ON THE CONSTRUCTION OF PERMUTATION COMPLEXES FOR PROFINITE GROUPS

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Abstract. Goerss, Henn, Mahowald and Rezk construct a complex of permutation modules for the Morava stabilizer group $\mathbb{G}_2$ at the prime 3. We describe how this can be done using techniques from homological algebra.

1. Introduction

In [5], Goerss, Henn, Mahowald and Rezk consider the special extended Morava stabilizer group $\mathbb{G}_2^1 = \mathbb{S}_2^1 \rtimes \mathrm{Gal}$ at the prime 3 and construct an exact sequence of compact modules

$$0 \to \mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{SD_{16}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3(\chi) \to \mathrm{Ind}_{SD_{16}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3(\chi) \to \mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \hat{\mathbb{Z}}_3 \to 0,$$

where G_{24} is a subgroup of order 24 etc. and $\hat{\mathbb{Z}}_3(\chi)$ is a copy of $\hat{\mathbb{Z}}_3$ on which SD_{16} acts via a character $\chi : SD_{16} \to \{\pm 1\}$. They then use this to construct a certain tower of spectra.

The aim of this note is to show how methods from the homological algebra and representation theory of these groups can help in the algebraic part of this construction.

2. Background

Let G be a profinite group and let R be a complete noetherian local ring with finite residue class field k of characteristic p. For example, R could be the p-adic integers.

We work in the category of compact $R[[G]]$-modules, $\mathcal{C}_R(G)$, (see [10] for definitions, properties and more references).

The next result is basic, but does not seem to have appeared in the literature.

Proposition 2.1. *If G is a virtual pro-p-group then the Krull-Schmidt property holds for (topologically) finitely generated modules in $\mathcal{C}_R(G)$, i.e. every such module can be expressed as a finite sum of indecomposable modules and this decomposition is essentially unique in the sense that the multiplicity of each isomorphism type is the same in any such decomposition.*

Proof. The proof is just an adaptation of the one for finite groups (see [1]), but a little care is needed.

Let $H \trianglelefteq_o G$ be an open normal pro-p subgroup. If M is a finitely generated $R[[G]]$-module then $k \otimes_{R[[H]]} M$ is finite dimensional and we can decompose M as a finite sum of indecomposable modules using induction on $\dim_k k \otimes_{R[[H]]} M$.

For this to work we need to know that our induction starts, that is that if $M \neq 0$ then $k \otimes_{R[[H]]} M \neq 0$. Let M' be a finite quotient of M as an H-module; there is a surjection

$k \otimes_{R[[H]]} M \to k \otimes_{R[[H]]} M'$. The action of H on M' factors through that of a finite p-group P, and in this case it is well known that $k \otimes_{R[P]} M' \neq 0$.

All we need to do now is to show that the endomorphism ring of a finitely generated indecomposable module is local, because then the uniqueness of decomposition follows formally (see e.g. [1] 1.4.3).

The proof is just a variant of the one for finite groups (see [1] 1.9). Let J be the Jacobson radical of $R[[G]]$. For any open normal subgroup N of G let I_N denote the augmentation ideal of $R[[N]]$. Given an endomorphism f of $M \in \mathcal{C}_R(G)$ we set $\mathrm{Im}(f^\infty) = \cap_{n=1}^\infty \mathrm{Im}(f^n)$ and $\mathrm{Ker}(f^\infty) = \{x \in M | \forall N \trianglelefteq_o G \, \forall n \geq 0 \, \exists m \geq 0 \, \text{such that} \, f^m(x) \in J^n M + I_N M\}$.

