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The Tate-Farrell cohomology of the Morava Stabilizer Group S_{p-1} with coefficients in E_{p-1}

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Abstract. We calculate the Tate-Farrell cohomology of the Morava stabilizer group S_{p-1} with coefficients in the moduli space E_{p-1} for odd primes p.

1. Introduction

We present a calculation motivated by homotopy theory, although our methods are algebraic and involve the Tate-Farrell cohomology of a profinite group with compact coefficients. As a reference to the background in homotopy theory we suggest [**4, 5**]. For the Tate-Farrell cohomology of profinite groups with coefficients in compact module we refer to [**15**], although most of the results are analogues of one for discrete groups, for which see [**2**].

Let p be an odd prime and $n \in \mathbb{N}$ and let R be the ring of integers of the unramified extension of $\hat{\mathbb{Q}}_p$ of degree n, (so R is isomorphic to the Witt vectors $W_{\mathbb{F}_{p^n}}$): the residue class field is $k \cong \mathbb{F}_{p^n}$. Let χ be the Frobenius automorphism of R and $\mathrm{Gal} = \langle \chi \rangle$ the Galois group. Let S_n denote the (full) nth Morava stabilizer group: this is the group of units in the R-algebra M generated by S subject to the relations $S^n = p$ and $rS = S\chi(r)$ for $r \in R$. The Galois group Gal acts on S_n simply by $\chi(rS^i) = \chi(r)S^i$, or equivalently by conjugation by S.

It is known that S_n is virtually a pro-p group of virtual cohomological dimension n^2 and type FP_∞ over $\hat{\mathbb{Z}}_p$.

If Γ_n denotes the commutative one-dimensional p-typical formal group law with p-series x^{p^n}, then S_n is isomorphic to the group of automorphisms of Γ_n over $\mathbb{F}_p$. It therefore acts on the ring of functions on the Lubin-Tate moduli space of $\star$-isomorphism classes of lifts of Γ_n, which is $E_{n,0} = R[[u_1, \ldots, u_{n-1}]]$, a profinite RS_n-module. We denote the category of profinite RG-modules by $\mathcal{C}_R(S_n)$ (and similarly with R replaced by $\hat{\mathbb{Z}}_p$). There is also an action of S_n on a graded version $E_{n,*} = E_{n,0}[u^{\pm 1}]$. This is graded by the power of u, normalized so that u has degree -2 (called the internal degree).

This combines with the action of Gal on $E_{n,*}$ via its action on the coefficients to give an action of the semi-direct product $S_n \rtimes \mathrm{Gal}$ on $E_{n,*}$, and so each $E_{n,r} \in \mathcal{C}_{\hat{\mathbb{Z}}_p}(S_n \rtimes \mathrm{Gal})$.

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We would like to calculate the ring $H^*(S_n, E_{n,*})^{\mathrm{Gal}}$, by which we mean the sum $\bigoplus_{r,s} H^r(S_n \rtimes \mathrm{Gal}, E_{n,s})$, since this is the initial term of a spectral sequence which converges to $\pi_* L_{K(n)}$, the homotopy groups of the localization of the sphere spectrum at the nth Morava K-theory (all at the prime p). Notice that, since Gal has order coprime to p, $H^*(S_n, E_{n,*})^{\mathrm{Gal}}$ is actually isomorphic to the invariants of $H^*(S_n, E_{n,*})$ under the action of Gal. What we will actually do is calculate the Tate-Farrell cohomology in the case $n = p - 1$: this is equal to the ordinary cohomology in degrees greater than n^2.

THEOREM 1.1. *For odd p and $n = p - 1$*

$$\begin{aligned}\hat{H}^*(S_n, E_{n,*})^{\mathrm{Gal}} &= \hat{H}^*(G, E_{n,*})^{\mathrm{Gal}} \otimes \Lambda(x_0, \dots, x_{n-1}) \\ &= \hat{H}^*(S_n, \hat{\mathbb{Z}}_p) \otimes \Lambda(\alpha) \otimes \mathbb{F}_p[\Delta^{\pm 1}] \\ &= \mathbb{F}_p[\Delta^{\pm 1}, \beta^{\pm 1}] \otimes \Lambda(\alpha, x_0, \dots, x_{n-1}).\end{aligned}$$

Here G is a finite group that will be defined later, and the generators will be defined in the course of the calculation.

