MACKEY FUNCTORS AND CONTROL OF FUSION

PETER SYMONDS

Abstract. We present an algebraic approach to Mislin's theorem that control of p-fusion is equivalent to inducing a mod-p cohomology isomorphism. There are consequences for the cohomology of p-permutation modules.

1. Introduction

In 1990 Guido Mislin published the following theorem.

Theorem 1.1. *(Mislin [4]) Let $H < G$ be finite groups. Then the restriction in mod-p cohomology* $\operatorname{res}^G_H : H^*(G, \mathbb{F}_p) \to H^*(H, \mathbb{F}_p)$ *is an isomorphism if and only if H controls p-fusion in G.*

(Actually Mislin proved a result for compact Lie groups.) Recall that H is said to control p-fusion in G if the following two conditions are satisfied:

(1) the index $|G/H|$ is coprime to p and
(2) if $Q \leq H$ is a p-subgroup and $g \in G$ is such that $Q^g \leq H$ also then $g = ch$ for some $c \in C_G(Q)$ and $h \in H$.

This is a fundamental concept in group theory.

It is not hard to see that control of fusion implies a cohomology isomorphism. What is surprising is that cohomology controls the subgroup structure of the group in such a strong way. Mislin's proof uses deep results from algebraic topology, in particular Carlsson's proof of Segal's Burnside ring conjecture [1]. Jon Alperin issued a challenge to produce an algebraic proof.

We do not achieve this. What we do is to show that Mislin's theorem is a consequence of the following algebraic statement, which we verify using topological methods.

First recall that Peter Webb showed that a global Mackey functor, such as cohomology, can be decomposed into its simple composition factors [10]. We will prove:

Theorem. (4.1) Cohomology, considered as a global Mackey functor, contains every simple cohomological global Mackey functor as a composition factor.

There is the following consequence, where k is a field of characteristic p and $\operatorname{Perm}^G_{P,V}$ denotes the p-permutation kG-module parametrized by the p-subgroup P and the simple $kN_G(P)$-module V.

Theorem. (5.3) $H^*(G, \operatorname{Perm}^G_{P,V}) \neq 0$ if and only if $C_G(P)$ acts trivially on V.

It appears that this is currently inaccessible by algebraic means.

2. Mackey Functors

Mackey functors have been described in many places, for example [8], [10] and [11], but we will attempt to provide a summary of what we need here.

We will always work over a field k of prime characteristic p.

Given a finite group G, define a category $\mathcal{S}(G)$ to have as objects the subgroups of G and as morphisms from K to H the symbols $(g; H, K)$, for $g \in G$ such that ${}^gK \leq H$. These compose according to the rule $(g; H, J)(h; J, K) = (gh; H, K)$.

To $(g; H, K)$ we associate the group homomorphism $K \to H$ given by $x \mapsto {}^gx$. However $(g_1; H, K) \neq (g_2; H, K)$ if $g_1 \neq g_2$ even though the associated group homomorphisms might be equal.

A Mackey functor on G is a pair of functors M^*, M_* from $\mathcal{S}(G)$ to finite-dimensional k-vector spaces. The first is contravariant and the second is covariant and they coincide on objects, i.e. $M^*(H) = M_*(H)$, and we denote this common value by $M(H)$.

We normally write $c_g = M_*((g; H, H))$, (regardless of H), $\operatorname{res}^H_K = M^*((1; H, K))$ and $\operatorname{tr}^H_K = M_*((1; H, K))$.

The pair of functors must satisfy three axioms:

(1) $M^*((g; {}^gH, H))M_*((g; {}^gH, H)) = \mathrm{Id}_{M(H)}$,
(2) c_h acts trivially on $M(H)$ when $h \in H$,
(3) The Mackey double coset formula for $K, L \leq G$

$$\operatorname{res}^H_L \operatorname{tr}^H_K = \sum_{h \in L \backslash H / K} \operatorname{tr}^L_{L \cap {}^hK} c_h \operatorname{res}^K_{L^h \cap K}.$$

There are many naturally occurring examples. If V is a kG-module then the fixed point functor $V^?$ is a Mackey functor, where we take the evident maps for c_g and res, and tr is defined by $\operatorname{tr}^H_K v = \sum_{h \in H/K} hv$. Here we see for the first time the convention of using the symbol ? to denote the place in the formula where we put the subgroup on which we are evaluating.

