeq 0$ *if P is not in $\mathcal{D}_p(G)$. (Where G_0 is the left adjoint of the forgetful functor $I : \mathrm{GCS} \to \mathrm{CS}$.)*

Proof. For (1), observe that the functors preserve projectives because their right adjoints are exact.

For (2), the adjoint properties show that we are just calculating $\hom_{\mathrm{CS}_{\mathcal{S}_p(N_G(P)/P)}}(\mathrm{St}^0_\bullet(N_G(P)/P), Q_{N_G(P)/P}^{N_G(P)} \mathrm{Res}_{N_G(P)}^G M)$, and this is 0 by 7.3.

Now (3) is a formal consequence of (2) if P is not centric, since any geometric M will have $PC_G(P)$ in the kernel of $M(P)$, and p divides the order of $PC_G(P)/P$, so p will divide the order of the kernel of $M(P)$ as an $N_G(P)/P$-module. If P is centric but not in $\mathcal{D}_p$ then P is not in $\mathcal{B}_p$, by 3.14, so the complex is acyclic. □

The next result is a version of one in [12].

Theorem 7.5. *If $L \in \mathrm{GCS}_{\mathcal{S}_p}(G)$ then $H^*_{\mathrm{CS}_{\mathcal{S}_p}(G)}(|\mathcal{S}_p|, L)(G) \cong H^*_{\mathrm{CS}_{\mathcal{S}_p}(G)}(|\mathcal{C}_p|, L)(G)$, (as R-modules).*

Proof. We work in $\mathrm{CS}_{\mathcal{S}_p}(G)$. Define $\mathcal{S}_p^{\geq n}$ to be the subclass of $\mathcal{S}_p$ of elements of order greater than or equal to n. Then $C_\bullet(|\mathcal{S}_p|^?)$ is filtered by the complexes $C_\bullet(|\mathcal{S}_p^{\geq n}|^?)$. Now the terms of the factor $C_\bullet(|\mathcal{S}_p^{\geq n}|^?)/C_\bullet(|\mathcal{S}_p^{\geq n+1}|^?)$ correspond to chains with bottom element of order n. There are no inclusions between such chains with the same bottom element, so this factor splits as a direct sum of pieces indexed by the conjugacy class of the bottom element P of the chain, and this piece is induced from $N_G(P)$.

$$\begin{aligned} C_\bullet(|\mathcal{S}_p^{\geq n}|^?)/C_\bullet(|\mathcal{S}_p^{\geq n+1}|^?) &\cong \bigoplus_{P \in \mathcal{S}_p/G,\ |P|=n} \mathrm{Ind}_{N_G(P)}^G \tilde{C}_{\bullet+1}(|(\mathcal{S}_p)_{>P}|^?) \\ &\cong \bigoplus_{P \in \mathcal{S}_p/G,\ |P|=n} \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \tilde{C}_{\bullet+1}(|(\mathcal{S}_p)_{>P}|^?) \\ &\simeq \bigoplus_{P \in \mathcal{S}_p/G,\ |P|=n} \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \mathrm{St}^0_{\bullet+1}(N_G(P)/P). \end{aligned}$$

The last equivalence is a consequence of the well-known fact that $|(\mathcal{S}_p(G))_{>P}|$ is equivariantly homotopy equivalent to $|(\mathcal{S}_p(N_G(P)))_{>P}|$, (by the assignment $Q \geq N_Q(P)$), and the latter is clearly isomorphic to $|\mathcal{S}^1_p(N_G(P)/P)|$.

If we apply G_0 then the summands with P not centric will vanish by 7.4. Note also that $(\mathcal{S}_p)_{>P} = (\mathcal{C}_p)_{>P}$ if P is centric.

Since the last line of the formulas above is a complex of projectives, it follows that the inclusion of $\mathcal{C}_p$ into $\mathcal{S}_p$ induces an isomorphism

$$\mathbb{E}\mathrm{xt}^*(C_\bullet(|(\mathcal{C}_p)^{\geq n}|^?)/C_\bullet(|(\mathcal{C}_p)^{\geq n+1}|^?), L) \cong \mathbb{E}\mathrm{xt}^*(C_\bullet(|(\mathcal{S}_p)^{\geq n}|^?)/C_\bullet(|(\mathcal{S}_p)^{\geq n+1}|^?), L).$$

We can now show, induction on n, the long exact sequence for $\mathbb{E}\mathrm{xt}$ and the five lemma, that there is an isomorphism

$$\mathbb{E}\mathrm{xt}^*(C_\bullet(|\mathcal{C}_p|^?)/C_\bullet(|\mathcal{C}_p^{\geq n}|^?), L) \cong \mathbb{E}\mathrm{xt}^*(C_\bullet(|\mathcal{S}_p|^?)/C_\bullet(|\mathcal{S}_p^{\geq n}|^?), L).$$

The case $n = 0$ is trivial and the case n large is the result claimed. □

8. The Subgroup Sequence

There are other ways of obtaining a chain complex in $\mathrm{CS}_{\mathcal{W}}(G)$ from a class $\mathcal{X}$ of subgroups, which do not factor through the geometric realisation. Although the examples that we will consider are simple and concrete it seems helpful to mention the general context.

We consider the category $\mathrm{ch}(\mathcal{X})$ of chains in $\mathcal{X}$ and inclusions. We need a contravariant functor $F : \mathrm{ch}(\mathcal{X}) \to \mathrm{PCS}_{\mathcal{W}}(G)$ together with a collection of conjugation maps $c_g : F(\sigma)(H) \to F({}^g\sigma)({}^gH)$ for each $g \in G, \sigma \in \mathrm{ch}(\mathcal{X}), H \in \mathcal{W}$. These conjugation maps must satisfy the usual properties $c_1 = \mathrm{Id}$, $c_{g_1}c_{g_2} = c_{g_1g_2}$ and they must commute with restriction in $\mathrm{PCS}_{\mathcal{W}}(G)$ and also with restriction in $\mathrm{ch}(\mathcal{X})$ (induced by inclusion). In addition we require that $F(\sigma)(H) = 0$ if $H \not\leq N_G(\sigma)$ and, for $h \in H \leq N_G(\sigma)$, we need $c_h = \mathrm{Id} : F(\sigma)(H) \to F(\sigma)(H)$.

