

COMPARISON OF THE DISCRETE AND CONTINUOUS COHOMOLOGY GROUPS OF A PRO- p GROUP

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1. INTRODUCTION

Let G be a profinite group and let M be a discrete G -module, i.e. M has the discrete topology and G acts on M continuously. The continuous, or Galois, cohomology groups of G with coefficients in M can be defined via continuous cochains from $G \times \cdots \times G$ to M . General references for the continuous cohomology of profinite groups are the classical monograph of Serre [11], and some chapters of the recent books [10] and [13] on profinite groups.

Of course, a profinite group G can also be regarded as an abstract group, and then the corresponding cohomology groups are obtained by considering all cochains, and not only the continuous ones. Topologists prefer to think of this as giving G the discrete topology, so we speak of discrete cohomology groups in this case. Thus for a discrete G -module M we can consider both the continuous and the discrete cohomology groups. In order to avoid confusion, we denote them by $H_{\text{cont}}^q(G, M)$ and $H_{\text{disc}}^q(G, M)$, respectively. Since the continuous cochains form a subgroup of the group of all cochains, it readily follows that there are natural homomorphisms $\varphi^q : H_{\text{cont}}^q(G, M) \rightarrow H_{\text{disc}}^q(G, M)$, which we call the *comparison maps* between the continuous and discrete cohomology groups.

Our interest is focused on pro- p groups. If P is a pro- p group, then the trivial P -module \mathbb{F}_p plays a prominent role in the study of the cohomology of P , since \mathbb{F}_p is the only discrete simple p -torsion P -module. As a consequence, in order to determine the continuous cohomological dimension of P , one has to look only at the cohomology groups of P with coefficients in \mathbb{F}_p . Let us write, for short, $H_{\text{cont}}^q(P)$ and $H_{\text{disc}}^q(P)$ instead of $H_{\text{cont}}^q(P, \mathbb{F}_p)$ and $H_{\text{disc}}^q(P, \mathbb{F}_p)$, respectively.

The following question was posed to one of the authors by G. Mislin.

Question 1. If P is a pro- p group, when is the comparison map $\varphi^q : H_{\text{cont}}^q(P) \rightarrow H_{\text{disc}}^q(P)$ an isomorphism?

According to Mislin [9], the answer to this question is affirmative for every q if P is poly- \mathbb{Z}_p by finite. If we look at small values of q , it is obvious that φ^0 is always an isomorphism. On the other hand, since $H_{\text{disc}}^1(P)$ coincides with the group of all homomorphisms from P to \mathbb{F}_p and $H_{\text{cont}}^1(P)$ is the subgroup of continuous homomorphisms, it is clear that φ^1 is an isomorphism if and only if every $f \in \text{Hom}(P, \mathbb{F}_p)$ is continuous. This happens if and only if every maximal subgroup of

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P is open, which in turn is equivalent to P being finitely generated. (See Exercise 6(d) in Section I.4.2 of [11], and Exercise 4.6.5 of [13].) Now we ask: for which finitely generated pro- p groups is also $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$ an isomorphism? As proved by Sury [12], this is the case if P is either soluble p -adic analytic or Chevalley p -adic analytic, and the following conjecture has been raised.

Conjecture (D. Segal, P. Symonds, N. Nikolov). For every p -adic analytic group P , the comparison map $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$ is an isomorphism.

We also address the corresponding problem for the homology of a profinite group G . In this case, we have to consider profinite G -modules. Recall that, as an abstract group, the homology of G with coefficients in M can be defined as the homology of the chain complex which is obtained by tensoring M with a projective resolution $X \rightarrow \mathbb{Z}$ of \mathbb{Z} over $\mathbb{Z}G$. (See Section III.1 of [3].) The homology of G as a profinite group is defined similarly, but this time we need to use a projective resolution $X' \rightarrow \widehat{\mathbb{Z}}$ of $\widehat{\mathbb{Z}}$ in the category of profinite modules over the completed group ring $\widehat{\mathbb{Z}}[[G]]$, and the complete tensor product $\widehat{\otimes}$ instead of the usual tensor product \otimes . (See Section 6.3 of [10] in this case.) Hence, again we have discrete homology groups $H_q^{\text{disc}}(G, M)$ and continuous homology groups $H_q^{\text{cont}}(G, M)$. By the Comparison Theorem (Theorem III.6.1 of [7]), there is a chain map $X \rightarrow X'$, which gives rise to a chain map $X \otimes_{\mathbb{Z}G} M \rightarrow X' \widehat{\otimes}_{\widehat{\mathbb{Z}}[[G]]} M$. This way we obtain natural homomorphisms $\varphi_q : H_q^{\text{disc}}(G, M) \rightarrow H_q^{\text{cont}}(G, M)$ between the corresponding homology groups, and we are interested in determining when φ_q is an isomorphism. As before, we concentrate on the homology groups of a pro- p group P with coefficients in the trivial module \mathbb{F}_p , which we denote by $H_q^{\text{disc}}(P)$ and $H_q^{\text{cont}}(P)$.

Our first main theorem is the following, which will be proved in Section 2 together with some other general results.

Theorem A. *Let P be a finitely generated pro- p group. If P is not finitely presented then $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$ is not surjective. Furthermore, if P is finitely presented then the following conditions are equivalent:*

- (i) $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$ is an isomorphism.
- (ii) $\varphi_2 : H_2^{\text{disc}}(P) \rightarrow H_2^{\text{cont}}(P)$ is an isomorphism.

Note that Theorem A is not in conflict with the conjecture above, since p -adic analytic groups are finitely presented [13, Proposition 12.2.3]. The following question arises.

Question 2. Does there exist a finitely presented pro- p group P for which $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$ is not an isomorphism?