For each open normal subgroup $N \trianglelefteq_o G$ define $M_N = R \otimes_{R[[N]]} M \cong M/I_N M \in \mathcal{C}_R(G/N)$. Then $M \cong \varprojlim M_N$. Since M is finitely generated, M_N is too. Now f induces an endomorphism f_N of M_N. Define $\mathrm{Im}(f_N^\infty) = \cap_{n=1}^\infty \mathrm{Im}(f_N^n)$ and $\mathrm{Ker}(f_N^\infty) = \{x \in M_N | \forall n \geq 0 \, \exists m \geq 0 \, \text{such that} \, f^m(x) \in J^n M_N\}$. From the finite group case of Fitting's Lemma we know that $M_N = \mathrm{Im}(f_N^\infty) \oplus \mathrm{Ker}(f_N^\infty)$.

But $\mathrm{Im}(f^\infty) \cong \varprojlim \mathrm{Im}(f_N^\infty)$ and $\mathrm{Ker}(f^\infty) \cong \varprojlim \mathrm{Ker}(f_N^\infty)$. Hence $M = \mathrm{Im}(f^\infty) \oplus \mathrm{Ker}(f^\infty)$.

Now suppose that M is indecomposable and let I be a maximal left ideal in $\mathrm{End}_{\mathcal{C}_R(G)}(M)$ and let a be an endomorphism not in I. Then $1 = ba + f$ for some $b \in \mathrm{End}_{\mathcal{C}_R(G)}(M)$ and $f \in I$. But f is not an isomorphism, so $M = \mathrm{Ker}(f^\infty)$ and $\mathrm{Im}(f^\infty) = 0$.

Now $(1 + f + \cdots + f^{n-1})ba = 1 - f^n$. Let $N \trianglelefteq_o G$ be some arbitrary open normal pro-p subgroup. Since M is finitely generated, for sufficiently large n we have $f^n(M) \subseteq JM + I_N M \subseteq JM$. Thus $1 - f^n$ is onto, by the profinite version of Nakayama's Lemma ([4] 1.4). Also if $(1 - f^n)(x) = 0$ then $x \in \mathrm{Im}(f^\infty) = 0$, so $1 - f^n$ is injective. Thus $1 - f^n$ is an isomorphism and a has a left inverse, c say.

But c_N must also be a right inverse to a_N on each M_N, so c is also a right inverse and a is an isomorphism, as required. □

Projective covers exist in $\mathcal{C}_R(G)$ ([9]), thus so do minimal projective resolutions.

If S is a simple module, let P_S denote the projective cover of S. The P_S are precisely the indecomposable projective modules, and any other projective is a product of them.

If there is an open normal pro-p subgroup $H \trianglelefteq_o G$, then any simple module for $R[[G]]$ is the inflation of one for $k[G/H]$ so, in particular, there are only finitely many simple modules up to isomorphism.

The next result is well known for finite groups.

Proposition 2.2. *Suppose that $M \in \mathcal{C}_R(G)$ is projective over R and let $\cdots \to P_r \to \cdots \to P_1 \to P_0 \to M$ be the minimal projective resolution of M. If S is a simple module then the multiplicity of P_S in P_r is $\dim_{\mathrm{End}(S)} \mathrm{Ext}^r_{R[[G]]}(M, S) = \dim_{\mathrm{End}(S)} H^r(G, (k \otimes_R M)^* \otimes_R S)$.*

Here S^* denotes the dual over k, or rather the contragredient.

(If k is a splitting field for G/H, where $H < G$ is open, normal and pro-p, then $\mathrm{End}(S) \cong k$.)

Proof. (cf. [11]) The multiplicity of P_S in P_r is $\dim_{\mathrm{End}(S)} \mathrm{Hom}_{R[[G]]}(P_r, S)$. The fact that the projective resolution is minimal implies that the differentials in the complex $\mathrm{Hom}_{R[[G]]}(P_\bullet, S)$ are zero.

