A RELATIVE VERSION OF WEBB'S THEOREM ON THE EQUIVARIANT CHAIN COMPLEX OF A SUBGROUP COMPLEX

PETER SYMONDS

Abstract. Draft, 17 Jan 2006. We prove a relative version of the theorem of Webb that the augmented chain complex of the p-subgroup complex of a finite group, considered as a complex of modules for the group, is homotopy equivalent to a complex of projectives. This allows us to take into account the group of automorphisms of the group.

Let p be a prime and R a complete p-local ring.

Theorem. *Let G be a finite group, $H \lhd G$, and let Δ be a CW-complex on which G acts admissibly. Suppose that the fixed point set Δ^P is R-acyclic for each p-subgroup $P \leq G$ that intersects H non-trivially. Let $\tilde{C}_\bullet(\Delta)$ denote the augmented CW-chain complex of Δ over R, considered as a complex of RG-modules.*

Then $\tilde{C}_\bullet(\Delta) \cong P_\bullet \oplus E_\bullet$, where $P_\bullet$ is a complex of trivial source RG-modules that are projective relative to subgroups that have trivial intersection with H, and $E_\bullet$ is split exact.

The case where $H = G$ is a celebrated theorem of Peter Webb [3].

The following corollary was conjectured to us by Jesper Grodal. It is basically the same as Webb's Theorem, but it incorporates the action of the automorphism group of the group. This can be useful in induction arguments.

Corollary. *Let Γ be a finite group and let Δ be the Brown complex of Γ (i.e. the geometric realization of the poset of chains of non-trivial p-subgroups). Thus $\mathrm{Aut}(\Gamma)$ acts on Δ and also on $\tilde{C}_\bullet(\Delta)$.*

Then $\tilde{C}_\bullet(\Delta) \cong P_\bullet \oplus E_\bullet$ as a complex of $R\,\mathrm{Aut}(\Gamma)$-modules, where $P_\bullet$ is a complex of $R\,\mathrm{Aut}(\Gamma)$-modules that are projective on restriction to Γ (via the map $\Gamma \to \mathrm{Inn}(\Gamma) \leq \mathrm{Aut}(\Gamma)$) and $E_\bullet$ is split exact.

Proof. We wish to apply the Theorem with $G = \mathrm{Aut}(\Gamma)$ and $H = \mathrm{Inn}(\Gamma)$.

Note that the kernel of the natural map $\Gamma \to H$ is the centre, $Z(\Gamma)$, of Γ. The condition that Δ^P be R-acyclic for each p-subgroup $P \leq G$ that intersects H non-trivially is satisfied because Δ^P is contractible via the inclusions $Q \leq Q(\widetilde{P \cap H}) \geq \widetilde{P \cap H}$, where Q is a non-trivial p-subgroup of Γ and $\widetilde{P \cap H}$ is the unique Sylow p-subgroup of the inverse image of $P \cap H$ in Γ, (see e.g. [1] Ch. 6).

We deduce that $\tilde{C}_\bullet(\Delta) \cong P_\bullet \oplus E_\bullet$, where $P_\bullet$ is a complex of trivial source $R\,\mathrm{Aut}(\Gamma)$-modules that are projective relative to subgroups that trivial intersection with $\mathrm{Inn}(\Gamma)$ and $E_\bullet$ is split exact. If the order of $Z(\Gamma)$ is coprime to p then the modules in $P_\bullet$ are also projective over Γ. Otherwise, if $Z_p \leq Z(\Gamma)$ denotes the Sylow p-subgroup then Δ is equivariantly conically contractible, by $Q \leq QZ_p \geq Z_p$. Thus $\tilde{C}_\bullet(\Delta)$ is homotopy equivalent to 0, i.e. it is itself split, so we can take $E_\bullet = \tilde{C}_\bullet(\Delta)$ and $P_\bullet = 0$. □

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Our proof of the theorem follows closely that of Webb's Theorem that we gave in [2]. In fact, it can be viewed as extracting from that paper a quick proof of Webb's theorem even for the reader who is not interested in the relative version.