The second cohomology groups $H_{\text{disc}}^2(P)$ and $H_{\text{cont}}^2(P)$ classify the extensions of P by \mathbb{F}_p : arbitrary extensions in the first case, and extensions which are again a pro- p group in the second. (See Section IV.3 of [3] in the discrete case and Section 6.8 of [10] in the continuous case.) Since φ^2 is not surjective by Theorem A, for every finitely generated but not finitely presented pro- p group P , there exists an extension

$$1 \longrightarrow \mathbb{F}_p \longrightarrow G \longrightarrow P \longrightarrow 1$$

of abstract groups such that G can not be given the structure of a pro- p group. A different, and more difficult, question is to construct such extensions explicitly. That is the purpose of the second main result in this paper.

Theorem B. *For every prime number p , and every power q of p , there is a explicit construction of an abstract group G with a normal subgroup Z of order q such that:*

- (i) *The quotient $P = G/Z$ can be endowed with a topology that makes this group a pro- p group. Furthermore, G can be chosen so that P is finitely generated.*
- (ii) *It is not possible to define a topology on G so that it becomes a pro- p group.*

Unfortunately, as we will see later, the groups we construct do not yield a finitely presented quotient P , and Question 2 remains open. However, we hope that some modification of this construction can be used to provide examples with P finitely presented. On the other hand, even though $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$ is not an isomorphism for the groups of Theorem B, it is not clear whether $\varphi_2 : H_2^{\text{disc}}(P) \rightarrow H_2^{\text{cont}}(P)$ is an isomorphism or not.

Let us explain briefly the main idea behind these constructions. Recall that a group G is called *monolithic* if the intersection T of all its non-trivial normal subgroups is non-trivial. The subgroup T is then called the *monolith* of G . For example, a nilpotent group is monolithic if and only if $Z(G)$ is monolithic. In particular, a finite p -group is monolithic if and only if it has a cyclic centre. On the other hand, an infinite pro- p group is never monolithic, since its open normal subgroups have trivial intersection. As a consequence, if G is an infinite monolithic group then it is impossible to define a topology on G that makes it a pro- p group. Thus in order to prove Theorem B it suffices to construct monolithic groups having a pro- p group as a quotient. In fact, we will obtain groups in which the centre is a cyclic p -group and it is precisely the central quotient that will yield a pro- p group. More precisely, in Section 3 we give a very general construction of monolithic groups whose central quotient turns out to be the unrestricted direct product of countably many copies of a certain finite p -group (thus this quotient is an infinitely generated pro- p group). Important ingredients of this construction are central products, inverse limits and ultraproducts. Then, in Section 4, we modify this construction with the help of the wreath product in order to get finitely generated examples.

2. GENERAL RESULTS AND PROOF OF THEOREM A

In this section, we prove Theorem A and some related results. First of all, we collect some tools from the (co)homology of profinite groups that will be needed. In the following, let G be a profinite group, and let M be a topological module on which G acts continuously. (Notice that, regardless of what kind of (co)homology we are considering, discrete or continuous, the action of G on all G -modules is assumed to be continuous.)

If M is discrete, the continuous cohomology of G with coefficients in M can be obtained from the cohomology of its finite quotients [10, Corollary 6.5.6]:

$$H_{\text{cont}}^q(G, M) = \varinjlim H^q(G/U, M^U),$$

where U runs over all open normal subgroups of G , and the direct limit is taken over the inflation maps. Here, M^U is the subgroup of U -invariant elements of M . As a consequence, if P is a pro- p group and M a p' -module for P (i.e. a p' -torsion P -module) then $H_{\text{cont}}^q(P, M) = 0$ for $q \geq 1$, since the cohomology of a finite p -group with coefficients in a p' -module is trivial for $q \geq 1$ [3, Corollary III.10.2].

In a similar fashion, if M is profinite, the continuous homology groups of G with coefficients in M are inverse limits [10, Corollary 6.5.8]:

$$H_q^{\text{cont}}(G, M) = \varprojlim H_q(G/U, M_U),$$

where M_U is the module of continuous coinvariants corresponding to U , i.e.

$$M_U = M / \langle gm - m \mid m \in M, g \in U \rangle,$$

and the inverse limit is taken over the coinflation maps. It follows that, for any pro- p group P and any p' -module M , the homology groups $H_q^{\text{cont}}(P, M)$ are trivial for all $q \geq 1$.

As a consequence, we have the following description of the comparison maps φ^q and φ_q .

Theorem 2.1. *For any profinite group G , the following hold:*

- (i) *If M is a discrete G -module, then the comparison map $\varphi^q : H_{\text{cont}}^q(G, M) \rightarrow H_{\text{disc}}^q(G, M)$ is the direct limit of the inflation maps $H^q(G/U, M^U) \rightarrow H_{\text{disc}}^q(G, M)$.*
- (ii) *If M is a profinite G -module, then the comparison map $\varphi_q : H_q^{\text{disc}}(G, M) \rightarrow H_q^{\text{cont}}(G, M)$ is the inverse limit of the coinflation maps $H_q^{\text{disc}}(G, M) \rightarrow H_q(G/U, M_U)$.*

There are two dualising functors that concern us: $*_{\text{disc}} = \text{Hom}_{\text{disc}}(-, \mathbb{Q}/\mathbb{Z})$ and $*_{\text{cont}} = \text{Hom}_{\text{cont}}(-, \mathbb{Q}/\mathbb{Z})$. They take G -modules to G -modules, and since we always deal with left G -modules, we are implicitly taking the contragredient representation, i.e. changing the right action into a left one by using the inverse map. Observe also that $*_{\text{disc}}$ and $*_{\text{cont}}$ agree on finite modules, in which case we just write $*$. The second of these functors is Pontryagin duality, which converts compact modules into discrete ones and vice versa. We know that $(M^{*\text{cont}})^{*_{\text{cont}}} \cong M$ for M discrete torsion or profinite, although $(M^{*\text{disc}})^{*_{\text{disc}}} \cong M$ if and only if M is finite. (In this paper, when we speak of finite modules, we tacitly understand that they are discrete or, what is the same in that case, profinite.)