More generally, homology $H_*(?, V)$ and cohomology $H^*(?, V)$ are both Mackey functors with the usual transfer or corestriction maps.

There is a natural concept of a morphism of Mackey functors, and we obtain an abelian category $\mathrm{Mack}(G)$ of Mackey functors on G.

The simple Mackey functors are described explicitly in [8]. Each one has a minimal subgroup H on which it is non-zero, unique up to conjugation, and the $kN_G(H)/H$-module V say obtained by evaluation at H is simple. In fact any pair consisting of a subgroup H and a simple $kN_G(H)/H$-module can occur, and these parametrize the simple Mackey functors, which we denote by $S^G_{H,V}$. There are explicit formulas for these functors. One that we will need is

$$S^G_{H,V}(J) = \bigoplus_{\substack{g \in J \backslash G / H \\ H \leq J^g}} \operatorname{tr}_1^{N_{J^g}(H)/H} V,$$

where tr is as for the fixed point functor.

We are particularly interested in global Mackey functors. Again, one of these consists of a pair of functors, but they are defined on the category of finite groups and the injective homomorphisms between them. They take values in finite-dimensional k-vector spaces.

If α denotes an injective homomorphism then we write $\alpha^* = M^*(\alpha)$ and $\alpha_* = M_*(\alpha)$.

We require the relation:

(1′) If α is an isomorphism then $\alpha^*\alpha_* = 1$.

Given any finite group the obvious forgetful map yields a pair of functors on $\mathcal{S}(G)$, and we require that this should be a Mackey functor, i.e. we impose axioms (2) and (3). Thus a global Mackey functor is a pair of functors that is globally defined and locally a Mackey functor. Examples are homology and cohomology with trivial coefficients. The category of global Mackey functors is denoted by Mack.

We will also deal with inflation functors. Again an inflation functor consists of a pair of functors defined on finite groups; but M^* is defined on all group homomorphisms, although M_* is still only defined on injective homomorphisms.

This time the first axiom is replaced by

(1″) Given a commutative diagram of finite groups with exact rows and injective columns

$$\begin{array}{ccccc} J' & \longrightarrow & G' & \xrightarrow{\alpha} & H' \\ \big\downarrow{\cong} & & \big\downarrow{\beta} & & \big\downarrow{\gamma} \\ J & \longrightarrow & G & \xrightarrow{\delta} & H \end{array}$$

we have $\beta_*\alpha^* = \delta^*\gamma_*$.

We keep axioms (2) and (3).

Any inflation functor is naturally a global Mackey functor. An example is cohomology with trivial coefficients. The category is denoted by Mack*.

The simple functors in Mack and Mack* were classified by Webb in [10]. Again there is a minimal subgroup H on with the functor is non-zero, unique up to isomorphism, and the value at H is a simple $k\,\mathrm{Out}(H)$-module V. All such pairs H, V can occur.

The simple functors in Mack are denoted by $\mathcal{S}_{H,V}$ and their projective covers by $\mathcal{P}_{H,V}$. The simple functors in Mack* are denoted by $\mathbb{S}_{H,V}$ and their projective covers by $\mathbb{P}_{H,V}$. Again there are explicit formulas, but we will not need them. Just notice that when $\mathbb{S}_{H,V}$ is considered as an object of Mack it has $\mathcal{S}_{H,V}$ as a subfunctor.

All the Mackey functors that we will deal with are cohomological, that is they satisfy a fourth axiom

(4) If $K \leq H$ then $\mathrm{tr}_K^H \mathrm{res}_K^H = |K/H|\,\mathrm{Id}$.

Cohomology with trivial coefficients is, of course, an example. The categories of cohomological Mackey functors will be denotes by coMack etc..