This naturally makes $F(\sigma)$ into an element $\tilde{F}(\sigma) \in \mathrm{CS}_{\mathcal{W}}(N_G(\sigma))$, extended to $H \not\leq N_G(\sigma)$ by 0. These $\tilde{F}(\sigma)$ and the restriction maps $F(\sigma) \to F(\tau)$ uniquely determine the structure defined above.

For a G-subset X of $\mathrm{ch}(\mathcal{X})$ we define $F(X) = \oplus_{\sigma \in X} F(\sigma)$. With the natural action of G via the c_g we have $F(X) \in \mathrm{CS}_{\mathcal{W}}(G)$. In particular, writing (σ) for the orbit of σ, we have $F((\sigma)) \cong \mathrm{Ind}_{N_G(\sigma)}^G \tilde{F}(\sigma)$.

Up until now we have always used the functor F^N, where $\tilde{F}^N(\sigma) = \bar{R} \in \mathrm{CS}_{\mathcal{W}}(N_G(\sigma))$, so $F^N((\sigma)) \cong \mathrm{Ind}_{N_G(\sigma)}^G \bar{R} \cong R[G/N_G(\sigma)^?]$.

Recall that for any chain σ in $\mathcal{X}$ we denote by σ_b the smallest element and by σ_t the largest.

The functor which represents the subgroup sequence is F^S, defined by

$$F^S(\sigma)(P) = \begin{cases} R & \text{if } \sigma_b \geq P, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $F^S((\sigma))(P)$ is the free R-module on the chains τ in the orbit of σ with $\tau_b \geq P$. The restrictions are the canonical inclusion maps.

For $P \leq G$, define $\bar{R}_P \in \mathrm{CS}_{\mathcal{W}}(N_G(P))$ by $\bar{R}_P = \varinjlim_{\mathcal{W}(P)}^{\mathcal{W}(N_G(P))} \bar{R}$, so that for $H \in \mathcal{W}$, $\bar{R}_P(H)$ is R if $H \leq P$ and 0 otherwise. (So, in fact, $\bar{R}_P = \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \bar{R}_1$.)

Then we have $F^S((\sigma)) = \mathrm{Ind}_{N_G(\sigma)}^G \mathrm{Res}_{N_G(\sigma)}^{N_G(\sigma_b)} \bar{R}_{\sigma_b}$, which we will abbreviate to $\mathrm{Ind}_{N_G(\sigma)}^G \bar{R}_{\sigma_b}$.

Lemma 8.1. $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}_{N_G(\sigma)}^G \bar{R}_{\sigma_b}, L) \cong ((\varprojlim_{\mathcal{W}(\sigma_b)} L)(\sigma_b))^{N_G(\sigma)}$, *and if* $\sigma_b \in \mathcal{W}$, *then this is also isomorphic to* $L(\sigma_b)^{N_G(\sigma)}$.

Proof.

$$\begin{aligned}\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}_{N_G(\sigma)}^G \bar{R}_{\sigma_b}, L) &\cong \hom_{\mathrm{CS}_{\mathcal{W}}(N_G(\sigma))}(\bar{R}_{\sigma_b}, L)\\ &\cong \hom_{\mathrm{CS}_{\mathcal{W}}(N_G(\sigma))}(\varinjlim_{\mathcal{W}(\sigma_b)}^{\mathcal{W}(N_G(\sigma_b))} \bar{R}, L)\\ &\cong \hom_{\mathrm{CS}_{\mathcal{W}(\sigma_b)}(N_G(\sigma))}(\bar{R}, L)\\ &\cong (\varprojlim_{\mathcal{W}(\sigma_b)} L)(\sigma_b)^{N_G(\sigma)}\end{aligned}$$

□

Now $C_\bullet^S(\mathcal{X})$ is defined to be the complex in $\mathrm{CS}_{\mathcal{W}}(G)$ with terms $F^S(\mathrm{ch}_n(\mathcal{X}))$ and the usual boundary maps arising from the semisimplicial structure of $\mathrm{ch}(\mathcal{X})$. There is also an augmented version $\tilde{C}_\bullet^S(\mathcal{X})$.

The next two results follow directly from the definitions and 8.1 respectively.

Lemma 8.2. *$C_\bullet^S(\mathcal{X})(H) \cong C_\bullet(|\mathcal{X}_{\geq H}|)$.*

Corollary 8.3. *If $L \in \mathrm{CS}_{\mathcal{W}}(G)$ and $\mathcal{X} \subseteq \mathcal{W}$ then $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\tilde{C}_\bullet^S(\mathcal{X}), L)$ is the second sequence of the introduction, the subgroup sequence, except that $L(G)$ is replaced by $\varprojlim_{\mathcal{W}}^G L$.*

Lemma 8.4. *If every $\mathcal{X}_{\geq H}$, $H \in \mathcal{W}$, is R-acyclic (e.g. $\mathcal{W} \subseteq \mathcal{X}$) then $\mathbb{E}\mathrm{xt}^n_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet^S(\mathcal{X}), L) \cong R^n \mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-, L)(\bar{R})$ in $\mathrm{CS}_{\mathcal{W}}(G)$. If $\mathcal{T}$ is taut with respect to $\mathcal{W}$ then this is also equal to $(\varprojlim_{\mathcal{W}}^{\mathcal{T}})^n L$.*