We have the following Duality Theorem (see [3, Proposition VI.7.1] and [10, Proposition 6.3.6]).

Theorem 2.2. *Let G be a profinite group. Then:*

- (i) $H_{\text{disc}}^q(G, M^{*\text{disc}}) \cong H_q^{\text{disc}}(G, M)^{*_{\text{disc}}}$, for every G -module M .
- (ii) $H_{\text{cont}}^q(G, M^{*\text{cont}}) \cong H_q^{\text{cont}}(G, M)^{*_{\text{cont}}}$, for every profinite G -module M .

This result allows us to relate the homology comparison map for a finite G -module with the cohomology comparison map for its dual.

Lemma 2.3. *Let G be a profinite group and let M be a finite G -module. Then the map $\varphi^q : H_{\text{cont}}^q(G, M^*) \rightarrow H_{\text{disc}}^q(G, M^*)$ is equal to the composition*

$$H_{\text{cont}}^q(G, M^*) \cong H_q^{\text{cont}}(G, M)^{*_{\text{cont}}} \xrightarrow{\delta} H_q^{\text{cont}}(G, M)^{*_{\text{disc}}} \xrightarrow{(\varphi_q)^*} H_q^{\text{disc}}(G, M)^{*_{\text{disc}}} \cong H_{\text{disc}}^q(G, M^*),$$

where δ is the injection map and the isomorphisms come from the Duality Theorem 2.2.

Proof. Every element of $H_{\text{cont}}^q(G, M^*)$ is in the image of the inflation map from $H^q(G/U, (M^*)^U)$, for some open normal subgroup U of M . All the maps involved commute with inflation. It is therefore sufficient to prove equality in the case when G is finite. But then all three maps φ^q , φ_q and δ are the identity and the result follows. \square

Theorem 2.4. *Let G be a profinite group and let M be a finite G -module. Consider the comparison maps $\varphi^q : H_{\text{cont}}^q(G, M^*) \rightarrow H_{\text{disc}}^q(G, M^*)$ and $\varphi_q : H_q^{\text{disc}}(G, M) \rightarrow H_q^{\text{cont}}(G, M)$. Then:*

- (i) *If φ^q is surjective then φ_q is injective, and if φ_q is surjective then φ^q is injective.*
- (ii) *Any two of the following conditions implies the third: φ_q is an isomorphism, φ^q is an isomorphism, and $H_q^{\text{cont}}(G, M)$ is finite.*

Proof. All the assertions follow easily from Lemma 2.3. If φ^q is surjective, then $(\varphi_q)^*$ must be surjective, and so φ_q is injective. On the other hand, if φ_q is surjective then $(\varphi_q)^*$ is injective, and thus φ^q is injective. This shows that (i) holds.

Let us now prove (ii). Since φ^q is the composition of δ , φ_q and two isomorphisms, it is clear that any two of these conditions implies the third: φ_q is an isomorphism, φ^q is an isomorphism, and δ is an isomorphism. Now, the injection $\delta : H_q^{\text{cont}}(G, M)^{*_{\text{cont}}} \rightarrow H_q^{\text{cont}}(G, M)^{*_{\text{disc}}}$ is an isomorphism if and only if $H_q^{\text{cont}}(G, M)$ is finite. \square

In what follows, we specialize to pro- p groups. Special attention is devoted to the trivial module \mathbb{F}_p , since it is the only discrete simple p -torsion module for a pro- p group. As a consequence, if P is a pro- p group and M a finite P -module of p -power order, all composition factors of M are isomorphic to \mathbb{F}_p .

The following result relates the cohomology of pro- p groups in low dimension with group presentations (see [11, Sections I.4.2 and I.4.3]).

Theorem 2.5. *Let P be a pro- p group. Then:*

- (i) *$\dim_{\mathbb{F}_p} H_{\text{cont}}^1(P)$ is equal to the number of topological generators of P .*
- (ii) *If P is finitely generated, then $\dim_{\mathbb{F}_p} H_{\text{cont}}^2(P)$ is equal to the number of relations in a presentation of P as a pro- p group with minimum number of generators.*

Let us now examine the behaviour of the comparison maps in low dimensions.

Theorem 2.6. *Let P be a pro- p group. Then:*

- (i) *φ^0 is an isomorphism for every discrete P -module.*
- (ii) *φ_0 is an isomorphism if either the module is finite or P is finitely generated.*

Proof. As already mentioned in the introduction, (i) is obvious. On the other hand, by [3, Section III.1] and [10, Lemma 6.3.3], we have

$$H_0^{\text{disc}}(P, M) = M/N \quad \text{and} \quad H_0^{\text{cont}}(P, M) = M/\overline{N},$$

where $N = \langle gm - m \mid m \in M, g \in P \rangle$. Thus φ_0 is an isomorphism precisely when $\overline{N} = N$. This is clearly the case if the module M is finite. Suppose then that P is finitely generated, by elements g_1, \dots, g_n , say. Since M is profinite in the case of homology, M is compact. Consequently the sub-abelian-group $T = \sum_{i=1}^n (g_i - 1)M$

is also compact, and thus it is closed in M and is a P -submodule. Since P acts trivially on the quotient M/T , it follows that T contains \overline{N} , and thus $\overline{N} = N$ also in this case. \square

Theorem 2.7. *For finitely generated pro- p groups and finite modules, φ^1 and φ_1 are isomorphisms.*

Proof. Let P be a finitely generated pro- p group and let M be a finite P -module. By [10, Lemma 6.8.6], $H_1^{\text{cont}}(P)$ is isomorphic to $P/\Phi(P)$, where $\Phi(P)$ is the Frattini subgroup of P . Thus $H_1^{\text{cont}}(P)$ is finite. Let W be an arbitrary composition factor of M . Then either W is isomorphic to \mathbb{F}_p and $H_1^{\text{cont}}(P, W)$ is finite, or W is a p' -module and $H_1^{\text{cont}}(P, W) = 0$. It follows that $H_1^{\text{cont}}(P, M)$ is also finite. According to Theorem 2.4 (ii), it suffices to prove the result for φ^1 .