Combining these facts, we find that the multiplicity that we are calculating is equal to $\dim_{\mathrm{End}(S)} \mathrm{Ext}^r_{R[[G]]}(M,S)$. But $\mathrm{Ext}^r_{R[[G]]}(M,S) \cong \mathrm{Ext}^r_{R[[G]]}(R, \mathrm{Hom}_R(M,S))$ (see e.g. [1] 3.1.8) and $\mathrm{Hom}_R(M,S) \cong (k \otimes_R M)^* \otimes_R S$. □

From now on we assume that G is of finite virtual cohomological dimension over R. The definition of Tate-Farrell cohomology appears in [8] for discrete coefficients and in [10] for compact ones, as does the next result. (See [3] for its basic properties in the case of an abstract group.)

Proposition 2.3. *For M in $\mathcal{C}_R(G)$ or $\mathcal{D}_R(G)$, the Tate-Farrell cohomology $\hat{H}^*(G,M)$ is isomorphic to the equivariant Tate-Farrell cohomology of the Quillen complex of G with coefficients in M.*

Corollary 2.4. *If G has p-rank 1 (i.e. no subgroups isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$) and only finitely many conjugacy classes of subgroups isomorphic to $\mathbb{Z}/p$ with representatives $C_1, \ldots, C_n$ then $\hat{H}^*(G,M) \cong \oplus_{i=1}^n \hat{H}^*(N_G(C_i), M)$ for any M in $\mathcal{C}_R(G)$ or $\mathcal{D}_R(G)$.*

A similar result for $M = k$ also appears in [7].

For $M, N \in \mathcal{C}_R(G)$ we can also define Tate-Farrell Ext groups $\hat{\mathrm{Ext}}^*_G(M,N)$. This allows us to define the stable category $\mathrm{St}_R(G)$ to have the same objects as $\mathcal{C}_R(G)$ but morphism groups $\hat{\mathrm{Ext}}^0_G(M,N)$. We write $\simeq$ for isomorphism in the stable category.

There is another description. We define the Heller translate Ω on $\mathcal{C}_R(G)$ by the short exact sequence $\Omega M \to P_M \to M$, where P_M denotes the projective cover of M. We also define $\underline{\mathrm{Hom}}_G(M,N)$ to be the quotient of $\mathrm{Hom}_{\mathcal{C}_R(G)}(M,N)$ by the submodule of all homomorphisms that factor through a projective module. Then $\hat{\mathrm{Ext}}^r_G(M,N) \cong \varinjlim_i \underline{\mathrm{Hom}}_G(\Omega^{r+i}M, \Omega^i N)$. In fact we only need to take $i \geq \mathrm{vcd}\, G$.

For the basic properties of the stable category see [1] for finite groups and [2] for infinite abstract groups. In particular, it is a triangulated category with the inverse of Ω as translation and the exact triangles coming from short exact sequences in $\mathcal{C}_R(G)$.

The next statement is basic to our approach, although it is just a corollary of Yoneda's Lemma.

Lemma 2.5. *If $f : A \to B$ induces an isomorphism $f^* : \hat{\mathrm{Ext}}^0_G(B,M) \to \hat{\mathrm{Ext}}^0_G(A,M)$ for all $M \in \mathcal{C}_R(G)$ then f is an isomorphism in the stable category.*

Definition 2.6. A module $M \in \mathcal{C}_R(G)$ is *cofibrant* if it is projective on restriction to some open subgroup of G.

In fact, if M is cofibrant then it is projective on restriction to any p-torsion free subgroup.

Notice that $\Omega^i M$ is always cofibrant if $i \geq \mathrm{vcd}\, G$. If M and N are cofibrant then $\hat{\mathrm{Ext}}^0_G(M,N) \cong \underline{\mathrm{Hom}}_G(M,N)$.

The definition is taken from [2], as is the next lemma. As the terminology suggests, this is part of a the structure of a closed model category, but we do not need that here.