The next lemma is easy to prove.

Lemma 2.1. *For Mackey functors in* Mack(G), Mack *or* Mack**:*

1. *If $L \to M \to N$ is exact then M is cohomological if and only if both L and N are cohomological.*
2. *If $|H/K|$ is coprime to p then* res_K^H *is injective.*

In fact (2) has a more precise version known as the method of stable elements.

Lemma 2.2. *If S is a Sylow p-subgroup of H and M is in* coMack *then* res_S^H *is injective and its image consists of the elements $x \in M(S)$ satisfying $c_h \mathrm{res}^S_{S^h\cap S} = \mathrm{res}^S_{S\cap {}^hS}$ for all $h \in H$.*

The simple cohomological Mackey functors (of whichever sort) are those with minimal subgroup H a p-group, see [9] and [10].

One way of understanding a Mackey functor in $\mathrm{Mack}(G)$ is to look at its simple composition factors. In fact Webb showed in [10] that this also makes sense for Mack and Mack^*. There might be infinitely many simple factors, but only finitely many of them show up at any particular evaluation and so there are only finitely many in any given isomorphism class.

We see from 2.1 that for a cohomological functor it makes no difference whether we take the composition factors in Mack or coMack.

One of the ideas of Peter Webb was that if we could calculate the composition factors of cohomology $H^r(?, k)$ in coMack or coMack^* then the explicit formulas for the simples would give an explicit formula for $H^r(G, k)$ for any finite group G.

Of course this would be a very difficult calculation, but we will show in theorem 4.1 that every simple does occur for some r.

3. Algebraic Results

For any group G, let $S_p(G)$ denote the set of p-subgroups of G and $C_p(G) = S_p(G)/G$ the set of their conjugacy classes. Set

$$\lambda_p(G) = \sum_{U \in C_p(G)} \frac{1}{|N_G(U)/C_G(U)|}$$

and, for any p-group Q,

$$\lambda_Q(G) = \sum_{\substack{U \in C_p(G) \\ U \cong Q}} \frac{1}{|N_G(U)/C_G(U)|}.$$

Lemma 3.1. *For any finite groups $H < G$, the following are equivalent.*

1. *H controls p-fusion in G,*
2. *Every conjugacy class of p-subgroups of G has intersection with $S_p(H)$ consisting of exactly one conjugacy class from $C_p(H)$ and for any $Q \in S_p(H)$ we have $N_G(Q) = N_H(Q)C_G(Q)$,*
3. *The index $|G/H|$ is coprime to p and $\lambda_p(G) = \lambda_p(H)$.*

Proof. (1) ⇒ (2) ⇒ (3) is clear.

For (3) ⇒ (1), note that every conjugacy class in $C_p(G)$ intersects $S_p(H)$. So

$$s = |C_p(H)| \geq |C_p(G)| = r \tag{†}$$

and we can choose representative subgroups $Q_1, \ldots, Q_r$ for $C_p(G)$, all in H, and enlarge this to representatives $Q_1, \ldots, Q_r, \ldots, Q_s$ for $C_p(H)$.

In addition

$$|N_G(Q_i)/C_G(Q_i)| \geq |N_H(Q_i)/C_H(Q_i)| \quad i = 1, \ldots, s. \tag{‡}$$

It follows that if

$$\sum_1^s \frac{1}{|N_G(Q_i)/C_G(Q_i)|} = \lambda_p(H) = \lambda_p(G) = \sum_1^r \frac{1}{|N_H(Q_i)/C_H(Q_i)|}$$

then all inequalities in † and ‡ above must be equalities, as required. □

For finite groups Q and G, let $\mathrm{Inj}(Q,H)$ denote the set of injective group homomorphisms $Q \to H$. Then H acts on the left by conjugation and $\mathrm{Aut}(Q)$ on the right by composition.

Definition 3.2. Given a finite p-group Q we now construct a global Mackey functor M_Q as follows.