We know that $H_{\text{cont}}^1(P, M)$ and $H_{\text{disc}}^1(P, M)$ classify continuous and discrete sections of $M \rtimes P \rightarrow P$, respectively. (See [10, Lemma 6.8.1] and [3, Proposition IV.2.3].) Since all abstract homomorphisms between finitely generated pro- p groups are continuous [5, Corollary 1.21], we conclude that φ^1 is an isomorphism. \square

We only need the following lemma in order to proceed to the proof of Theorem A. This result is adapted from Exercise 1 in [11, Section I.2.6].

Lemma 2.8. *Let P be a pro- p group and let n be an integer. Then the following conditions are equivalent:*

- (i) $\varphi_q : H_q^{\text{disc}}(P, M) \rightarrow H_q^{\text{cont}}(P, M)$ is an isomorphism for $q \leq n$ and a surjection for $q = n + 1$, for every finite P -module M of p -power order.
- (ii) $\varphi_q : H_q^{\text{disc}}(P, M) \rightarrow H_q^{\text{cont}}(P, M)$ is an injection for $q \leq n$, for every finite P -module M of p -power order.
- (iii) $\varphi_q : H_q^{\text{disc}}(P) \rightarrow H_q^{\text{cont}}(P)$ is an injection for $q \leq n$ and a surjection for $q = n + 1$.

There is also a cohomology version, which is obtained by interchanging everywhere discrete and continuous, and injection and surjection.

Proof of Theorem A. Since P is finitely generated, we know from Theorems 2.6 and 2.7 that φ_0 and φ_1 are isomorphisms for finite coefficients. It follows from Lemma 2.8 that φ_2 is surjective for coefficients in \mathbb{F}_p . Then φ^2 is injective by Theorem 2.4.

Suppose first that P is not finitely presented. By Theorem 2.5, the cohomology group $H_{\text{cont}}^2(P)$ is infinite. If φ^2 is surjective for \mathbb{F}_p , then it is an isomorphism, and consequently $H_2^{\text{cont}}(P)$ is finite, by combining (i) and (ii) of Theorem 2.4. Since $\mathbb{F}_p^* \cong \mathbb{F}_p$, it follows from Theorem 2.2 that $H_{\text{cont}}^2(P) \cong H_2^{\text{cont}}(P)^*$. Hence $H_{\text{cont}}^2(P)$ is also finite, which is a contradiction.

Assume finally that P is finitely presented. By Theorem 2.5, $H_{\text{cont}}^2(P)$ is finite. Since $|H_2^{\text{cont}}(P)| = |H_{\text{cont}}^2(P)|$, the equivalence of (i) and (ii) follows from Theorem 2.4. \square

As a consequence, we see that extensions of the sort constructed in this paper must always exist for pro- p groups that are finitely generated but not finitely presented; the problem is to construct them explicitly. It is still possible that φ_2 is an isomorphism for these groups. This also explains our interest in finding a finitely presented example.

Remark 1. Bousfield [2] has shown that if F is a free pro- p group on at least two generators then $H_2^{\text{disc}}(F) \oplus H_3^{\text{disc}}(F)$ is uncountable. On the other hand, it is

known [10, Theorem 7.7.4] that $H_{\text{cont}}^q(F) = 0$ for $q \geq 2$, and by Theorem 2.2, also $H_q^{\text{cont}}(F) = 0$. Hence, at least one of the maps φ_2 and φ_3 is not an isomorphism.

Remark 2. There is an interesting analogy with what happens when a compact Lie group is considered as a discrete group (see [8], [6]).

At this point, it is not difficult to give a proof of Mislin's result mentioned in the introduction, Theorem 2.10 below. For this purpose, we study the comparison maps for the group \mathbb{Z}_p .

Theorem 2.9. *Let M be a finite \mathbb{Z}_p -module. Then the comparison maps $\varphi^q : H_{\text{cont}}^q(\mathbb{Z}_p, M) \rightarrow H_{\text{disc}}^q(\mathbb{Z}_p, M)$ and $\varphi_q : H_q^{\text{disc}}(\mathbb{Z}_p, M) \rightarrow H_q^{\text{cont}}(\mathbb{Z}_p, M)$ are isomorphisms for all q .*

Proof. We already know that the comparison maps are isomorphisms in degrees 0 and 1, so we assume $q \geq 2$ in the remainder of the proof. In that case, we are going to prove that $H_{\text{cont}}^q(\mathbb{Z}_p, M) = H_q^{\text{disc}}(\mathbb{Z}_p, M) = 0$. It then follows from the Duality Theorem 2.2 that also $H_q^{\text{cont}}(\mathbb{Z}_p, M) = H_{\text{disc}}^q(\mathbb{Z}_p, M) = 0$, and φ^q and φ_q are trivially isomorphisms. Since any finite module is the direct sum of a p -module and a p' -module, we may assume that either M is a p -module or a p' -module.

Let us first deal with H_{cont}^q . If M is a p -module then $H_{\text{cont}}^q(\mathbb{Z}_p, M) = 0$, since the cohomological p -dimension of \mathbb{Z}_p is 1 [10, Theorem 7.7.4]. On the other hand, as mentioned at the beginning of this section, $H_{\text{cont}}^q(\mathbb{Z}_p, M) = 0$ if M is a p' -module.