Lemma 2.7. *If $M \simeq N$ in $\mathrm{St}_R(G)$ and M and N are cofibrant then there exist projective modules P and Q such that $M \oplus P \cong N \oplus Q$ in $\mathcal{C}_R(G)$. If M and N are finitely generated then P and Q can be chosen to be finitely generated.*

Proof. Let $H \trianglelefteq_o G$ be open normal of finite cohomological dimension. The inclusion of the fixed points induces a map $R \to R[G/H]$, which is split over H. This induces a map $M \to R[G/H] \otimes M \cong \mathrm{Ind}_H^G M$, which is also split over H and where $Q = \mathrm{Ind}_H^G M$ is projective, and finitely generated if M is.

Consider the map $M \to Q \oplus N$, where the first component is the map constructed above and the second is a stable isomorphism. This map is split over H, so the cokernel, call it P, is cofibrant, and finitely generated if M and N are.

The long exact sequence for $\hat{\mathrm{Ext}}_G^*(P, -)$ yields that $0 = \hat{\mathrm{Ext}}_G^0(P, P) \cong \underline{\mathrm{Hom}}_G(P, P)$, so P is projective and the short exact sequence splits. □

3. The Calculation

We set $R = \hat{\mathbb{Z}}_3$, $k = \mathbb{F}_3$. The Morava stabilizer group $\mathbb{S}_2$ at the prime 3 can be split as a product $\mathbb{S}_2^1 \times \hat{\mathbb{Z}}_3$, where $\mathbb{S}_2^1$ is the kernel of the reduced norm. There is a natural action of the Galois group $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$, and we will consider the special extended Morava stabilizer group $\mathbb{G}_2^1 = \mathbb{S}_2^1 \rtimes \mathrm{Gal}$.

Let S_2^1 be the Sylow 3-subgroup of $\mathbb{S}_2^1$. It is normal in $\mathbb{G}_2^1$ and $\mathbb{G}_2^1 = S_2^1 \rtimes SD_{16}$, where SD_{16} is a subgroup isomorphic to the special dihedral group of order 16. In fact, if ϕ denotes the generator of Gal (of order 2) there is an element $\omega \in S_2^1$ of order 8 such that SD_{16} is generated by ϕ and ω. There is just one finite 3-subgroup, up to conjugation. It is cyclic of order 3 and we denote it by C_3. It is contained in a subgroup G_{24} of order 24, but there is no subgroup of order 48. We can, however, choose conjugacy class representatives so that $SD_{16} \cap G_{24} = Q_8$, a quaternion group of order 8 generated by $\omega\phi$, which commutes with C_3, and ω^2, which does not. We refer to [5] for the details.

As a consequence, the simple modules in $\mathcal{C}_{\hat{\mathbb{Z}}_3}(G)$ correspond to the simple modules for SD_{16} over $\mathbb{F}_3$. In particular there is a character χ corresponding to the map $SD_{16} \to SD_{16}/Q_8 \cong \{\pm 1\}$, so $\chi(\phi) = \chi(\omega) = -1$. Define a module N_1 by

$$0 \to N_1 \to \mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \hat{\mathbb{Z}}_3 \to 0,$$

where the right hand arrow is the natural augmentation.

Let S be a simple module and apply $\mathrm{Ext}^*_{\mathbb{G}_2^1}(-, S)$. We obtain the long exact sequence

$$\cdots \to \mathrm{Ext}^*_{\mathbb{G}_2^1}(\hat{\mathbb{Z}}_3, S) \to \mathrm{Ext}^*_{\mathbb{G}_2^1}(\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3, S) \to \mathrm{Ext}^*_{\mathbb{G}_2^1}(N_1, S) \to \cdots .$$

The arrow on the left is just $H^*(\mathbb{G}_2^1, S) \xrightarrow{\mathrm{res}} H^*(G_{24}, S)$, which is equivalent to $H^*(S_2^1, S)^{SD_{16}} \xrightarrow{\mathrm{res}} H^*(C_3, S)^{C_8}$ or $(H^*(S_2^1) \otimes S)^{SD_{16}} \xrightarrow{\mathrm{res}} (H^*(C_3) \otimes S)^{C_8}$ or, more naturally, $(H^*(S_2^1) \otimes S)^{SD_{16}} \xrightarrow{\mathrm{res}} ((H^*(C_3) \oplus H^*(C_3')) \otimes S)^{SD_{16}}$, where C_3' is the conjugate of C_3 by ω. (Where no coefficients for the cohomology are indicated they are just $\mathbb{F}_3$.)