Proof. This is a consequence of 4.2, in view of 8.2. □

Lemma 8.5. *In any Abelian category, let $C_\bullet$ be a chain complex and suppose that it has a finite filtration such that each of the factors is homotopy equivalent to a complex of projectives $P_\bullet^i$ and such that in each degree the filtration splits to give a direct sum decomposition. Then $C_\bullet$ is homotopy equivalent to a complex of projectives: ignoring the boundary maps, this complex can be taken to be $\oplus_i P_\bullet^i$.*

Proof. By induction, we may reduce to the case where there are only two composition factors, so we have a short exact sequence $X_\bullet^1 \xrightarrow{a} C_\bullet \to X_\bullet^2$, which is split in each degree and where $X_\bullet^i \simeq P_\bullet^i, i = 1, 2$. Since a is split in each degree we know that $X_\bullet^2 \simeq \mathrm{cone}(a)$, so $X_\bullet^1 \xrightarrow{a} C_\bullet \to X_\bullet^2$ extends to a triangle in the homotopy category.

Thus we have a triangle $P_{\bullet+1}^2 \xrightarrow{b} P_\bullet^1 \to C_\bullet \to P_\bullet^2$, so $C_\bullet \simeq \mathrm{cone}(b)$, which is a complex of projectives. □

We say that a coefficient system is *based at* $\mathcal{X}$ if it is a summand of a sum of terms of the form $R[G/X^?]$ with $X \in \mathcal{X}$.

Proposition 8.6. *Suppose that every prime dividing $|G|$ except perhaps p is invertible in R, that $\mathcal{X} \subseteq \mathcal{S}_p(G)$ and that $\mathcal{X}$ contains the Sylow p-subgroups of G.*

If for every $H \in \mathcal{X}$ that is not a Sylow p-subgroup of G and every p-subgroup K with $H < K \leq N_G(H)$ we have $(\mathcal{X}_{>H})^K$ is R-acyclic, then $C_\bullet^S(\mathcal{X})$ is, as a complex in $\mathrm{CS}_{\mathcal{S}_p}(G)$, homotopy equivalent to a complex of projectives, based at $\mathcal{X}$.

If we ignore the boundary maps, then this complex can be taken to be

$$\bigoplus_{P \in \mathcal{X}/G} \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} \mathrm{St}^0_{\bullet-1}(N_G(P)/P).$$

Proof. Define $\mathcal{X}^{\geq n}$ to be the subclass of $\mathcal{X}$ of elements of order greater than or equal to n. Then $C_\bullet^S(\mathcal{X})$ is filtered by the complexes $C_\bullet^S(\mathcal{X}^{\geq n})$. Now the terms of the factor $C_\bullet^S(\mathcal{X}^{\geq n})/C_\bullet^S(\mathcal{X}^{\geq n+1})$ correspond to chains with bottom element of order n. There are no inclusions between such chains with the same bottom element, so this

factor splits as a direct sum of pieces indexed by the conjugacy class of the bottom element P of the chain, and this piece is induced from $N_G(P)$.

$$\begin{aligned} C^S_\bullet(\mathcal{X}^{\geq n})/C^S_\bullet(\mathcal{X}^{\geq n+1}) &\cong \bigoplus_{P\in\mathcal{X}/G,\,|P|=n} \mathrm{Ind}^G_{N_G(P)}\,\tilde{C}_{\bullet-1}(|\mathcal{X}_{>P}|^?)\otimes\bar{R}_P \\ &\cong \bigoplus_{P\in\mathcal{X}/G,\,|P|=n} \mathrm{Ind}^G_{N_G(P)}\,\mathrm{Inf}^{N_G(P)}_{N_G(P)/P}\,\tilde{C}_{\bullet-1}(|\mathcal{X}_{>P}|^?)\otimes\bar{R}_1 \\ &\simeq \bigoplus_{P\in\mathcal{X}/G,\,|P|=n} \mathrm{Ind}^G_{N_G(P)}\,\mathrm{Inf}^{N_G(P)}_{N_G(P)/P}\,\mathrm{St}^0_{\bullet-1}(N_G(P)/P), \end{aligned}$$

by 6.8. These terms are projective, based at $\mathcal{X}$, if P is not maximal in $\mathcal{X}$. If P is Sylow then we just have $\mathrm{Ind}^G_{N_G(P)}\bar{R}_P$ in degree 0. But this is projective in $\mathrm{CS}_{\mathcal{W}}(N_G(P))$ by 2.5.

Now we can apply 8.5. □

Corollary 8.7. *([12]) Suppose that every prime dividing $|G|$ except perhaps p is invertible in R, that $\mathcal{X}\subseteq\mathcal{W}\subseteq\mathcal{S}_p(G)$ and that $\mathcal{X}$ contains the Sylow p-subgroups of G. Suppose also that, for every $H\in\mathcal{W}$, $\mathcal{X}_{\geq H}$ is R-acyclic and that for every non-Sylow $H\in\mathcal{X}$ and $K\in\mathcal{S}_p$ with $H\lhd K$ we have that $(\mathcal{X}_{>H})^K$ is R-acyclic.*

Then $C^S_\bullet(\mathcal{X})$ is homotopy equivalent to a projective resolution of $\bar{R}$ in $\mathrm{CS}_{\mathcal{W}}(G)$ by projectives based at $\mathcal{X}$.

So, for $L\in\mathrm{CS}_{\mathcal{W}}(G)$, we have $H^n\,\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(C^S_\bullet(\mathcal{X}),L)\cong R^n\,\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-,L)(\bar{R})$ in $\mathrm{CS}_{\mathcal{T}}(G)$.

In particular, the homology of the subgroup sequence of the introduction (after removing the first term) is $(\varprojlim^G_{\mathcal{W}})^n L$.