We use the machinery of coefficient systems that was developed in [2], but we sketch the relevant parts.

Given a finite group G, let $\mathcal{W}$ be a class of subgroups that is closed under conjugation. We let $\mathcal{S}$ denote the class of all subgroups. A coefficient system over R is just an additive contravariant functor from the category of G-sets with stabilizers in $\mathcal{W}$ to R-modules. The morphisms are the natural transformations of functors and we denote the category by $\mathrm{CS}_{\mathcal{W}}(G)$.

Sometimes it is easier to think of this in a slightly different, but equivalent, way. A coefficient system L gives an R-module $L(H)$ for each subgroup $H \in \mathcal{W}$ together with restriction maps $L(H) \to L(K)$ for each $K \leq H$, $K, H \in \mathcal{W}$. There are also conjugation maps $c_g : L(H^g) \to L(H)$ for $H \in \mathcal{W}$ and all these maps satisfy certain obvious relations. In particular, the conjugation maps make $L(H)$ into an $RN_G(H)$-module and H is required to act trivially, so $L(H)$ is actually an $RN_G(H)/H$-module.

For $\mathcal{V} \subseteq \mathcal{W}$ there is a forgetful map $\mathrm{Res}^{\mathcal{W}}_{\mathcal{V}} : \mathrm{CS}_{\mathcal{W}}(G) \to \mathrm{CS}_{\mathcal{V}}(G)$. We denote its left adjoint by $\varinjlim^{\mathcal{W}}_{\mathcal{V}}$.

For any G-set X, let $R[X^?]$ denote the coefficient system with evaluation on H equal to $R[X^H]$, the free R-module on the fixed points under H. It is easily verified that, for any other coefficient system L, $\mathrm{Hom}_{\mathrm{CS}_{\mathcal{W}}(G)}(R[(G/H)^?], L) \cong L(H)$ by $f \mapsto f(eH \in R[(G/H)^H])$ provided that $H \in \mathcal{W}$.

It follows that:

(1) $R[(G/H)^?]$ is projective in $\mathrm{CS}_{\mathcal{W}}(G)$ provided that $H \in \mathcal{W}$,
(2) If $\mathcal{V} \subseteq \mathcal{W}$ and $H \in \mathcal{V}$ then $\varinjlim^{\mathcal{W}}_{\mathcal{V}} \mathrm{Res}^{\mathcal{W}}_{\mathcal{V}} R[(G/H)^?] \cong R[(G/H)^?]$.

If Δ is a CW-complex on which G acts admissibly (i.e. G permutes the cells and the stabilizer of a cell stabilizes all the points in the cell) then we construct a chain complex $C_\bullet(\Delta^?)$ in $\mathrm{CS}_{\mathcal{S}}(G)$: the term in degree n is $R[\Delta_n^?]$, where Δ_n denotes the G-set of the n-cells. The boundary maps are defined in the usual way. There is also an augmented version $\tilde{C}_\bullet(\Delta^?)$ and a relative version $C_\bullet((\Delta, \Delta')^?)$ when $\Delta' \subseteq \Delta$.

The evaluation of $C_\bullet(\Delta)$ at H is just the usual CW-chain complex $C_\bullet(\Delta^H)$, which is naturally a complex of permutation $N_G(H)/H$-modules. In particular, if we evaluate at the trivial subgroup 1 we recover the usual CW-chain complex $C_\bullet(\Delta)$ as a complex of permutation RG-modules.