REMARK 1.2. It would be natural to regard $E_{n,*}$ as $\hat{\bigoplus}_s E_{n,s}$, the sum in $\mathcal{C}_R(S_n)$, but $H^r(S_n, \hat{\bigoplus}_s E_{n,s}) \cong \Pi_s H^r(S_n, E_{n,s})$. Since only the homogeneous parts appear in the spectral sequence, the difference is immaterial, but we conform to the conventional usage.

We will need the following corollary of [**15**] 7.3 and the remark following it. It is what we would expect from the theory for discrete groups in [**2**]. Similar results for profinite groups but with discrete coefficients also appear in [**13**] and [**12**].

THEOREM 1.3. *Let G be a profinite group of finite virtual cohomological dimension over $\hat{\mathbb{Z}}_p$. Suppose that G has no subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$ and only a finite number of conjugacy classes of subgroups isomorphic to $\mathbb{Z}/p$, which we denote by $\mathcal{S}(p)$. Let M be a module in $\mathcal{C}_{\hat{\mathbb{Z}}_p}(G)$. Then the Tate-Farrell cohomology satisfies*

$$\hat{H}^*(G, M) \cong \bigoplus_{P \in \mathcal{S}(p)} \hat{H}^*(N_G(P), M).$$

2. Trivial Coefficients

From now on $n = p - 1$ and we want to define a certain subgroup G of S_n of order pn^2. This is done in [**7**], but we need to be precise about some of the details so we give the construction here. $M_{\hat{\mathbb{Q}}_p} = \hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} M$ is a division algebra with center $\hat{\mathbb{Q}}_p$. It has a valuation ν such that $\nu(S) = \frac{1}{n}$, and S_n is the set of elements of valuation 0.

The general theory of division algebras tells us that the field $\hat{\mathbb{Q}}_p(\omega; \omega^p = 1)$ can be embedded in $M_{\hat{\mathbb{Q}}_p}$ and the image is self-centralizing ([**11**] 28.10, 31.10). Furthermore, by the Skolem-Noether Theorem ([**11**] 7.21), any two embeddings differ only by conjugation in $M_{\hat{\mathbb{Q}}_p}$.

In fact we can find a pth root of unity ω in $\hat{\mathbb{Z}}_p[S] \subset M$ as follows: $1 + S$ is a pth root of unity modulo p and we can lift it by Hensel's Lemma ([**14**] II Prop. 7). For convenience we will regard $\hat{\mathbb{Q}}_p(\omega) = \hat{\mathbb{Q}}_p(S)$. It is easy to see that the only elements of $M_{\hat{\mathbb{Q}}_p}$ invariant under conjugation by S are in $\hat{\mathbb{Q}}_p(S)$, so $\hat{\mathbb{Q}}_p(\omega)$ is self centralizing.

If $\phi : \hat{\mathbb{Q}}_p(\omega) \to M_{\hat{\mathbb{Q}}_p}$ is another embedding then, by the discussion above, there is a $u \in M_{\hat{\mathbb{Q}}_p}$ such that $u\phi(x)u^{-1} = x$ for $x \in \hat{\mathbb{Q}}_p(\omega)$. But S^iU will serve as well as u, since S is central in $\hat{\mathbb{Q}}_p(\omega)$, and we can choose i to give this element valuation 0. Thus any two embeddings of $\hat{\mathbb{Z}}_p[\omega]$ in M are conjugate by an element of S_n and so any two elements of S_n of order p are conjugate in S_n.

Now let $e \in \mathbb{Z}$ be a generator of $(\mathbb{Z}/p)^*$, so $\hat{\mathbb{Z}}_p[\omega^e] = \hat{\mathbb{Z}}_p[\omega] \subset M$. The Galois automorphism $\omega \mapsto \omega^e$ must be realized by conjugation by some $v \in S_n$, as has just been shown, i.e. $v^{-1}\omega v = \omega^e$. Now v^n centralizes $\hat{\mathbb{Z}}_p[\omega]$, so $v^n \in \hat{\mathbb{Z}}_p[\omega]$. Clearly v^n is invariant under conjugation by v, so $v^n \in \hat{\mathbb{Z}}_p$.