On a group H,

$$M_Q(H) = k \otimes_{kH} k[\mathrm{Inj}(Q,H)].$$

If $\phi : K \hookrightarrow H$ then

$$\phi^*(f) = \sum_{\substack{h \in K\backslash H \\ c_h f(Q) \le \phi(K)}} \phi^{-1} c_h f,$$

where $f \in \mathrm{Inj}(Q,H)$ and c_h denotes conjugation by h. Also

$$\phi_*(f') = \phi f,$$

for $f' \in \mathrm{Inj}(Q,K)$.

(Here $k[\mathrm{Inj}(Q,H)]$ denotes the free k-module on the set $\mathrm{Inj}(Q,H)$.)

It is easy to check that M_Q is a global Mackey functor. For example, if $H, K \le G$ then

$$\mathrm{res}^G_K \mathrm{tr}^G_H f = \sum_{\substack{g \in K\backslash G \\ {}^g f(Q) \le K}} c_g f = \sum_{\substack{g \in K\backslash G/H \\ {}^g f(Q) \le K}} \sum_{h \in K^g \cap H\backslash H} c_g c_h f = \sum_{g \in K\backslash G/H} \mathrm{tr}^K_{K\cap {}^g H} c_g \mathrm{res}^H_{K^g\cap H} f.$$

Lemma 3.3. *M_Q is cohomological.*

Proof. This is true for quite general reasons (cf. [7]) but we give an elementary proof.

We need to show that for $K < H$ we have $\mathrm{tr}^H_K \mathrm{res}^H_K f = |H/K| f$, where $f : Q \to H$. That is $\sum_{\substack{h \in K\backslash H \\ {}^h f(Q) \le K}} f = |H/K| f$. This would certainly be true *without* the second condition in the sum.

If $K < J < H$ then $\mathrm{tr}^H_K \mathrm{res}^H_K = \mathrm{tr}^H_J (\mathrm{tr}^J_K \mathrm{res}^J_K) \mathrm{res}^H_J$ and $|H/K| = |H/J||J/K|$. Thus it suffices to prove the case when $K < H$ is maximal. There are two cases, and in both we identify $f(Q)$ with Q for convenience:

(a) $N_H(K) = H$: this subdivides into two more cases:

(i) All conjugates of Q lie in K and the result is clear from the formulas.

(ii) No conjugate of Q is in K, so $\mathrm{tr}^H_K \mathrm{res}^H_K f = 0$. But then K can not contain a Sylow p-subgroup of H, so p divides $|H/K|$.

(b) $N_H(K) = K$: Rewrite the condition ${}^h Q \le K$ as $Q \le K^h$. In this case $\{K^h | h \in K\backslash H\}$ is precisely the conjugacy class of K in H (without repetition).

Q acts on this set by conjugation. We claim that the fixed point set consists exactly of the K^h containing Q. Thus the size of the rest is divisible by p and the result follows.

Clearly if $Q \le K^h$ then K^h is fixed under conjugation by Q. Conversely, if Q fixes K^h then $K^h Q$ is a subgroup of H. If $Q \not\le K^h$ then $K^h Q = H$, by our hypothesis that K is maximal. Thus K^h, and hence K, is normal, a contradiction. □

Lemma 3.4. *Suppose that $H < G$ is of index coprime to p and that $L \to M \to N$ is a short exact sequence of Mackey functors on G such that N is cohomological. Then $\mathrm{res}^G_H : M(G) \to M(H)$ is an isomorphism if and only if both $\mathrm{res}^G_H : L(G) \to L(H)$ and $\mathrm{res}^G_H : N(G) \to N(H)$ are isomorphisms.*

Proof. The *if* part is just the Five Lemma, so it is the *only if* part that concerns us. The map $\mathrm{res}^G_H : N(G) \to N(H)$ is injective by 2.1. The rest follows by an easy diagram chase. □

Theorem 3.5. *Let $\{M^i : i \in I\}$ be a set of cohomological global Mackey functors. Let G be a finite group and $H < G$. Suppose that every simple cohomological global Mackey functor with minimal subgroup isomorphic to a subgroup of G occurs as a composition factor of some M^i.*

The following are equivalent:

(1) $\mathrm{res}^G_H : M^i(G) \to M^i(H)$ *is an isomorphism for every* $i \in I$ *and* $|G : H|$ *is coprime to* p.
(2) $\mathrm{res}^G_H : M(G) \to M(H)$ *is an isomorphism for every cohomological global Mackey functor* M.
(3) $\mathrm{res}^G_H : S(G) \to S(H)$ *is an isomorphism for every* simple *cohomological global Mackey functor* S.
(4) H *controls* p*-fusion in* G.