Now, for any finite $\mathbb{F}_3 SD_{16}$-module A, $\dim_{\mathrm{End}(S)}(A \otimes S)^{SD_{16}} \cong \dim_{\mathrm{End}(S^*)} \mathrm{Hom}_{SD_{16}}(S^*, A)$ is just the multiplicity of the dual S^* as a summand of A (A is completely reducible). So we are just decomposing the $\mathbb{F}_3 SD_{16}$-modules and identifying the map $\rho : H^*(S_2^1) \xrightarrow{\mathrm{res}} H^*(C_3) \oplus H^*(C_3')$. But this factors as

$$H^*(S_2^1) \xrightarrow{\mathrm{res}} H^*(C_{S_2^1}(C_3)) \oplus H^*(C_{S_2^1}(C_3')) \xrightarrow{\mathrm{res}} H^*(C_3) \oplus H^*(C_3').$$

A standard calculation ([5, 7, 9]) shows that $C_{S_2^1}(C_3) \cong \hat{\mathbb{Z}}_3 \times C_3$. Its cohomology is just $H^*(C_{S_2^1}(C_3)) \cong H^*(\mathbb{Z}_3) \otimes H^*(C_3) \cong E(a_1) \otimes (\mathbb{F}_3[y_1] \otimes E(x_1)) \cong \mathbb{F}_3[y_1] \otimes E(x_1, a_1)$, where E denotes an exterior algebra, a_1, x_1 are in degree 1 and y_1 is in degree 2. The restriction to C_3 just kills a_1. For $C_{S_2^1}(C_3')$ the result is similar, but we use the subscript 2 for the generators, which we take to be the images of those in the first case under conjugation by ω.

Henn [7] shows that the first of the maps above is injective. Its image is generated as an algebra by $x_1, x_2, y_1, y_2, (x_1a_1 - x_2a_2), y_1a_1, y_2a_2$. The action of SD_{16} can now be calculated and is given in [5]:

$$\omega_*(x_i) = -(-1)^i x_{i+1}, \quad \omega_*(y_i) = -(-1)^i y_{i+1}, \quad \omega_*(a_i) = -(-1)^i a_{i+1},$$

$$\phi_*(x_i) = -x_{i+1}, \quad \phi_*(y_i) = -y_{i+1}, \quad \phi_*(a_i) = -a_{i+1},$$

(where the subscripts are taken modulo 2).

The map ρ is also explicitly calculated in [6].

From this we can read off that ρ is surjective, except in degree 0, where the cokernel is $\mathbb{F}_3(\chi)$ as an SD_{16}-module. It is also injective in degrees 0 and 1. In degree 2 the kernel is generated by $x_1a_1 - x_2a_2$, which gives a copy of $\mathbb{F}_3(\chi)$ again. In degree 3 the kernel is generated by y_1a_1 and y_2a_2, so consists of two simples: one trivial generated by $y_1a_1 + y_2a_2$ and a copy of $\mathbb{F}_3(\chi)$ generated by $y_1a_1 - y_2a_2$.