Let $e' \in \hat{\mathbb{Z}}_p$ be an nth root of unity such that $e' \equiv v^n \pmod{p}$. Then $e'v^{-n} \equiv 1 \pmod{p}$ so has an nth root in $\hat{\mathbb{Z}}_p$ (constructed from the binomial power series). Let $v' = v(e'v^{-n})^{\frac{1}{n}}$, so $v'^{-1}\omega v' = \omega^e$ and also $v'^n = e'$, so $v'^{n^2} = 1$. In fact v' has order n^2, since if it had order $\frac{n^2}{r}$ then $v'^{\frac{n}{r}}$ would be in $\hat{\mathbb{Z}}_p$, contradicting $v'^{-1}\omega v' = \omega^e$.

But R itself contains the n^2th roots of unity so, by the Skolem-Noether Theorem again, there is an $x \in M_{\hat{\mathbb{Q}}_p}$ such that $xv'x^{-1} \in R$. Let $a = x\omega x^{-1}$ and $b = xv'x^{-1}$ and define the group $G = \langle a, b\rangle \cong \mathbb{Z}/p \rtimes \mathbb{Z}/n^2 \leq S_n$.

When we want to think of b as an element of R rather than G we will denote it by η.

The reason for the last conjugation in the definition of G is to ensure that our formulas are consistent with those of [**8, 9**].

Some more work with division algebras will show that any subgroup of S_n that is isomorphic to G is, in fact, conjugate to G in S_n, but we refer the reader to [**7**].

The significance of G is that it is a maximal finite subgroup of S_n and, up to conjugacy, the only one of order divisible by p. Again we refer the reader to [**7**].

Lemma 2.1. *([**8**]) $e' \equiv e \pmod{p}$.*

Proof. Working in S_n modulo S^2 we have $a = 1+uS$ for some $u \not\equiv 0 \pmod{S}$. Now

$$1 + euS = a^e = b^{-1}ab = 1 + u\eta^{-1}S\eta = 1 + u\eta^n S = 1 + ue'S. \qquad \square$$

As mentioned above, $\hat{\mathbb{Q}}_p(a)$ is self-centralizing in $M_{\hat{\mathbb{Q}}_p}$, so $C = C_{S_n}(a)$ is equal to the units in $\hat{\mathbb{Z}}_p[a]$. In additive notation these are of the form $\hat{\mathbb{Z}}_p^n \times \mathbb{Z}/p \times \mathbb{Z}/n$. This is because the p-torsion is known from the construction and the p'-torsion corresponds to the units in the residue class field $\mathbb{F}_p$. To see that the free part has rank n, let U_1 denote the units congruent to 1 modulo p: then U_1 is isomorphic to $p\hat{\mathbb{Z}}_p[a]$ under the inverse exponential and logarithm power series ([**3**] 54.2). U_1 is of finite index in C, so C also has free rank n.

Notice that these maps respect the action of b, so $U_1 \cong \hat{\mathbb{Z}}_p[a]$. Also, since $\langle b\rangle$ is abelian of order coprime to p, any $\hat{\mathbb{Z}}_p\langle b\rangle$-lattice is a sum of eigenspaces for b, and hence any sublattice of finite index is isomorphic to the original. In particular $U_1 \cong C/(\text{torsion})$. Thus $C/(\text{torsion})$ is isomorphic to $\hat{\mathbb{Z}}_p[a]$ as a $\hat{\mathbb{Z}}_p\langle b\rangle$-lattice, so is a sum of rank 1 lattices; one for each nth root of unity on which b acts as multiplication by that root of unity. Finally, since $\hat{\mathbb{Z}}_p[a]$ is free as a $\hat{\mathbb{Z}}_p\langle b\rangle$-module, the map $C \to C/(\text{torsion})$ is split over $\langle b\rangle$. We have shown:

LEMMA 2.2. *As a $\hat{\mathbb{Z}}_p\langle b\rangle$-module, $C \cong \hat{\mathbb{Z}}_p\langle b\rangle \oplus \langle a\rangle \oplus \mathbb{Z}/p$.*

Now $N = N_{S_n}(\langle a\rangle) = \langle C, b\rangle \cong (\hat{\mathbb{Z}}_p[a] \times \langle a\rangle) \rtimes \langle b\rangle$.