Proof. By 8.6 we see that $C^S_\bullet(\mathcal{X})$ is homotopy equivalent to a complex of projectives in $\mathrm{CS}_{\mathcal{S}_p}(G)$ based at $\mathcal{X}$. This remains projective on restriction to $\mathcal{W}$ by 2.8. Its cohomology is $\bar{R}$, by 8.2. This proves the claim about the projective resolution; for the rest use 8.3. □

For the rest of this section we continue to suppose that every prime dividing $|G|$ except perhaps p is invertible in R.

Corollary 8.8. *Suppose that $\mathcal{W}\subseteq\mathcal{S}_p$ is non-empty and closed under supergroups in $\mathrm{Wess}(S_p)$ and that $\mathrm{Wess}(S_p)\cap\mathcal{W}\subseteq\mathcal{X}\subseteq\mathcal{W}$. Then the hypotheses of 8.7 are satisfied.*

Note that we could replace $\mathrm{Wess}(\mathcal{S}_p)$ by $\mathcal{B}_p$ to obtain a simpler statement. (That the two are the same is a conjecture of Quillen.) We could also use a weaker definition of Wess, in terms of R-acyclicity instead of contractibility.

Proof. We need to check the conditions of 8.7.

Since $\mathcal{W}$ is closed under supergroups in $\mathrm{Wess}(\mathcal{S}_p)$, it contains the Sylow p-subgroups of G. Thus these are in $\mathrm{Wess}(\mathcal{S}_p)\cap\mathcal{W}$ and so in $\mathcal{X}$. So the Sylow p-subgroups are the only maximal elements of $\mathcal{X}$ and certainly remain maximal in $\mathcal{W}$.

Notice that, for any class of subgroups $\mathcal{V}$ and any $H,K\in\mathcal{S}_p$, we have that $\mathrm{Wess}(\mathcal{V}^K_{>H})=\mathrm{Wess}(\mathcal{V}^K)\cap\mathcal{V}_{>H}$.

Also, if K normalises J, then $(\mathcal{S}^K_p)_{>J}$ contracts to JK unless $K\leq J$. It follows that $\mathrm{Wess}(\mathcal{S}^K_p)=\mathrm{Wess}(\mathcal{S}_p)\cap(S_p)_{\geq K}$.

Now, if $H,K\in\mathcal{S}_p$ and $H\lhd K$, $H\neq K$ then $\mathrm{Wess}((\mathcal{S}^K_p)_{>H})=\mathrm{Wess}(\mathcal{S}_p)\cap(\mathcal{S}_p)_{\geq K}$. If also $H\in\mathcal{W}$ then, since $\mathcal{W}$ is closed under supergroups in $\mathrm{Wess}(\mathcal{S}_p)$, we have $\mathrm{Wess}((\mathcal{S}^K_p)_{>H})\subseteq\mathcal{W}^K_{>H}$. But the left hand side is clearly in $\mathrm{Wess}(\mathcal{S}_p)\cap\mathcal{W}$, so by hypothesis is in $\mathcal{X}$, and thus $\mathrm{Wess}((\mathcal{S}^K_p)_{>H})\subseteq\mathcal{X}^K_{>H}$.

Thus we can apply 3.10 to the inclusion $\mathcal{X}^K_{>H} \subseteq (\mathcal{S}^K_p)_{>H}$ to see that we have a homotopy equivalence on the geometric realisations. But we have just seen that the right hand side is contractible, so $\mathcal{X}^K_{>H}$ is contractible.

A similar proof shows that $\mathcal{X}_{\geq H}$ is contractible. □

Corollary 8.9. *If $\mathcal{W}$ is closed under intersections, $M \in \mathrm{MF}_{\mathcal{W}}(G)$ is injective relative to $\mathcal{W}$ and also the hypotheses of 8.7 are satisfied, then the subgroup sequence is split as a complex of Mackey functors.*

Proof. Just as in the proof of 4.3, the condition of relative injectivity allows us to reduce to the case of a group $H \in \mathcal{W}$. But $\bar{R}$ is projective in $\mathrm{CS}_{\mathcal{W}}(G)$, so $\tilde{C}^S_\bullet(\mathcal{X})$ splits. □

9. Change of Class of Groups

The results are based on those in [12].

As before, we continue to suppose that every prime dividing the order of $|G|$, except perhaps p, is invertible in R.

Corollary 9.1. *In the circumstances of 8.8, $(\varprojlim^G_{\mathcal{W}})^n \cong (\varprojlim^G_{\mathcal{B}_p\cap\mathcal{W}})^n \operatorname{Res}^{\mathcal{W}}_{\mathcal{B}_p\cap\mathcal{W}}$ on $\mathrm{CS}_{\mathcal{W}}(G)$.*

Proof. Use 8.7 and 8.8 with $\mathcal{X} = \mathcal{B}_p \cap \mathcal{W}$ and notice that the terms of the subgroup sequence only evaluate L on groups in $\mathcal{X}$. □

Corollary 9.2. $(\varprojlim^G_{\mathcal{S}_p})^n \cong (\varprojlim^G_{\mathcal{B}_p})^n \operatorname{Res}^{\mathcal{S}_p}_{\mathcal{B}_p}$ *on* $\mathrm{CS}_{\mathcal{S}_p}(G)$.

We extract for future use the main feature of the proof of 8.6.