The theorem remains valid if we replace global Mackey functor *by* inflation functor *throughout.*

Proof. (1) ⇒ (3): If the minimal subgroup of S is not isomorphic to a subgroup of G then both $S(G)$ and $S(H)$ are zero and (3) holds.

Otherwise S is a composition factor of some M^i and so there are two short exact sequences of cohomological global Mackey functors $L \to M^i \to N$ and $K \to L \to S$. Now we apply lemma 3.4 twice.

(3) ⇒ (2) by induction on the number of composition factors with minimal subgroup isomorphic to a subgroup of G.

(2) ⇒ (1): We only need to check that $|G : H|$ is coprime to p. Let P be a Sylow p-subgroup of G. Then $\mathcal{S}_{P,k}(G) \cong k$ by 2.2 so, by hypothesis, $\mathcal{S}_{P,k}(H) \cong k$ and P must be isomorphic to a subgroup of H.

(4) ⇒ (2) by the method of stable elements 2.2.

(2) ⇒ (4): For any p-group Q consider the cohomological global Mackey functor M_Q of 3.2. We have $\lambda_Q(G) = \dim M_Q(G) = \dim M_Q(H) = \lambda Q(H)$. By summing over the isomorphism classes of p-subgroups of G we obtain $\lambda_p(G) = \lambda_p(H)$. Now apply lemma 3.1.

This proof remains valid if we work with inflation functors instead, except for (2) ⇒ (4). For this we could show that the dual of M_Q is naturally an inflation functor.

Alternatively just notice that $\mathbb{S}_{P,V}$, considered as a global Mackey functor, contains $\mathcal{S}_{P,V}$ as a composition factor. So we can use the simple cohomological inflation functors $\mathbb{S}_{P,V}$ as the M^i in the global Mackey functor version of the theorem and use (1) ⇒ (4). □

4. Composition Factors of $H^*(?, k)$

Now we show that the functors $H^i(?, k)$, $i \in \mathbb{N}_0$ satisfy condition (1) of 3.5, completing a proof of Mislin's Theorem. However our methods are topological, making crucial use of Carlsson's proof of the Segal Conjecture, [1], as does Mislin's original proof.

Theorem 4.1. *Considering cohomology as an inflation functor, every simple cohomological inflation functor occurs as a composition functor of some $H^i(?, k)$.*

The same is true with global Mackey functor *instead of* inflation functor.

Proof. First we prove the case with $k = \mathbb{F}_p$. We need to show that $\mathrm{Hom}(\mathbb{P}_{P,V}, H^*(?, \mathbb{F}_p)) \neq 0$ for each $\mathbb{P}_{P,V}$. Our proof is based on one by Harris and Kuhn, [3].

Recall that there are inflation functors $A(?, P) = \mathrm{Hom}_{\Omega^+_{\mathbb{F}_p}}(?, P)$, where $\Omega^+_{\mathbb{F}_p}$ is a certain category defined in terms of G-sets in [10], [11]. They have the property that $\mathrm{Hom}_{\mathrm{Mack}^*}(A(?, P), M) \cong M(P)$ for any inflation functor M, so in particular they are projective and also $\mathrm{End}_{\mathrm{Mack}^*}(A(?, P)) \cong A(P, P)$. In this way we can regard $M(P)$ as an $A(P, P)$-module in a manner consistent with morphisms of inflation functors.