Consider $N_{S_n \rtimes \mathrm{Gal}}(G)/N_{S_n}(G)$. Every conjugate of G in $S_n \rtimes \mathrm{Gal}$ is in S_n so, as mentioned above, is conjugate to G in S_n. It follows that $N_{S_n \rtimes \mathrm{Gal}}(G)/N_{S_n}(G) \cong \mathrm{Gal}$. Choose $c \in N_{S_n \rtimes \mathrm{Gal}}(G)$ to have image $\chi \in \mathrm{Gal}$: we may also arrange for c to act trivially on $\langle a\rangle$ and for $\langle c\rangle$ to have no pro-p part.

Finally, for similar reasons, $N' = N_{S_n \rtimes \mathrm{Gal}}(\langle a\rangle) = \langle N, c\rangle$.

First we calculate the Tate-Farrell cohomology with trivial coefficients. We use Λ to denote an exterior algebra over $\mathbb{F}_p$.

PROPOSITION 2.3. For p odd and $n = p-1$:

$$\hat{H}^*(S_n, \hat{\mathbb{Z}}_p) = \hat{H}^*(G, \hat{\mathbb{Z}}_p) \otimes \Lambda(x_0, \ldots, x_{n-1}) = \mathbb{F}_p[\beta^{\pm 1}] \otimes \Lambda(x_0, \ldots, x_{n-1}),$$

$$\hat{H}^*(S_n, \mathbb{F}_p) = \hat{H}^*(G, \mathbb{F}_p) \otimes \Lambda(x_0, \ldots, x_{n-1}) = \mathbb{F}_p[\beta^{\pm 1}] \otimes \Lambda(\alpha, x_0, \ldots, x_{n-1}),$$

where $|\beta| = 2n$, $|x_i| = 1-2i$ and $|\alpha| = -1$.

PROOF. By Theorem 1.3 we find that $\hat{H}^*(S_n, \hat{\mathbb{Z}}_p) \cong \hat{H}^*(N, \hat{\mathbb{Z}}_p)$. Notice that $\hat{H}^*(N, \hat{\mathbb{Z}}_p) \cong \hat{H}^*(\hat{\mathbb{Z}}_p^n \times \langle a\rangle, \hat{\mathbb{Z}}_p)^{\langle b\rangle}$, since b has order coprime to p.

By the Künneth Theorem in Tate-Farrell cohomology ([**2**] X 3 ex. 4), we have $\hat{H}^*(C, \hat{\mathbb{Z}}_p) \cong \hat{H}^*(\langle a\rangle, \hat{\mathbb{Z}}_p) \otimes H^*(\hat{\mathbb{Z}}_p^n, \mathbb{F}_p)$.

It is well known that $\hat{H}^*(\langle a\rangle, \hat{\mathbb{Z}}_p) = \mathbb{F}_p[\zeta^{\pm 1}]$, where $|\zeta| = 2$. To find the action of b on ζ use dimension shifting to see that $H^2(\langle a\rangle, R) \cong H^1(\langle a\rangle, k) \cong \mathrm{Hom}(\langle a\rangle, k)$. Then b acts on the latter by sending f to $(x \mapsto bf(b^{-1}xb))$, so $b(\zeta) = e\zeta$.

Now a basis $y_0, \ldots, y_{n-1}$ of $H^1(\hat{\mathbb{Z}}_p^n, \mathbb{F}_p)$ can be chosen so that $b(y_i) = e^i y_i$. We finish by calculating the invariants under b using the last part of Lemma 4.1 and setting $x_i = \zeta^{-i} \otimes y_i$.