Thus the minimal projective resolution of N_1 starts

$$\cdots \to P_{\mathbb{F}_3} \oplus P_{\mathbb{F}_3(\chi)} \to P_{\mathbb{F}_3(\chi)} \to P_{\mathbb{F}_3(\chi)} \to N_1 \to 0.$$

Now $P_{\mathbb{F}_3(\chi)} \cong \operatorname{Ind}_{SD_{16}}^G \hat{\mathbb{Z}}_3(\chi)$, because the latter is projective and, for any simple S, $\operatorname{Hom}_G(\operatorname{Ind}_{SD_{16}}^G \hat{\mathbb{Z}}_3(\chi), S) \cong \operatorname{Hom}_{SD_{16}}(\hat{\mathbb{Z}}_3(\chi), S)$, which is non-zero only for $S \cong \mathbb{F}_3(\chi)$ and then it has dimension 1. So if we define $N_3 = \Omega^2 N_1$ we have an exact sequence

$$0 \to N_3 \to \operatorname{Ind}_{SD_{16}}^G \hat{\mathbb{Z}}_3(\chi) \to \operatorname{Ind}_{SD_{16}}^G \hat{\mathbb{Z}}_3(\chi) \to N_1 \to 0,$$

where N_3 has projective cover $P_{\mathbb{F}_3} \oplus P_{\mathbb{F}_3(\chi)}$.

If we work stably we can obtain $\Omega^2 N_1$ another way. Recall that C_3 is the only cyclic subgroup of order 3 in $\mathbb{G}_2^1$ up to conjugacy. Write $N = N_{\mathbb{G}_2^1}(C_3)$; because Q_8 normalizes C_3 it also normalizes $C_{S_2^1}(C_3)$, and since the centralizer can be of index at most 2 in the normalizer we see that $N \cong C_3 \times \hat{\mathbb{Z}}_3 \rtimes Q_8$.

From 2.4 we see that res : $\hat{H}^*(\mathbb{G}_2^1, M) \to \hat{H}^*(N, M)$ is an isomorphism, or equivalently that the augmentation map $\epsilon : \operatorname{Ind}_N^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \hat{\mathbb{Z}}_3$ induces an isomorphism $\hat{\operatorname{Ext}}^*_{\mathbb{G}_2^1}(\hat{\mathbb{Z}}_3, M) \to \hat{\operatorname{Ext}}^*_{\mathbb{G}_2^1}(\operatorname{Ind}_N^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3, M)$, for any $M \in \mathcal{C}_{\hat{\mathbb{Z}}_3}(G)$. It follows from 2.5 that ϵ is a stable isomorphism.

So stably our complex starts $\operatorname{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \operatorname{Ind}_N^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3$, which is $\operatorname{Ind}_N^{\mathbb{G}_2^1}$ applied to the natural augmentation map $\operatorname{Ind}_{G_{24}}^N \hat{\mathbb{Z}}_3 \to \hat{\mathbb{Z}}_3$ over N.

But the subgroup $D < G_{24}$ generated by C_3 and $\omega\phi$ is normal in N, so N acts on $\operatorname{Ind}_{G_{24}}^N \hat{\mathbb{Z}}_3$ via its image $N/D \cong \hat{\mathbb{Z}}_3 \rtimes C_2$, the infinite virtually 3-adic dihedral group, so we can resolve to obtain

$$(\dagger) \qquad 0 \to \operatorname{Ind}_{G_{24}}^N \hat{\mathbb{Z}}_3(\theta) \to \operatorname{Ind}_{G_{24}}^N \hat{\mathbb{Z}}_3 \to \hat{\mathbb{Z}}_3 \to 0,$$

where $\theta : Q_8 \to \{\pm 1\}$ is the character with $\theta(\omega\phi) = 1$ and $\theta(\omega^2) = -1$.

This can be seen systematically using cohomology, as before. More explicitly, the non-zero map on the left is determined by $1 \otimes \theta \mapsto (g - g^{-1}) \otimes 1$, where g is a generator of the group $\hat{\mathbb{Z}}_3$ and θ is considered as a basis element of $\hat{\mathbb{Z}}_3(\theta)$. The sequence is exact because on restriction to $\hat{\mathbb{Z}}_3$ it is just a variation on the standard projective resolution for $\hat{\mathbb{Z}}_3$.