Corollary 9.3. *Suppose that $\mathcal{W} \subseteq \mathcal{S}_p$ is closed under supergroups in $\mathcal{B}_p$. Then $C^S_\bullet(\mathcal{W})$ and $C^S_\bullet(\mathcal{B}_p \cap \mathcal{W})$ are both homotopy equivalent to a projective resolution of $\bar{R}$ in which all the terms are based at $\mathcal{B}_p \cap \mathcal{W}$.*

Proof. Both complexes have cohomology $\bar{R}$ in degree 0 and 0 elsewhere, by 8.2. They are homotopy equivalent to a complex of projectives, by 8.6, and this also shows that $C^S_\bullet(\mathcal{B}_p \cap \mathcal{W})$ is homotopy equivalent to a complex of projectives based at $\mathcal{B}_p \cap \mathcal{W}$. □

Proposition 9.4. *Suppose that $\mathcal{D}_p \cap \mathcal{W} \subseteq \mathcal{X} \subseteq \mathcal{W}$ and that $\mathcal{X}, \mathcal{W} \subseteq \mathcal{S}_p$ are closed under supergroups in $\mathcal{B}_p$. Then $G_0C^S_\bullet(\mathcal{X}) \simeq G_0C^S_\bullet(\mathcal{W})$: on restriction to $\mathcal{C}_p \cap \mathcal{W}$ these are homotopy equivalent to a projective resolution of $\bar{R}$ in $\mathrm{CS}_{\mathcal{C}_p\cap\mathcal{W}}(G)$ in which all the terms are summands of sums of terms of the form $G_0R[G/H^?]$, for $H \in \mathcal{D}_p \cap \mathcal{W}$.*

Proof. Consider the inclusion map $i : G_0C^S_\bullet(\mathcal{X}) \to G_0C^S_\bullet(\mathcal{W})$. We claim that both sides are filtered by complexes which are homotopy equivalent to sums of complexes of the form $G_0 \operatorname{Ind}^G_{N_G(P)} \operatorname{Inf}^{N_G(P)}_{N_G(P)/P} \mathrm{St}^0_{\bullet-1}(N_G(P)/P)$, where P appears (once for its conjugacy class) if and only if it is in $\mathcal{X}$ (respectively $\mathcal{W}$). This is true before applying G_0 from the proof of 8.6 and 8.8, and remains true afterwards by 8.5 since all the complexes in the filtration are homotopy equivalent to complexes of projectives in $\mathrm{CS}_{\mathcal{W}}(G)$.

Now if $P \in \mathcal{X}$ then the P terms are the same in both $C^S_\bullet(\mathcal{X})$ and $C^S_\bullet(\mathcal{W})$. If $P \notin \mathcal{X}$ then $P \notin \mathcal{D}_p \cap \mathcal{W}$, so the P terms in $C^S_\bullet(\mathcal{X})$ are 0 and those in $C^S_\bullet(\mathcal{W})$are homotopy equivalent to 0 by 7.4. It follows that i is a homotopy equivalence.

Notice that G_0 and $\operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}}$ commute. Also $\operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}}$ is clearly exact and $G_0 : \mathrm{CS}_{\mathcal{C}_p\cap\mathcal{W}}(G) \to \mathrm{GCS}_{\mathcal{C}_p\cap\mathcal{W}}(G)$ is exact because it has the explicit description $(G_0L)(P) = H_0(C_G(P)/Z(P), L(P))$ on $P \in \mathcal{C}_p$, and $C_G(P)/Z(P)$ has order coprime to p.

Finally, $\operatorname{Res}^{\mathcal{W}}_{\mathcal{W}} G_0 C^S_\bullet(\mathcal{W}) \cong G_0 \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} C^S_\bullet(\mathcal{B}_p \cap \mathcal{W})$ has homology just $\bar{R}$ in degree 0, by 9.3 and the exactness property mentioned above. Also the previous part of the proof shows that $C^S_\bullet(\mathcal{W})$ is homotopy equivalent to a complex of projectives of the form claimed, and these remain projective on restriction. □

Recall that I denotes the inclusion functor $\mathrm{GCS} \to \mathrm{CS}$.

Corollary 9.5. *In the circumstances of 9.4, for any $L \in \mathrm{GCS}_{\mathcal{W}}(G)$ we have that $H^n \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(C^S_\bullet(\mathcal{X}), IL) \cong R^n \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-, IL)(\bar{R})$.*

In particular, the homology of the subgroup sequence of the introduction (after removing the first term) is $(\varprojlim^G_{\mathcal{W}})^n IL$.

Proof. We know that $H^n \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(C^S_\bullet(\mathcal{W}), IL) \cong R^n \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-, IL)(\bar{R})$ from 8.6. Now $\operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(C^S_\bullet(W), IL) \cong \operatorname{Hom}_{\mathrm{GCS}_{\mathcal{W}}(G)}(G_0 C^S_\bullet(W), L) \cong \operatorname{Hom}_{\mathrm{GCS}_{\mathcal{W}}(G)}(G_0 C^S_\bullet(X), L) \cong \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(C^S_\bullet(X), IL)$ by 9.4. □

Corollary 9.6. *In the circumstances of 9.4, $(\varprojlim^G_{\mathcal{W}})^n I \cong (\varprojlim^G_{\mathcal{C}_p\cap\mathcal{W}})^n \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} I \cong (\varprojlim^G_{\mathcal{D}_p\cap\mathcal{W}})^n \operatorname{Res}^{\mathcal{W}}_{\mathcal{D}_p\cap\mathcal{W}} I$ on $\mathrm{GCS}_{\mathcal{W}}(G)$.*

Proof. By 9.5 and 8.1 we see that $(\varprojlim^G_{\mathcal{W}})^n L \cong \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(C^S_\bullet(\mathcal{W}), L) \cong \hom_{\mathrm{CS}_{\mathcal{X}}(G)}(C^S_\bullet(\mathcal{W}), L) \cong (\varprojlim^G_{\mathcal{X}})^n L$. □

Corollary 9.7. *$(\varprojlim^G_{\mathcal{S}_p})^n I \cong (\varprojlim^G_{\mathcal{C}_p})^n \operatorname{Res}^{\mathcal{S}_p}_{\mathcal{C}_p} I \cong (\varprojlim^G_{\mathcal{D}_p})^n \operatorname{Res}^{\mathcal{S}_p}_{\mathcal{D}_p} I$ on $\mathrm{GCS}_{\mathcal{S}_p}(G)$.*

Finally, we relate higher limits in CS and in GCS in certain circumstances.