The calculation for $\mathbb{F}_p$ coefficients is almost identical. □

REMARK 2.4. Because c centralizes C we have $\hat{H}^*(N', \hat{\mathbb{Z}}_p) \cong \hat{H}^*(N, \hat{\mathbb{Z}}_p)$ and hence $\hat{H}^*(S_n \rtimes \mathrm{Gal}, \hat{\mathbb{Z}}_p) \cong \hat{H}^*(S_n, \hat{\mathbb{Z}}_p)$. Since the action of Gal on S_n is via conjugation by S, which is not inner in S_n, this is not immediately obvious.

3. Coefficients in E_n

Next we calculate $\hat{H}^*(\langle a\rangle, E_{n,*})$ following the method of Nave [**8, 9**], which in turn is based on unpublished work of Hopkins and Miller. This is also treated in detail for the prime 3 in [**6**].

First we need a change of basis.

LEMMA 3.1. *([**8, 9**]) There are elements $z, z_1, \ldots, z_{n-1} \in E_{n,0}$ such that, where $\mathfrak{m}$ denotes the ideal $(p, u_1, \ldots, u_{n-1})$ in $E_{n,0}$:*

(1) $z \equiv cu \mod (p, \mathfrak{m}^2)$ *for some c a unit in R,*
(2) $z_i \equiv c_i u u_i \mod (p, u_1, \ldots, u_{i-1}, \mathfrak{m}^2)$ *for some c_i a unit in R.*
(3) $(1 + a + \cdots + a^{p-1})z = 0$,
(4) $b(z) = \eta z$,
(5) $(a-1)z = z_{n-1}$ *and* $(a-1)z_{i+1} = z_i$ *for* $1 \le i < n-1$.

It follows from (1) and (2) that $E_{n,*} = R[[z^{-1}z_1, \ldots, z^{-1}z_{n-1}]][z^{\pm 1}]$.

Let V be the R-submodule of $E_{n,-2}$ spanned by $\{z, z_1, \ldots, z_{n-1}\}$. It follows from (3),(4) and (5) that V is an RG-submodule. Let $\delta = \Pi_{i=0}^{p-1} a^i(z)$: then $a(\delta) = \delta$ and $b(\delta) = \eta^p \delta = e\eta\delta$.

Consider the symmetric algebra $S[V] \subset E_{n,*}$. We claim that, as RG-modules,

$$(\dagger) \qquad R[V]_r = \begin{cases} 0 & r \text{ odd}, \\ \delta^{-r'} R \oplus (\text{proj}) & r = 2pr' \leq 0, \\ \delta^{-r'} V \oplus (\text{proj}) & r = 2(pr'-1) \leq 0, \\ (\text{proj}) & \text{otherwise}. \end{cases}$$

and

$$R[V]_{r-2p} = \delta R[V]_r \oplus (\text{proj}) \text{ for } r < 0.$$

Here (proj) indicates a projective summand. We will write this in the condensed form $R[V] = B \oplus (\text{proj})$, where $B = \bigoplus_i \delta^i (R \oplus V)$.

Recall that if G is a finite group of order not divisible by p^2 and $M \in \mathcal{C}_R(G)$ is projective in $\mathcal{C}_R$ then the isomorphism class of M is uniquely determined by its reduction modulo p, $k \otimes_R M$. This is true for a cyclic group of order p by the classification of $R\mathbb{Z}/p$-lattices, (see [**10**], [**3**] 34.31), and this classification extends to $\mathcal{C}_R(\mathbb{Z}/p)$. The general case follows by a transfer argument.

Thus we only have to check the claim over kG. But it is true over $k\langle a\rangle$ from the calculation of the symmetric algebra by Almkvist and Fossum [**1**]. The general case follows because both $\delta^{-r'}R$ and $\delta^{-r'}V$ are defined over G, and the quotients by them must still be projective over G since this depends only on the restriction to the Sylow p-subgroup. Being projective they force the extension to split, and our claim is proved.