From now on we are in the context of the Theorem, so H is a normal subgroup of G; let F be a Sylow p-subgroup of G and let $\mathcal{V}$ denote the class of subgroups that have non-trivial intersection with H. Let Δ_S denote the subcomplex of Δ consisting of cells $\delta \in \Delta$ such that $\mathrm{Stab}_F(\delta) \in \mathcal{V}$. There is an action of F on Δ_S. There is a short exact sequence of chain complexes in $\mathrm{CS}_{\mathcal{S}}(F)$

$$0 \to \tilde{C}_\bullet(\Delta_S^?) \to \tilde{C}_\bullet(\Delta^?) \xrightarrow{q} C_\bullet((\Delta, \Delta_S)^?) \to 0. \tag{3}$$

Consider $\mathrm{Res}^{\mathcal{S}}_{\mathcal{V}} \tilde{C}_\bullet(\Delta_S^?)$, a chain complex in $\mathrm{CS}_{\mathcal{V}}(F)$. It is exact, by hypothesis, and consists of projective modules, by (1) and the definition of $\mathcal{V}$. Thus it is split exact.

But $\tilde{C}_\bullet(\Delta^?_S) \cong \varinjlim^{\mathcal{S}}_{\mathcal{V}} \operatorname{Res}^{\mathcal{S}}_{\mathcal{V}} \tilde{C}_\bullet(\Delta^?_S)$, by (2), so is also split exact. It now follows from (3) that the map q is a quasi-isomorphism; since it is a map between complexes of projectives, q must be a homotopy equivalence.

We now need an easy lemma from homological algebra (see [2] 6.5). In this lemma and from now on all complexes are bounded and finitely generated over R in each degree (i.e. each evaluation in each degree is finitely generated).

Lemma. *Let $f : C_\bullet \to P_\bullet$ be a homotopy equivalence of chain complexes, where $P_\bullet$ is a complex of projectives. Then $C_\bullet \oplus S_\bullet \cong P_\bullet \oplus E_\bullet$, where $S_\bullet$ and $E_\bullet$ are split exact complexes and $S_\bullet$ is also a complex of projectives.*

We deduce that in our context $\tilde{C}_\bullet(\Delta^?) \oplus S_\bullet \cong P_\bullet \oplus E_\bullet$. Now evaluate at the trivial group to obtain a complex of RF-modules $\tilde{C}_\bullet(\Delta) \oplus S_\bullet(1) \cong P_\bullet(1) \oplus E_\bullet(1)$, where $P_\bullet(1)$ is a complex of permutation RF-modules with stabilizers that are not in $\mathcal{V}$, so they intersect H trivially.

Now induce from F to G. This preserves the properties of being split, exact or projective, so we obtain an isomorphism of complexes of RG-modules $\operatorname{Ind}^{\mathcal{S}}_{\mathcal{V}} \operatorname{Res}^{\mathcal{S}}_{\mathcal{V}} \tilde{C}_\bullet(\Delta) \oplus S'_\bullet \cong P'_\bullet \oplus E'_\bullet$, where $P'_\bullet$ is a complex of permutation RG-modules which are projective relative to subgroups that intersect H trivially and $E'_\bullet$ is split exact.

But $\tilde{C}_\bullet(\Delta)$ is a summand of $\operatorname{Ind}^G_F \operatorname{Res}^G_F \tilde{C}_\bullet(\Delta)$ by the maps

$$c \mapsto \sum_{g \in G/F} g \otimes g^{-1}c, \quad h \otimes c \mapsto |G:F|^{-1}hc, \quad c \in \tilde{C}_\bullet(\Delta), h \in G.$$

Thus $\tilde{C}(\Delta)$ is a summand of $P'_\bullet \oplus E'_\bullet$. The Krull-Schmidt property applies to complexes of RG-modules (since R is complete and the complexes are bounded and finitely generated in each degree), so there is a summand $P_\bullet$ of $P'_\bullet$ and a summand $E_\bullet$ of $E'_\bullet$ such that $\tilde{C}_\bullet(\Delta) \cong P_\bullet \oplus E_\bullet$, as required to complete the proof.

Remark. We only need the ring R to be complete in the last paragraph of the proof; the preceding statements are true for any p-local ring. In particular, $\tilde{C}_\bullet(\Delta)$ is homotopy equivalent to a bounded complex of trivial source RG-modules that are projective relative to subgroups that have trivial intersection with H.

References

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School of Mathematics, University of Manchester, P.O. Box 88, Manchester M60 1QD, England

E-mail address: Peter.Symonds@manchester.ac.uk