Similarly, since G_{24} has a quotient $G_{24}/\langle\omega\phi\rangle \cong D_6$, the dihedral group of order 6, we also have an exact sequence

$$0 \to \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{Q_8}^{G_{24}} \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{Q_8}^{G_{24}} \hat{\mathbb{Z}}_3(\theta) \to \hat{\mathbb{Z}}_3(\theta) \to 0,$$

with middle map determined by $1 \otimes 1 \mapsto (c - c^{-1}) \otimes \theta$ where c is a generator of C_3.

Inducing this to N gives

$$(\ddagger) \qquad 0 \to \mathrm{Ind}_{G_{24}}^{N} \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{Q_8}^{N} \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{Q_8}^{N} \hat{\mathbb{Z}}_3(\theta) \to \mathrm{Ind}_{G_{24}}^{N} \hat{\mathbb{Z}}_3(\theta) \to 0.$$

Now splice $\dagger$ and $\ddagger$ together at $\mathrm{Ind}_{G_{24}}^{N} \hat{\mathbb{Z}}_3(\theta)$ and induce up to $\mathbb{G}_2^1$ to obtain

$$0 \to \mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{Q_8}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{Q_8}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3(\theta) \to \mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to \mathrm{Ind}_{N}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \to 0.$$

The second and third non-zero terms are projective, so stably $\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \simeq N_3$. But $\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3$ is cofibrant by construction and, on restriction to an open torsion free subgroup, N_3 is a third syzygy hence also cofibrant, so, by 2.7, there are finitely generated projective modules P and Q such that $\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \oplus P \cong N_3 \oplus Q$.

Let S be a simple $\hat{\mathbb{Z}}_3[[\mathbb{G}_2^1]]$-module (recall that these correspond to simple SD_{16}-modules). Then $\mathrm{Hom}_{\mathbb{G}_2^1}(\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3, S) \cong \mathrm{Hom}_{G_{24}}(\hat{\mathbb{Z}}_3, S)$. For this to be non-zero we need $\mathrm{Res}_{G_{24}}^{\mathbb{G}_2^1} S \cong \mathbb{F}_3$, so S must be either $\mathbb{F}_3$ or $\mathbb{F}_3(\chi)$; in both cases the dimension of the Hom group is 1.

It follows that the projective cover of $\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3$ is $P_{\mathbb{F}_3} \oplus P_{\mathbb{F}_3(\chi)}$. Now, taking projective covers in $\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \oplus P \cong N_3 \oplus Q$, we obtain $P_{\mathbb{F}_3} \oplus P_{\mathbb{F}_3(\chi)} \oplus P \cong P_{\mathbb{F}_3} \oplus P_{\mathbb{F}_3(\chi)} \oplus Q$, so $P \cong Q$ and thus $\mathrm{Ind}_{G_{24}}^{\mathbb{G}_2^1} \hat{\mathbb{Z}}_3 \cong N_3$, by 2.1.

Remark. This construction generalizes to $\mathbb{G}_{p-1}^1$ for larger primes p. It is simpler to discuss if we restrict to the Sylow p subgroup. We now have $N = C_p \times \hat{\mathbb{Z}}_p^{p-2}$. Since $\hat{\mathbb{Z}}_p^{p-2}$ has cohomological dimension $p-2$, we could take its projective resolution to the penultimate term and inflate to N. We then splice on a part induced from a partial projective resolution of $\hat{\mathbb{Z}}_3$ over C_p that is long enough to make the last term cofibrant. It is not clear whether this has any significance in the homotopy theory.

Remark. The Tate-Farrell cohomology of $\mathbb{G}_{p-1}^1$ is easy to compute (see [9]). It is the low-dimensional cohomology that is difficult to calculate, but that is precisely what is needed to identify the projective modules in the complex. If we are satisfied with a complex with unknown projectives then the construction is much easier and only depends on the structure of N.

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