If we invert δ we obtain a dense subset $R[V][\delta^{-1}] \subset E_{n,*}$. As an RG-module this still has the same form $B \oplus (\text{proj})$, by the second identity in †. In fact this form is preserved by completion:

Proposition 3.2. As a sum of compact modules for RG, $E_{n,*} = B \oplus (\text{proj})$.

Proof. Let $(z^{-1}z_1, \ldots, z^{-1}z_{n-1})$ denote the ideal generated (topologically) by the given elements in $E_{n,0}$. It is easy to check that

$$E_{n,-2r} = R[V]_{-2r} \oplus (z^{-1}z_1, \ldots, z^{-1}z_{n-1})^{r+1} E_{n,-2r}, \quad r > 0$$

and also

$$R[V]_{-2r} = B_{-2r} \oplus P_{-2r}, \quad r > 0$$

for some projective P_{-2r}. Thus, for $r + pt > 0$,

$$\begin{aligned} E_{n,-2r} &= \delta^t E_{n,-2(r+pt)} \\ &= \delta^t R[V]_{-2(r+pt)} \oplus \delta^t (z^{-1}z_1, \ldots, z^{-1}z_{n-1})^{r+1} E_{n,-2(r+pt)} \\ &= B_{-2r} \oplus P_{-2(r+pt)} \oplus (z^{-1}z_1, \ldots, z^{-1}z_{n-1})^{r+1} E_{n,-2r}. \end{aligned}$$

Now $R[V][\delta^{-1}]_{-2r} = B_{-2r} \oplus \varinjlim P_{-2(r+pt)}$ and $E_{n,2r} = B_{-2r} \oplus \varprojlim P_{-2(r+pt)}$ as $t \to \infty$. As a consequence, if $\varinjlim P_{-2(r+pt)} = \oplus_i Q_i$, as a sum of indecomposable projective RG-modules then $E_{n,2r} = B_{2r} \oplus \Pi_i Q_i$. □

We say that $x \in \hat{H}^r(-, E_{n,s})$ has bidegree $|x| = (r, s)$. We will also write $\hat{H}^*(G, E_{n,*})^{\text{Gal}}$ for $\hat{H}^*(\langle G, c\rangle, E_{n,*}) \cong \hat{H}^*(G, E_{n,*})^{\langle c\rangle}$. This is not so strange since $\hat{H}^*(S_n, E_{n,*})^{\text{Gal}} \cong \hat{H}^*(\langle S_n, c\rangle, E_{n,*}) \cong \hat{H}^*(S_n, E_{n,*})^{\langle c\rangle}$.

Corollary 3.3. ([**8**]) The Tate cohomology is given by

$$\hat{H}^*(\langle a\rangle, E_{n,*}) = k[\delta^{\pm 1}, \zeta^{\pm 1}] \otimes \Lambda(\nu),$$

where $|\delta| = (0, -2p)$, $|\zeta| = (2, 0)$, $|\alpha| = (1, -2)$ and b acts by

$$b(\delta) = e\eta\delta, \quad b(\zeta) = e\zeta, \quad b(\nu) = e\eta\nu.$$

As a consequence

$$\hat{H}^*(G, E_{n,*}) = k[\Delta^{\pm 1}, \beta^{\pm 1}] \otimes \Lambda(\alpha),$$
$$\hat{H}^*(G, E_{n,*})^{\mathrm{Gal}} = \mathbb{F}_p[\Delta^{\pm 1}, \beta^{\pm 1}] \otimes \Lambda(\alpha).$$

where $|\Delta| = (0, 2pn^2)$, $|\beta| = (2, 2pn)$ and $|\alpha| = (1, 2n)$.

Proof. The first calculation is an easy consequence of 3.2 (we identify δ with its image in $\hat{H}^0(\langle a \rangle, E_{n,-2p})$).

$$\begin{aligned}\hat{H}^*(\langle a \rangle, E_{n,*}) &= \hat{H}^*(\langle a \rangle, B)\\ &= \bigoplus_{r \in \mathbb{Z}} \hat{H}^*(\langle a \rangle, \delta^r(R \oplus V))\\ &= \bigoplus_{r \in \mathbb{Z}} \delta^r k[\zeta^{\pm 1}] \otimes \Lambda(\nu) \quad \text{(a well-known calculation)}\\ &= k[\delta^{\pm 1}, \zeta^{\pm 1}] \otimes \Lambda(\nu).\end{aligned}$$

The action of b on δ is from the definition of δ and e and that on ζ was found in the proof of 2.3.