Lemma 9.8. *Suppose that $\mathcal{X} \subseteq \mathcal{C}_p$ and $M, N \in \mathrm{GCS}_{\mathcal{X}}(G)$. Then $R^n \hom_{\mathrm{GCS}_{\mathcal{X}}(G)}(-, M) \cong R^n(\hom_{\mathrm{CS}_{\mathcal{X}}(G)}(-, IM)) \circ I$ on $\mathrm{GCS}_{\mathcal{X}}(G)$.*

Proof. Because $\hom_{\mathrm{GCS}_{\mathcal{W}}(G)}(-, M) \cong \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(-, IM) \circ I$ we can deduce $R^n \hom_{\mathrm{GCS}_{\mathcal{W}}(G)}(-, M) \cong R^n \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(-, IM) \circ I$ provided that I is exact (which it clearly is) and preserves injectives.

But the right adjoint of I is G_0, which is exact on $\mathrm{CS}_{\mathcal{X}}(G)$ (see proof of 9.4), so I does preserve injectives.

Now observe that hom is balanced. □

Corollary 9.9. *Suppose that every prime dividing $|G|$ except perhaps for p is invertible and that $\mathcal{W}$ is closed under supergroups in $\mathcal{B}_p$. Then $(\varprojlim^G_{\mathcal{W}})^n \cong (\varprojlim^G_{\mathcal{W}})^n I$ on $\mathrm{GCS}_{\mathcal{C}_p}(G)$ (where the first higher limit is in $\mathrm{GCS}_{\mathcal{W}}(G)$ and the second is in $\mathrm{CS}_{\mathcal{W}}(G)$).*

Proof. Using 2.3 and 9.6 and the fact that $\hom_{\mathrm{GCS}}(M, N) \cong \hom_{\mathrm{CS}}(IM, IN)$ for $M, N \in \mathrm{GCS}$, we find that, for $N \in \mathrm{GCS}_{\mathcal{W}}(G)$, $\hom_{\mathrm{GCS}_{\mathcal{W}}(G)}(-, N) \cong \varprojlim^G_{\mathcal{W}} \operatorname{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(-, N) \cong \varprojlim^G_{\mathcal{C}_p\cap\mathcal{W}} \operatorname{Hom}_{\mathrm{GCS}_{\mathcal{C}_p\cap\mathcal{W}}(G)}(\operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} -, \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} N) \cong \hom_{\mathrm{GCS}_{\mathcal{C}_p\cap\mathcal{W}}(G)}(\operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} -, \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} N) \cong \hom_{\mathrm{GCS}_{\mathcal{C}_p\cap\mathcal{W}}(G)}(-, \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} N) \circ \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}}$ on $\mathrm{GCS}_{\mathcal{W}}(G)$.

But $\operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}}$ is exact and preserves projectives in GCS, just as in the proof of 3.12. From this we obtain that $R^n \hom_{\mathrm{GCS}_{\mathcal{W}}(G)}(-, N) \cong R^n(\hom_{\mathrm{GCS}_{\mathcal{C}_p\cap\mathcal{W}}(G)}(-, \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} N)) \circ \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}}$ on $\mathrm{GCS}_{\mathcal{W}}(G)$.

Similarly, since $\operatorname{Hom}_{\mathrm{CS}}(M, IN)$ is naturally in GCS for $M \in \mathrm{CS}, N \in \mathrm{GCS}$, we obtain that $R^n \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(-, IN) \cong R^n(\hom_{\mathrm{CS}_{\mathcal{C}_p\cap\mathcal{W}}(G)}(-, \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} IN)) \circ I \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}}$ on $\mathrm{CS}_{\mathcal{W}}(G)$.

Now $R^n \hom_{\mathrm{GCS}_{\mathcal{C}_p\cap\mathcal{W}}(G)}(-, \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} N) \cong R^n(\hom_{\mathrm{CS}_{\mathcal{C}_p\cap\mathcal{W}}(G)}(-, \operatorname{Res}^{\mathcal{W}}_{\mathcal{C}_p\cap\mathcal{W}} IN)) \circ I$ on $\mathrm{GCS}_{\mathcal{C}_p\cap\mathcal{W}}(G)$, by 9.8. We deduce that $R^n \hom_{\mathrm{GCS}_{\mathcal{W}}(G)}(-, N) \cong R^n \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(-, IN)$ on $\mathrm{GCS}_{\mathcal{W}}(G)$. □

Remark. In general the higher limits in CS and GCS are not the same. For example if $\mathcal{W}$ consists only of the trivial group then the higher limits in $\mathrm{CS}_{\{1\}}(G)$ are the cohomology groups $H^*(G,-)$, whilst the higher limits in $\mathrm{GCS}_{\{1\}}(G)$ vanish.

10. The Centraliser Sequence

The construction is analogous to that of the subgroup sequence and we will be brief.

We use the functor F^C defined on a chain σ by

$$F^C(\sigma)(P) = \begin{cases} R & \text{if } C_G(\sigma_t) \geq P, \\ 0 & \text{otherwise.} \end{cases}$$

this implies that $F^C((\sigma))(P)$ is the free R-module on the chains τ in the orbit of σ with $C_G(\tau_t) \geq P$ or, equivalently, $F^C((\sigma)) \cong \mathrm{Ind}_{N_G(\sigma)}^G \bar{R}_{C_G(\sigma_t)}$.