For the case of H_q^{disc} , we use a result of Cartan (see [4] or Theorem V.6.4 in [3]), stating in particular that $H_*^{\text{disc}}(G, R) \cong \bigwedge^*(G \otimes R)$ for any torsion-free abelian group G and any principal ideal domain R on which G acts trivially. Here, $H_*^{\text{disc}}(G, R)$ is viewed as an R -algebra under the Pontryagin product and \bigwedge^* indicates the exterior algebra. Since $q \geq 2$, it follows from this result that $H_q^{\text{disc}}(\mathbb{Z}_p, F) = 0$ for every field F of prime order with trivial action of \mathbb{Z}_p . As a consequence, $H_q^{\text{disc}}(\mathbb{Z}_p, M) = 0$ if M is either a p -module, or a p' -module on which \mathbb{Z}_p acts trivially, since in both cases all composition factors of M are cyclic of prime order with trivial action.

For a general p' -module, we can consider an open subgroup U of \mathbb{Z}_p acting trivially on M , since the action is continuous and M is finite. Then $U \cong \mathbb{Z}_p$ and $H_q^{\text{disc}}(U, M) = 0$, as we have just shown. Since $|\mathbb{Z}_p : U|$ is a p -power, this index is invertible in M and we deduce from [3, Proposition III.10.1] that $H_q^{\text{disc}}(\mathbb{Z}_p, M) = 0$. This concludes the proof of the theorem. \square

Theorem 2.10. *Let P be a pro- p group which is poly- \mathbb{Z}_p by finite, and let M be a finite P -module. Then $\varphi^q : H_{\text{cont}}^q(P, M) \rightarrow H_{\text{disc}}^q(P, M)$ and $\varphi_q : H_q^{\text{disc}}(P, M) \rightarrow H_q^{\text{cont}}(P, M)$ are isomorphisms for all q .*

Proof. By hypothesis, there is a series $P = P_0 \geq P_1 \geq \dots \geq P_r = 1$ of closed subgroups of P such that $P_{i+1} \trianglelefteq P_i$ and P_i/P_{i+1} is either finite or isomorphic to \mathbb{Z}_p for all i . We argue by induction on r . According to Theorem 2.9, φ^q is an isomorphism on each factor P_i/P_{i+1} . We can now assemble these isomorphisms by using the Lyndon-Hochschild-Serre spectral sequence $H^r(G/H, H^s(H)) \Rightarrow H^{r+s}(G)$ in both cohomology theories and the Comparison Theorem for spectral sequences. The proof for homology goes along the same lines. \square

3. PROOF OF THEOREM B: EXAMPLES WITH P NOT FINITELY GENERATED

Let E be a monolithic finite p -group. As mentioned in the introduction, this means simply that E has cyclic centre, so there is certainly a vast choice for E . Just note that, for any non-trivial power q of p , it is possible to take E so that its centre has order q . For example, since the group of units $\mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$ has a cyclic subgroup of order p^{n-1} if $p > 2$ and $n \geq 2$ and of order 2^{n-2} if $p = 2$ and $n \geq 3$, it is easy to construct semidirect products of two cyclic p -groups which have a cyclic centre of any desired order, and the group can be even of class 2 if we wish.

Let us write E_n to denote the direct product of n copies of E . Inside the centre Z_n of E_n we consider the following subgroup:

$$F_n = \{(x^{(1)}, \dots, x^{(n)}) \in Z_n \mid x^{(1)} \dots x^{(n)} = 1\}.$$

Note that F_n is generated by all tuples of the form $(1, \dots, 1, x, 1, \dots, 1, x^{-1}, 1, \dots, 1)$. Thus the quotient $K_n = E_n/F_n$ can be viewed as the “natural” central product of n copies of E , and it has the following property.

Theorem 3.1. *The group K_n is monolithic, with centre $L_n = Z_n/F_n \cong Z(E)$.*

Proof. Since $Z(E)$ is cyclic, it suffices to prove the second assertion or, equivalently, that if $x = (x^{(i)}) \in E_n$ satisfies that $[x, E_n] \leq F_n$ then $x^{(i)} \in Z(E)$ for all $i = 1, \dots, n$. This follows easily arguing by way of contradiction. \square

Consider now the projection map

$$\begin{array}{ccc} \varphi_n & : & E_n & \longrightarrow & E_{n-1} \\ & & (x^{(1)}, \dots, x^{(n)}) & \longmapsto & (x^{(1)}, \dots, x^{(n-1)}). \end{array}$$

Note that $\varphi(Z_n) = Z_{n-1}$, so we get a surjective homomorphism $\psi_n : E_n/Z_n \longrightarrow E_{n-1}/Z_{n-1}$. Since there is a natural isomorphism $K_n/L_n \cong E_n/Z_n$, we may suppose that the homomorphisms ψ_n are defined from K_n/L_n onto K_{n-1}/L_{n-1} . Thus we can consider the projective system $\{K_n/L_n, \psi_n\}$ over \mathbb{N} . We then define

$$K = \{(k_n) \in \prod_{n \in \mathbb{N}} K_n \mid (k_n L_n) \in \varprojlim K_n/L_n\}.$$

Note that $L = \prod_{n \in \mathbb{N}} L_n$ is a subgroup of K .

Before giving the last step of our construction, we digress for a while in order to introduce the concepts of non-principal ultrafilters, ultraproducts and ultrapowers.

Definition 3.2. A *filter* over a non-empty set I is a non-empty family \mathcal{U} of subsets of I such that:

- (i) The intersection of two elements of \mathcal{U} also lies in \mathcal{U} .
- (ii) If $P \in \mathcal{U}$ and $P \subseteq Q$ then also $Q \in \mathcal{U}$.
- (iii) The empty set does not belong to \mathcal{U} .