Consider the composition of ring homorphisms

$$A(P, P) \to \{(BP_+)^\wedge_p, (BP_+)^\wedge_p\} \otimes \mathbb{F}_p \to \mathrm{End}(H^*(P, \mathbb{F}_p))^{\mathrm{opp}},$$

where $\{\}$ denotes homotopy classes of stable maps.

In [3] 2.14 it is shown that the first has nilpotent kernel, by the Segal Conjecture, and so does the second, by Whitehead's Theorem. In particular all idempotents survive.

Now $\mathbb{P}_{P,V}$ is a direct summand of $A(?, P)$, because $\mathrm{Hom}_{\mathrm{Mack}^*}(A(?, P), \mathbb{S}_{P,V}) \cong \mathbb{S}_{P,V}(P) = V \neq 0$. Let $e \in A(P, P)$ be the corresponding idempotent. We find

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mack}^*}(\mathbb{P}_{P,V}, H^*(?, \mathbb{F}_p)) &= \mathrm{Hom}_{\mathrm{Mack}^*}(eA(?, P), H^*(?, \mathbb{F}_p)) \\ &\cong \mathrm{Hom}_{\mathrm{Mack}^*}(A(?, P), eH^*(?, \mathbb{F}_p)) \\ &\cong eH^*(P, \mathbb{F}_p) \\ &\neq 0.\end{aligned}$$

as required.

To obtain the result over a general field k note that any simple kP-module V is a composition factor of $W \otimes_{\mathbb{F}_p} k$ for some simple $\mathbb{F}_p P$-module W. Then $\mathbb{S}_{P,V}$ is a composition factor of $S_{P,W} \otimes_{\mathbb{F}_p} k$.

But we have just shown that the latter is a composition factor of some $H^i(?, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k \cong H^i(?, k)$.

For the global Mackey functor version just note that $\mathbb{S}_{P,V}$, regarded as a global Mackey functor, has $\mathcal{S}_{P,V}$ as a composition factor. □

Corollary 4.2. *If we consider $H^*(?, k)$ as a Mackey functor on a particular group G only then we still find as a composition factor every simple cohomological $S^G_{P,V}$ for which $C_G(P)$ acts trivially on V, and only these.*

Proof. The global simple $\mathcal{S}_{P,W}$, restricted to G, contains every simple S^G_{P,W_i} as a composition factor, where W_i is a composition factor of W regarded as a $kN_G(P)$-module via the conjugation map $N_G(P) \to \mathrm{Out}(P)$. This map has kernel $PC_G(P)$, so every simple module V for $N_G(P)/PC_G(P)$ will occur as some W_i.

Of course, since $C_G(P)$ acts trivially on $H^*(P,k)$, it must act trivially on $S^G_{P,V}(P) = V$ for any composition factor $S^G_{P,V}$. □

5. Permutation Modules

It was shown by Thévenaz and Webb [9] that if $P^{\mathrm{co},G}_{P,V}$ denotes the projective cover of $S^G_{P,V}$ as a *cohomological* Mackey functor then the evaluation at the trivial subgroup $P^{\mathrm{co},G}_{P,V}(1)$ is the p-permutation kG-module $\mathrm{Perm}^G_{P,V}$, which is defined by taking the projective cover of V as an $N_G(P)/P$-module, inflating to $N_G(P)$ and taking the Green correspondent.

Proposition 5.1. *For any kG-module M and $r \geq 0$,*

$$\mathrm{Hom}_{\mathrm{coMack}(G)}(P^{\mathrm{co},G}_{P,V}, H^r(?,M)) \cong \mathrm{Ext}^r_{kG}(\mathrm{Perm}^G_{P,V}, M).$$

Proof. Both sides have long exact sequences in M and vanish on injective M for $r > 0$. They agree when $r = 0$ by the following lemma, and the result follows by general homological algebra. □

Lemma 5.2. *For F a Mackey functor on G over k and M a kG-module, evaluation at 1 yields an isomorphism*

$$\mathrm{Hom}_{\mathrm{Mack}(G)}(F, M^?) \cong \mathrm{Hom}_{kG-\mathrm{mod}}(F(1), M).$$

Proof. We construct the inverse map. Given $f : F(1) \to M$ then $F(H) \stackrel{\mathrm{res}}{\to} F(1) \stackrel{f}{\to} M$ has image in M^H, so we obtain a map $F(H) \to M^H$. These piece together to give a map of Mackey functors $F \to M^?$. □

We can now deduce a result that does not mention Mackey functors.