Lemma 10.1. $\hom_{\mathrm{CS}_{\mathcal{W}}(G)}(\mathrm{Ind}_{N_G(\sigma)}^G \bar{R}_{C_G(\sigma_t)}, L) \cong ((\varprojlim_{\mathcal{W}(C_G(\sigma_t))} L)(C_G(\sigma_t)))^{N_G(\sigma)}$, *and if* $C_G(\sigma_t) \in \mathcal{W}$, *then this is also isomorphic to* $L(C_G(\sigma_t))^{N_G(\sigma)}$.

Let $C_\bullet^C(\mathcal{X})$ be the complex of coefficient systems obtained from the class of subgroups $\mathcal{X}$ using F^C, and $\tilde{C}_\bullet^C(\mathcal{X})$ the augmented version.

Corollary 10.2. *If* $L \in \mathrm{CS}_{\mathcal{W}}(G)$ *and for each* $X \in \mathcal{X}$ *we have* $C_G(X) \in \mathcal{W}$, *then* $L(G) \to \hom_{\mathrm{CS}_{\mathcal{W}}(G)}(C_\bullet^C(\mathcal{X}), L)$ *is the third sequence of the introduction, the centraliser sequence.*

Lemma 10.3. *If every* $\mathcal{X}(C_G(H))$, $H \in \mathcal{W}$, *is* R*-acyclic (e.g.* $\mathcal{X} = \mathcal{A}_p^1$ *or* $\mathcal{S}_p^1$ *and for any* $H \in \mathcal{W}$, $Z_p(H) \neq 1$*) then* $H^*(C_\bullet^C(\mathcal{X})) \cong \bar{R}$ *and* $\mathbb{E}\mathrm{xt}_{\mathcal{W}}^n(C_\bullet^C(\mathcal{X}), L) \cong (\varprojlim_{\mathcal{X}}^{\mathcal{W}})^n L$ *in* $\mathrm{CS}_{\mathcal{W}}(G)$.

Proof. This is a consequence of 4.2, since $C_\bullet^C(\mathcal{X})(H) \cong C_\bullet(|\mathcal{X}(C_G(H))|)$. □

Proposition 10.4. *If* $\mathcal{A}_p^1 \subseteq \mathcal{X} \subseteq S_p$ *and* $\mathcal{W}$ *contains the centraliser of every element of* $\mathcal{X}$ *then* $C_\bullet^C(\mathcal{X})$ *is homotopy equivalent to a projective resolution of* $\bar{R}$ *in* $\mathrm{CS}_{\mathcal{W}}(G)$.

Proof. The complex has the correct homology, by 10.3.

$$\begin{aligned} C_\bullet^C(\mathcal{X}^{\leq n})/C_\bullet^C(\mathcal{X}^{\leq n-1}) &\cong \bigoplus_{P \in \mathcal{X}/G, |P|=n} \mathrm{Ind}_{N_G(P)}^G \tilde{C}_\bullet(|\mathcal{X}_{<P}|) \otimes \bar{R}_{C_G(P)} \\ &\cong \bigoplus_{P \in \mathcal{X}/G, |P|=n} \mathrm{Inf}_{N_G(P)/C_G(P)}^{N_G(P)} \tilde{C}_{\bullet+1}(|\mathcal{X}_{<P}|)^0. \end{aligned}$$

So we just have to show that $\tilde{C}_{\bullet+1}(|\mathcal{X}_{<P}|)^0$ is homotopy equivalent to a complex of projectives, or equivalently that $\tilde{C}_\bullet(|\mathcal{X}_{<P}|)$ is homotopy equivalent to a complex of projective $N_G(P)/C_G(P)$-modules.

If P is trivial or cyclic of order p then everything is projective, since $N_G(P)/C_G(P)$ is trivial.

For the other cases we check the conditions of 6.4.

If P is elementary abelian of rank greater than 1, then, for any p-subgroup H of $N_G(P)/C_G(P)$, let $E \leq P$ be the subgroup of elements centralised by H. Let F be a subgroup of P of index p, containing E and normalised by H. Now if $X \in \mathcal{X}_{<P}$ is normalised by H then $X \cap F \neq 1$: this is seen by considering the codimensions of vector spaces over $\mathbb{F}_p$ if the rank of X is at least 2, and if X has rank 1 then it must be in E. We see that $|\mathcal{X}_{<P}|^H$ contracts to F by $X \to X \cap F \to F$.

Otherwise, for any p-subgroup H of $N_G(P)/C_G(P)$, let $1 \neq E \leq \Phi(P) \cap Z_p(P)$ be centralised by H. Then we see that $|\mathcal{X}_{<P}|^H$ contracts to E by $X \to X.E \to E$. □

Corollary 10.5. *([12]) Suppose that $\mathcal{A}_p^1 \subseteq \mathcal{X} = \mathcal{S}_p^1$ and that $\mathcal{X}$ is closed under products with elementary abelian groups (i.e. if $X \in \mathcal{X}$ and $Y < G$, $Y = X \times E$, $E \in \mathcal{A}_p$, then $Y \in \mathcal{X}$). Suppose also that $\mathcal{W}$ contains the centraliser of every element of $\mathcal{X}$ and also that for any $H \in \mathcal{W}$, $Z_p(H) \neq 1$. Then the homology of the centraliser sequence of the introduction (after removing the first term) is $(\varprojlim^G_{\mathcal{W}})^n L$.*

11. Projective Resolutions and the Steinberg Complex

Notice that 8.6 describes a projective resolution of $\bar{R}$ in $\mathrm{CS}_{\mathcal{S}_p}(G)$ in terms of Steinberg complexes. In particular $\bar{R}$ has finite projective dimension, originally a result of Bouc [5], (it also appears without proof in [14]).