The filter \mathcal{U} is called *principal* if it consists of all supersets of a fixed subset of I , and it is called an *ultrafilter* if it is maximal in the set of all filters over I ordered by inclusion.

Let \mathcal{U} be a non-principal ultrafilter over \mathbb{N} in the remainder. Then \mathcal{U} enjoys the following two properties (see [1] for the proofs):

- (P1) If $\mathbb{N} = P_1 \cup \dots \cup P_k$ is a finite disjoint union then there exists an index i such that $P_i \in \mathcal{U}$ and $P_j \notin \mathcal{U}$ for all $j \neq i$.

(P2) All cofinite subsets of \mathbb{N} belong to \mathcal{U} .

Given a family $\{H_n\}_{n \in \mathbb{N}}$ of groups we can form the *ultraproduct* of this family by the non-principal ultrafilter \mathcal{U} : this is the quotient of the unrestricted direct product $\prod_{n \in \mathbb{N}} H_n$ by the subgroup D consisting of all tuples whose support does not lie in \mathcal{U} . If all groups H_n are equal to the same group H , we speak of the *ultrapower* of H by \mathcal{U} . The following property is an easy consequence of (P1) and will play a fundamental role in our discussion:

(P3) If H is a finite group then the ultrapower of H by \mathcal{U} is isomorphic to H .

Now, in order to complete the construction of the group G that will fulfil the conditions of Theorem B, we consider the subgroup

$$D = \{(k_n) \in L \mid \text{supp}(k_n) \notin \mathcal{U}\}$$

of L and define $G = K/D$.

Theorem 3.3. *Let G be defined as above. Then:*

- (i) G is nilpotent and $Z(G) = L/D \cong Z(E)$ is a cyclic p -group. Thus G is monolithic and cannot be given the structure of a pro- p group.
- (ii) $P = G/Z(G)$ is a pro- p group.

Proof. (i) If E has nilpotency class c , it is clear from our construction that G is nilpotent of class at most c . Let us now see that $Z(G) = L/D$. The inclusion \supseteq is clear. In order to see the reverse inclusion, let $(k_n) \in K$ be such that $[(k_n), K] \leq D$ and let us see that $(k_n) \in L$, i.e. that $k_n \in L_n$ for all n . By way of contradiction, suppose that $k_s \notin L_s$ for some s and assume that s is minimal with this property. Write $k_s = aF_s$ and $a = (a^{(1)}, \dots, a^{(s)})$. Then $a^{(i)} \notin Z(E)$ for some index i . We claim that $i = s$. Otherwise write $k_{s-1} = bF_{s-1}$ with $b = (b^{(1)}, \dots, b^{(s-1)})$. Since $\psi_s(k_s L_s) = k_{s-1} L_{s-1}$, we have $\varphi_s(a) \equiv b \pmod{Z_{s-1}}$ and consequently $b^{(i)} \equiv a^{(i)} \pmod{Z(E)}$. It follows that $b^{(i)} \notin Z(E)$ and $k_{s-1} \notin L_{s-1}$, which is a contradiction with the minimality of s . Thus $i = s$ and $a^{(s)} \notin Z(E)$. Then there exists an element $y^{(s)} \in E$ such that $[a^{(s)}, y^{(s)}] \neq 1$. Now we define

$$y_n = \begin{cases} (1, \dots, 1), & \text{if } n < s, \\ (1, \dots, 1, y^{(s)}, 1, \dots, 1), & \text{if } n \geq s, \end{cases}$$

and $k'_n = y_n F_n$ for all $n \in \mathbb{N}$. Then the tuple (k'_n) lies in K .

Let us see that the support of $[(k_n), (k'_n)]$ is $[s, +\infty) \cap \mathbb{N}$. Once we prove this, it follows from (P2) that this commutator is not in D , contrary to our assumption that $[(k_n), K] \leq D$. Suppose then $n \geq s$ and let us check that $[k_n, k'_n] \neq 1$. For this purpose, write $k_n = xF_n$ and $x = (x^{(1)}, \dots, x^{(n)})$. Arguing as above, we have $x^{(s)} \equiv a^{(s)} \pmod{Z(E)}$. Hence $[x^{(s)}, y^{(s)}] = [a^{(s)}, y^{(s)}] \neq 1$ and

$$[x, y_n] = (1, \dots, 1, [x^{(s)}, y^{(s)}], 1, \dots, 1) \notin F_n.$$

It follows that $[k_n, k'_n] \neq 1$, as desired.

Finally, note that $Z(G) = L/D$ is nothing but the ultrapower of $Z(E)$ by the non-principal ultrafilter \mathcal{U} . Now (P3) yields that $Z(G) \cong Z(E)$ is cyclic.

(ii) It follows from (i) that $G/Z(G) \cong K/L$. Now the result follows from the isomorphism $K/L \cong \varprojlim K_n/L_n$, which is an immediate consequence of the First

Isomorphism Theorem if we consider the natural projection of K onto $\varprojlim K_n/L_n$. \square

4. PROOF OF THEOREM B: EXAMPLES WITH P FINITELY GENERATED

In this section we combine the ideas from Section 3 with the wreath product construction in order to obtain the groups claimed in Theorem B having a (topologically) finitely generated quotient which is a pro- p group. First we are going to combine central and wreath products as follows. Let H be a group and let C be a finite group of order n . Consider the regular wreath product $W = H \wr C$, let $B = H^n$ be the corresponding base group and let $Z = Z(H)^n$. We define the following subgroup of Z :

$$F = \{(x_1, \dots, x_n) \in Z \mid x_1 \dots x_n = 1\}.$$

Then $F \trianglelefteq W$ and we call the quotient W/F the *wreath-central product* of H and C . Our next theorem states that this construction behaves well when H is a monolithic finite p -group. We need a lemma for this.