Theorem 5.3. *$H^*(G, \mathrm{Perm}^G_{P,V}) \neq 0$ if and only if $C_G(P)$ acts trivially on V.*

Proof. For any left kH-module V let V^* denote the contragredient left kG-module $\mathrm{Hom}_{k-\mathrm{mod}}(V,k)$. Then, using 5.1, we find that

$$\begin{aligned} H^*(G, \mathrm{Perm}^G_{P,V}) &\cong \mathrm{Ext}^*_{kG}(k, \mathrm{Perm}^G_{P,V}) \\ &\cong \mathrm{Ext}^*_{kG}((\mathrm{Perm}^G_{P,V})^*, k) \\ &\cong \mathrm{Ext}^*_{kG}(\mathrm{Perm}^G_{P,V^*}, k) \\ &\cong \mathrm{Hom}_{\mathrm{MF}_k(G)}(P^{\mathrm{co},G}_{P,V^*}, H^*(?,k)). \end{aligned}$$

Now apply corollary 4.2. □

Remark. This result tells us something about the discarded terms in the Green correspondence. For example suppose that $P < H < G$ and $N_G(P) < H$. Then we have $\mathrm{Ind}^G_H \mathrm{Perm}^H_{P,V} = \mathrm{Perm}^G_{P,V} \oplus U$, where U is a sum of p-permutation modules $\mathrm{Perm}^G_{Q_i,W_i}$ with $Q_i \lneqq P$. By considering cohomology we see that if $C_G(P)$ acts non-trivially on V then each $C_G(Q_i)$ acts non-trivially on W_i.

Remark. Geoff Robinson's strategy for finding an algebraic proof of Mislin's theorem was essentially to try to prove the above theorem algebraically, see [5].

We can only find an algebraic proof in the local case (as could Robinson).

Proposition 5.4. *If $P \lhd G$ then we can prove algebraically that if $C_G(P)$ acts trivially on V then $H^*(G, \mathrm{Perm}^G_{P,V}) \neq 0$.*

Proof. The first part of the previous proof can still be used, so we need to show that $H^*(?, k)$, considered as a Mackey functor on G, contains the simple S^G_{P,V^*} whenever $C_G(P)$ acts trivially on V.

Suppose that $Q \lneqq P$ and W is a simple $kN_G(Q)/Q$-module. Then from the explicit formula for $S^G_{Q,W}$ we see that

$$S^G_{Q,W}(P) \cong \bigoplus_{g \in G/PN_G(Q)} \mathrm{tr}_1^{N_P({}^gQ)/{}^gQ}\, {}^gW \cong S^G_{Q,W}(P) \cong \bigoplus_{g \in G/PN_G(Q)} \mathrm{tr}_1^{N_P(Q)/Q} W.$$

But $N_P(Q)$ is strictly bigger than Q, yet it is normal in $N_G(Q)$, so acts trivially on W. Thus $\mathrm{tr}_1^{N_P(Q)/Q} W = 0$ and so $S^G_{Q,W}(P) = 0$. Thus all of $H^i(P, k)$ is accounted for by the $S^G_{P,V}$, and the multiplicity of $S^G_{P,V}$ in $H^i(?, k)$ is is that of V in $H^i(P, k)$ regarded as a G/P-module.

But, regarded as a $k\,\mathrm{Out}(P)$-module, $H^*(P, k)$ contains every simple, by [2] or [6]. Thus, as a kG/P-module, it contains every simple on which $C_G(P)$ acts trivially. □

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Department of Mathematics, U.M.I.S.T., P.O. Box 88, Manchester M60 1QD, England
E-mail address: Peter.Symonds@umist.ac.uk