For the action on $\nu \in H^1(\langle a \rangle, V)$ it is easy to verify that the quotient map $V \to V/\operatorname{rad}(V) \cong kz$ induces an isomorphism on H^1, so $H^1(\langle a \rangle, V) \cong zH^1(\langle a \rangle, k)$ as a $\langle b \rangle$-module, and this combines the action on z with that found in calculating the action on ζ in 2.3.

Thus $\hat{H}^*(G, E_{n,*}) \cong \hat{H}^*(\langle a \rangle, E_{n,*})^{\langle b \rangle}$ and the invariants can be calculated using lemma 4.1 below. They are generated by $\Delta = \delta^{-n^2}$, $\beta = \zeta\delta^{-n}$ and their inverses and $\alpha = \delta^{-1}\nu$.

Finally, notice that c acts on the R-module $\hat{H}^r(G, E_{n,s})$ according to the formula $c(\ell x) = \chi(\ell)c(x)$, $\ell \in R, x \in \hat{H}^r(G, E_{n,s})$. This cohomology group is either k or 0. We claim that in the former case the invariants are isomorphic to $\mathbb{F}_p$.

To see this let $0 \neq x \in \hat{H}^r(G, E_{n,s})$ be a generator over k. Then $c(x) = \lambda x$ for some $\lambda \in k$ and $x = c^n(x) = \lambda^{\frac{p^n-1}{p-1}} x$, so $\lambda^{\frac{p^n-1}{p-1}} = 1$. It follows that $\lambda = \omega^{p-1}$ for some $\omega \in k$ and the fixed points under c are $\mathbb{F}_p\omega^{-1}x$.

Since the generators Δ, β, α can be replaced by any non-zero element of the $\hat{H}^r(G, E_{n,s})$ that they appear in, we may assume that they are all invariant under c and hence generate the invariants under c. □

Proof. of 1.1. As before we use Theorem 1.3 to see that $\hat{H}^*(S_n, E_{n,*})^{\mathrm{Gal}} \cong \hat{H}^*(N', E_{n,*})$.

Recall that, for any short exact sequence $I \to J \to K$ of profinite groups of finite virtual cohomological dimension with K torsion-free, there is a spectral sequence $H^*(K, \hat{H}^*(I, M)) \Rightarrow \hat{H}^*(J, M)$ ([**2**] X 3 ex. 5).

Apply this to $C = \hat{\mathbb{Z}}_p^n \times \langle a \rangle$ to obtain $H^*(\hat{\mathbb{Z}}_p^n, \hat{H}^*(\langle a \rangle, E_{n,*})) \Rightarrow \hat{H}^*(C, E_{n,*})$. If we fix both r and s then $\hat{H}^*(\langle a \rangle, E_{n,r})$ is either k or 0 so $\hat{\mathbb{Z}}_p^n$, being a pro-p group, must act trivially. Thus the E_2-term is isomorphic to $\hat{H}^*(\langle a \rangle, E_{n,*}) \otimes H^*(\hat{\mathbb{Z}}_p^n, \mathbb{F}_p) \cong \hat{H}^*(\langle a \rangle, E_{n,*}) \otimes \Lambda(y_0, \ldots, y_{n-1})$.

We claim that this spectral sequences collapses, so that we have $\hat{H}^*(C, E_{n,*}) \cong \hat{H}^*(\langle a\rangle, E_{n,*}) \otimes \Lambda_{\mathbb{F}_p}(y_0, \dots, y_{n-1})$. To see this notice that, from the proof of 3.3, the map $E_{n,r} \to E_{n,r}/\mathfrak{m}E_{n,r} \cong k$ induces an injection on $\hat{H}^*(\langle a\rangle, -)$. The corresponding spectral sequence with coefficients k collapses, by the Künneth Theorem, so ours must too.