Conversely we can calculate Steinberg complexes from a projective resolution $L_\bullet \to \bar{R}$ by

$$\mathrm{St}_\bullet(N_G(P)/P) \simeq (\varinjlim_{P<Q\leq N_G(P)} \mathrm{Res}^G_{N_G(P)} L)(P).$$

For each complex in the homotopy class of $\mathrm{St}_\bullet(G)$ consider the highest degree in which the complex is non-zero, and define $\sigma(G)$ to be the minimum of these.

If R is p-complete then we have the Krull-Schmidt property, so there is actually a smallest representative of $\mathrm{St}_\bullet(G)$.

Similarly, define $\rho(G)$ to be the shortest possible length of a projective resolution of $\bar{R}$ in $\mathrm{CS}_{\mathcal{S}_p}(G)$.

So $\rho(G) = \max_{P\in\mathcal{S}_p} \sigma(N_G(P)/P) + 1 = \max_{P\in\mathcal{B}_p} \sigma(N_G(P)/P) + 1$.

Also $\sigma(G) \leq p\text{-rank}(G) - 1$

and $\sigma(G) \leq$maximum length of a chain in $B_p(G) - 1$.

Thus $\rho(G) \leq \max_{P\in\mathcal{B}_p(G)} p\text{-rank}(N_G(P))$

and $\rho(G) \leq$maximum length of a chain in $\mathcal{B}_p(G)$.

In fact the Steinberg complex also controls the difference between higher limits over $\mathcal{S}_p$ and $\mathcal{S}_p^1$.

To see this, for any RG-module V let $V_1 \in \mathrm{CS}_{\mathcal{S}_p}(G)$ denote the coefficient system which takes the value V on 1 and 0 elsewhere.

Proposition 11.1. *Assume that R is a field k, of characteristic p. For a fixed group G, the following are equivalent:*

1. $(\varprojlim^G_{\mathcal{S}_p})^i V_1 = 0$, *for all $i \geq 0$ and all kG-modules V;*
2. $(\varprojlim^G_{\mathcal{S}_p})^i (kG)_1 = 0$, *for all $i \geq 0$;*
3. $\mathrm{St}_\bullet(G) \simeq 0$;
4. *The canonical map yields* $(\varprojlim^G_{\mathcal{S}_p})^i \cong (\varprojlim^G_{\mathcal{S}_p^1})^i \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{S}_p^1}$, *for all $i \geq 0$, on* $\mathrm{CS}_{\mathcal{S}_p}(G)$;
5. $\varprojlim^G_{\mathcal{S}_p^1}(kG)^? \cong k$ *and* $(\varprojlim^G_{\mathcal{S}_p^1})^i (kG)^? = 0$ *for $i \geq 1$.*

Proof. It is clear that (1) $\Rightarrow$ (2).

Now $\hom_{\mathrm{CS}_{\mathcal{S}_p}(G)}(-, V_1)$ vanishes on any projective based at a non-trivial p-subgroup, by 2.8. Thus 8.6 shows that if $P_\bullet \to \bar{R}$ is a projective resolution then $\hom_{\mathrm{CS}_{\mathcal{S}_p}(G)}(P_\bullet, V_1)$ is homotopy equivalent to $\hom_{\mathrm{CS}_{\mathcal{S}_p}(G)}(\mathrm{St}^0_{\bullet+1}(G), V_1)$, which, in turn, is isomorphic to $\mathrm{Hom}_{kG}(\mathrm{St}_{\bullet+1}(G), V)$. So $(\varprojlim^G_{\mathcal{S}_p})^i V_1$ is equal to the cohomology of $\mathrm{Hom}_{kG}(\mathrm{St}_{\bullet+1}(G), V)$.

Clearly now (3) $\Rightarrow$ (1). Also $\mathrm{Hom}_{kG}(\mathrm{St}_{\bullet+1}(G), kG)$ will detect any non-exactness in $\mathrm{St}_\bullet(G)$, so (2) $\Rightarrow$ (3).

For any $L \in \mathrm{CS}_{\mathcal{S}_p}(G)$, let L^1 denote the cokernel of the inclusion $L(1)_1 \to L$. We claim that $(\varprojlim^G_{\mathcal{S}_p})^i L^1 \cong (\varprojlim^G_{\mathcal{S}_p^1})^i \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{S}_p^1} L$. This is because $L^1 \cong \varprojlim^{\mathcal{S}_p}_{\mathcal{S}_p^1} \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{S}_p^1} L$.

But $\varprojlim$ preserves injectives (because its left adjoint is exact) and is itself exact, so $\mathrm{Ext}^i_{\mathrm{CS}_{\mathcal{S}_p}(G)}(\bar{R}, L^1) \cong \mathrm{Ext}^i_{\mathrm{CS}_{\mathcal{S}_p}(G)}(\bar{R}, \varprojlim^{\mathcal{S}_p}_{\mathcal{S}^1_p} \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{S}^1_p} L) \cong \mathrm{Ext}^i_{\mathrm{CS}_{\mathcal{S}^1_p}(G)}(\bar{R}, \mathrm{Res}^{\mathcal{S}_p}_{\mathcal{S}^1_p} L)$.

The long exact Ext sequence for $L(1)_1 \to L \to L^1$ now shows that (1) $\Leftrightarrow$ (4). Similarly (2) $\Leftrightarrow$ (5), because $kG^?$ is injective on $\mathcal{S}_p$ so the higher limits vanish. $\square$

Remark. Quillen conjectured that $|\mathcal{S}^1_p(G)|$ is contractible if and only if G contains a non-trivial normal p-subgroup [18]. In fact, according to [12], no counterexample seems to be known if the contractibility condition is replaced by $\mathbb{F}_p$-acyclicity.

Notice that $\mathbb{F}_p$-acyclicity is equivalent to condition (3) of 11.1, so we see that the (stronger) conjecture is equivalent to the statement:
G contains a non-trivial normal p-subgroup if and only if the conditions of 11.1 are satisfied.

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