Lemma 4.1. *Let H be a non-abelian group and let C be a non-trivial finite group. Let $W = H \wr C$ be the regular wreath product of H and C , and let B and Z be as above. Then $[B, w] \not\leq Z$ for all $w \in W \setminus B$.*

Proof. Write $w = dx$ with $1 \neq d \in C$ and $x = (x_c)_{c \in C} \in B$. Choose an element $h \in H \setminus Z(H)$ and let $b \in B$ be the tuple having the element h at the position corresponding to the identity of C and 1 elsewhere. Then the value of the commutator $[b, w] = b^{-1}b^w = b^{-1}(b^d)^x$ at the position corresponding to d is $h^{x^d} \notin Z(H)$. Thus $[b, w] \notin Z$. \square

Theorem 4.2. *Let H be a non-abelian finite p -group which is monolithic and let J be the wreath-central product of H with another finite p -group C . Then $Z(J) \cong Z(H)$ and hence J is again monolithic.*

Proof. Keep the notation before the lemma and let $K = B/F$, which is the natural central product of $|C|$ copies of H . Arguing as in the proof of Theorem 3.1, we get $Z(K) = Z/F \cong Z(H)$. Thus it suffices to show that $Z(J) = Z(K)$. Notice that the inclusion \subseteq follows from the previous lemma. On the other hand, since $[Z, W] \leq F$ by the definition of the action of the wreath product, we also have $Z(K) \leq Z(J)$ and we are done. \square

Let now E be a finite non-abelian monolithic p -group. According to the last theorem, the wreath-central product J_n of E with a cyclic group $\langle \alpha_n \rangle$ of order p^n is again monolithic. Let us write E_n for the direct product of p^n copies of E (not n copies as in the previous section), $Z_n = Z(E_n)$ and

$$F_n = \{(x^{(1)}, \dots, x^{(p^n)}) \in Z_n \mid x^{(1)} \dots x^{(p^n)} = 1\}.$$

We know from Theorem 3.1 that the quotient $K_n = E_n/F_n$ is monolithic, with centre $L_n = Z_n/F_n$. Note that $J_n = K_n \rtimes \langle \alpha_n \rangle$ is a semidirect product with kernel K_n and complement $\langle \alpha_n \rangle$, and that $J_n/L_n = K_n/L_n \rtimes \langle \alpha_n \rangle$.

Now we further assume that E has class 2 and that $|Z(E)| = q$ is a previously chosen power of p . (Recall the beginning of Section 3.) We consider the map

$\varphi_n : E_n \longrightarrow E_{n-1}$ defined by the following rule: if $x = (x^{(i)}) \in E_n$ then the i -th component of the image $\varphi_n(x)$ is given by the product

$$\prod_{r \equiv i \pmod{p^{n-1}}} x^{(r)},$$

where the factors appear in increasing order of their indices. Note that φ_n is not a group homomorphism, for this we would need E to be abelian. However, since $x \equiv y \pmod{Z_n}$ implies that $\varphi_n(x) \equiv \varphi_n(y) \pmod{Z_{n-1}}$, φ_n induces a map $\psi_n : E_n/Z_n \longrightarrow E_{n-1}/Z_{n-1}$. Since E_n/Z_n is abelian (this is why we need E to be of class 2), ψ_n is a homomorphism. As in the previous section, due to the existence of a natural isomorphism $K_n/L_n \cong E_n/Z_n$, we may suppose that ψ_n is defined from K_n/L_n onto K_{n-1}/L_{n-1} .

Theorem 4.3. *By means of the rule $\psi_n(\alpha_n) = \alpha_{n-1}$, the homomorphism ψ_n extends to a homomorphism $J_n/L_n \longrightarrow J_{n-1}/L_{n-1}$.*

Proof. Let $x \in E_n$ and let y be the conjugate of x by α_n , so that $y^{(i)} = x^{(i-1)}$, where $i-1$ is taken modulo p^n (between 1 and p^n). If $a = \varphi_n(x)$ and $b = \varphi_n(y)$ then

$$\begin{aligned} b^{(i)} &= \prod_{r \equiv i \pmod{p^{n-1}}} y^{(r)} = \prod_{r \equiv i \pmod{p^{n-1}}} x^{(r-1)} \\ &\equiv \prod_{r \equiv i-1 \pmod{p^{n-1}}} x^{(r)} = a^{(i-1)} \pmod{Z(E)}, \end{aligned}$$

which means that bZ_n is the conjugate of aZ_n by α_{n-1} . This already implies the result. \square

Thus we can consider the projective systems $\{K_n/L_n, \psi_n\}$ and $\{J_n/L_n, \psi_n\}$ over \mathbb{N} . We then define

$$K = \{(k_n) \in \prod_{n \in \mathbb{N}} K_n \mid (k_n L_n) \in \varprojlim K_n/L_n\}$$

and

$$J = \{(j_n) \in \prod_{n \in \mathbb{N}} J_n \mid (j_n L_n) \in \varprojlim J_n/L_n\}.$$

If $L = \prod_{n \in \mathbb{N}} L_n$, we have $L \leq K \leq J$. As in the previous section, we consider a non-principal ultrafilter \mathcal{U} over \mathbb{N} and define the subgroup D of L consisting of all tuples whose support is not in \mathcal{U} . Let $G = J/D$, $H = K/D$ and $Z = L/D$. Note that $Z \cong Z(E)$ is a central cyclic subgroup of G of order q .