Now compute the invariants under b using Lemma 4.1. The result is that $\hat{H}^*(\langle a\rangle, E_{n,*})^{\langle b\rangle} \otimes \Lambda_{\mathbb{F}_p}(x_0, \dots, x_{n-1})$, where the x_i are as in 2.3.

Finally, c acts only on the first factor, so taking the invariants under c just replaces $\hat{H}^*(G, E_{n,*})$ by $\hat{H}^*(G, E_{n,*})^{\mathrm{Gal}}$. □

4. Invariants

The following lemma is elementary, but systematic use of it simplifies the invariant calculations above. For example in the proof of 3.3, first calculate the invariants $k[\delta^{\pm 1}, \zeta^{\pm 1}]^{\langle b\rangle} = (k[\delta^{\pm 1}] \otimes k[\zeta^{\pm 1}])^{\langle b\rangle}$ and then $(k[\delta^{\pm 1}, \zeta^{\pm 1}] \otimes \Lambda(\nu))^{\langle b\rangle}$.

Lemma 4.1. *Let H be a finite abelian group and let R be a commutative integral domain such that $|H|$ is invertible in R and that contains a root of unity of order the exponent of H. Suppose that A and B are two RH-modules such that A is a graded-commutative R algebra and the action of H is compatible with this structure. H acts on $A \otimes_R B$ diagonally.*

Let C be the set of isomorphism classes of homomorphisms from H to $R^\times$. (This can be identified, perhaps not canonically, with the characters of H.) Then there are decompositions of RH-modules $A = \oplus_{c\in C} A_c$ and $B = \oplus_{c\in C} B_c$, where $A_c = \{a \in A | ha = c(h)a, h \in H\}$ and similarly for B. Let $C_A = \{c \in C | A_c \neq 0\}$ and similarly for B.

Suppose that for each $c \in C_A$ there is a homogeneous element $a_c \in A_c$ that is invertible in A. Then

$$(A \otimes B)^H = \bigoplus_{d\in C_A} A^H a_{d^{-1}} \otimes B_d,$$

$$(A \otimes B)_c = \bigoplus_{d\in C_A} A^H a_{d^{-1}} \otimes B_{cd},$$

and if $C_B \subseteq C_A$ then

$$(A \otimes B)_c = a_c (A \otimes B)^H.$$

Suppose that B is also a graded commutative R-algebra and H acts compatibly with this structure. Then $A \otimes B$ is also a graded-commutative R-algebra in the usual way, and H acts as a group of automorphisms.

(1) *If, for each $c \in C_A \cap C_B$, there is a homogeneous element $b_c \in B_c$ that is invertible in B, then $(A \otimes B)^H$ is a free $A^H \otimes B^H$-module with basis $\{a_{c^{-1}} \otimes b_c : c \in C'\}$.*

Furthermore if the monomials in $c_1, \dots, c_r \in C_A \cap C_B$ yield all the $c \in C_A \cap C_B$ then $(A \otimes B)^H$ is generated as a ring by A^H, B^H and the $a_{c_i^{-1}} \otimes b_{c_i}$.

(2) *If B is generated as an R-algebra by $d_1, \dots, d_s$, where $d_i \in B_{c_{d_i}}$ for some $c_{d_i} \in C_A \cap C_B$, then $(A \otimes B)^H$ is generated as a ring by A^H and the $a_{c_{d_i}^{-1}} \otimes d_i$.*

If the d_i freely generate B as a graded-commutative R-algebra then the $a_{c_{d_i}^{-1}} \otimes d_i$ freely generate $(A \otimes B)^H$ over A^H. (So if $B = \Lambda_R(d_1, \ldots, d_s)$ then $(A \otimes B)^H = A^H \otimes_R \Lambda_R(a_{c_{d_1}^{-1}} \otimes d_1, \ldots, a_{c_{d_s}^{-1}} \otimes d_s)$.)

Proof. This is left as an exercise for the reader. Notice that $(A \otimes B)^H = \oplus_{c \in C'} A_{c^{-1}} \otimes B_c$ and $A_c a_{c'} = A_{cc'}$. □

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