Theorem 4.4. *The group H is monolithic with centre Z .*

Proof. As in the proof of Theorem 3.3, we suppose that $[(k_n), K] \leq D$ but that $k_n \notin L_n$ for some n . We claim that the set $S = \{n \in \mathbb{N} \mid k_n \notin L_n\}$ is of the form $[s, +\infty) \cap \mathbb{N}$ for some s . For this purpose, it suffices to see that if $n \in S$ and $m \geq n$ then also $m \in S$. This is an immediate consequence of the definition of K : since $(k_n) \in K$, we have $(\psi_{n+1} \circ \cdots \circ \psi_m)(k_n L_n) = k_n L_n \neq L_n$ and consequently $k_m \notin L_m$.

Write $k_n = x_n F_n$ for all $n \in \mathbb{N}$. Since $k_s \notin L_s$, it follows that the tuple $x_s \in E_s$ has an element $v \notin Z(E)$ at some position $i(s)$. Let us choose $w \in E$ such that $[v, w] \neq 1$. Now we construct recursively a sequence $\{i(n)\}_{n \geq s}$ such that

$i(n) \equiv i(n-1) \pmod{p^{n-1}}$ and the element at position $i(n)$ of x_n does not lie in the centraliser $C_E(w)$. Indeed, assume that $i = i(n-1)$ is already chosen and suppose by way of contradiction that there is no $r \equiv i \pmod{p^{n-1}}$ with the required property. Then $x_n^{(r)} \in C_E(w)$ for all $r \equiv i \pmod{p^{n-1}}$ and it follows from the definition of φ_n that also $x_{n-1}^{(i)} \in C_E(w)$, which is a contradiction. We define

$$y_n = \begin{cases} (1, \dots, 1), & \text{if } n < s, \\ (1, i(n)^{-1}, 1, w, 1, \dots, 1), & \text{if } n \geq s, \end{cases}$$

and $k'_n = y_n F_n$ for all $n \in \mathbb{N}$. Then the tuple (k'_n) lies in K and from this point onwards we can mimic the proof of Theorem 3.3 to get the final contradiction that $[(k_n), (k'_n)] \notin D$. \square

We are now ready to prove Theorem B in the finitely generated case. If P is a finitely generated (topological) group, let us write $d(P)$ to denote the minimum number of (topological) generators of P .

Theorem 4.5. *The group G is monolithic and $P = G/Z$ is a finitely generated pro- p group. More precisely, $d(P) = d(E) + 1$.*

Proof. By considering the natural projection of J onto $\varprojlim J_n/L_n$ it follows that J/L is isomorphic to this inverse limit. Thus $P = G/Z \cong \varprojlim J/L$ is a pro- p group. Also, since $J_n/L_n \cong E/Z(E) \wr C_{p^n}$, we have $d(J_n/L_n) = d(E) + 1$ for all n and consequently also $d(P) = d(E) + 1$. (The last assertion follows basically from Proposition 4.2.1 in [13], check its proof.)

Let us see now that $G/H \cong \mathbb{Z}_p$. Note that ψ_n induces an epimorphism $\bar{\psi}_n : J_n/K_n \rightarrow J_{n-1}/K_{n-1}$, hence we can consider the projective system $\{J_n/K_n, \bar{\psi}_n\}$. Since $J_n/K_n = \langle \bar{\alpha}_n \rangle$ and ψ_n maps α_n to α_{n-1} , we have $\varprojlim J_n/K_n \cong \mathbb{Z}_p$. Since the intersection of the kernels of the natural homomorphisms $\pi_n : J \rightarrow J_n/K_n$ is precisely K , it follows that $G/H \cong J/K \cong \mathbb{Z}_p$, as claimed.

Finally, let us prove that G is monolithic. Since we already know that H is monolithic, it suffices to see that $N \cap H \neq 1$ for every non-trivial normal subgroup N of G . Obviously, we may suppose that N is not contained in H . Since $G/H \cong \mathbb{Z}_p$ is abelian, we deduce that $[G, N] \leq H \cap N$. Assume by way of contradiction that $[G, N] = 1$. If we write $N = M/D$ with $D < M \not\leq K$, we obtain that $[J, M] \leq D$. Let W_n be the wreath product $E \wr \langle \alpha_n \rangle$, i.e. $W_n = E_n \rtimes \langle \alpha_n \rangle$. Note that $J_n = W_n/F_n$. Then to each subgroup of J there corresponds a subgroup in W_n , by first projecting onto J_n and then taking its preimage in W_n . Of course, the subgroup corresponding to J is W_n . Let us call M_n and T_n the subgroups corresponding to M in J_n and W_n , respectively. Since M is not contained in $K = \bigcap_{n \in \mathbb{N}} \ker \pi_n$, we have $M_n \not\leq K_n$ for some n . Consequently, for that value of n , T_n is not contained in the base group E_n of the wreath product W_n . It follows from Lemma 4.1 that $[W_n, T_n] \not\leq Z_n$, while the condition $[J, M] \leq D$ implies that $[W_n, T_n] \leq Z_n$. This contradiction proves the result. \square

In fact, the proof above shows that the subgroup Z is the centre of G , so that the quotient G/Z which yields a pro- p group is in fact the central quotient of G . Just note that from the condition $[G, N] = 1$ we have deduced that $N \leq H$.

Consequently $Z(G) \leq Z(H) = Z$, as proved in Theorem 4.4. The reverse inclusion is obvious.

Finally, observe that the pro- p group $P = G/Z$ in the last theorem is not finitely presented. This is a consequence of the following result [13, Corollary 12.5.10]: if P is a finitely presented soluble pro- p group having a closed normal subgroup Q such that $P/Q \cong \mathbb{Z}_p$, then Q is finitely generated. In our case, P is soluble and, as seen in the proof of Theorem 4.5, for the subgroup $Q = H/Z$ we have $P/Q \cong G/H \cong \mathbb{Z}_p$. Since $Q \cong K/L \cong \varprojlim K_n/L_n$ is not finitely generated, it follows that P is not finitely presented.

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