A SPLITTING PRINCIPLE FOR MODULAR GROUP REPRESENTATIONS

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Abstract. We have shown previously how a complex representation of a finite group can be split into a virtual sum of representations induced from 1-dimensional representations in a natural way (sometimes known as explicit Brauer induction). Here we consider the modular case.

1. Introduction

In [11] we showed how a complex representation of a finite group could be split into a virtual sum of representations induced up from 1-dimensional representations of subgroups in a natural way, by considering the projective space of the representation. Here we consider the modular case. Of course, by using a Brauer lift we could reduce the problem to the complex case, but this approach misses the special features of the modular case. In particular we want to be able to control the vertices of the subrepresentations involved.

The aim is to write a representation V of a finite group G as a virtual sum

$$V \cong \sum l_{J,\phi} \operatorname{Ind}_J^G \phi,$$

where J runs through the subgroups and ϕ is a 1-dimensional representation of J. This should be natural with respect to restriction to subgroups, which is defined on the right hand side by applying the double coset formula for $\operatorname{Res}\operatorname{Ind}$ formally to each term and collecting like terms. In order to make this explicit we introduce certain constructions on Mackey functors in section 2. In contrast to the complex case, we need to allow the coefficients to be rational, with denominator a power of the characteristic p.

The complex case was first treated by Snaith [9] and Boltje [2]. A more geometric approach was given by the author [11]. For more details and applications see [10].

Following [11] we see that such a formula can be obtained by considering the projective space associated to V and its canonical line bundle. In fact there are two versions of projective space that we can use in characteristic p. Either the usual version $\mathbb{P}(V)$, or one in which the points correspond to both a line through the origin and a complementary hyperplane, which we denote by $\mathbb{P}_\oplus(V)$.

Both versions yield a natural and additive formula which is correct at the level of Brauer characters. However, if V is indecomposable with vertex P then the $\mathbb{P}_\oplus$ version only sums over J with Sylow p-subgroup conjugate to a subgroup of P.

We would also like to have equality in the Green ring, but since the right hand side is a priori a virtual trivial source module this is clearly only possible if V is a trivial source module. We show that in fact there is always equality in the $\mathbb{P}_\oplus(V)$ version if V is a trivial source module, and that furthermore in this case the coefficients are integers.

These are old notes, but with renewed interest in the subject they seem worth recording. Some of these results have also been obtained by Boltje [3] in a different context, but the geometric situation presented here is of interest in itself and applies quite generally to any equivariant vector bundle. In particular we see the importance of the Brauer map in the modular case.

2. Constructions on Mackey functors

We will always work over some fixed ring R, usually $\mathbb{Z}$ or $\mathbb{Q}$, and all groups will be finite. A Mackey functor M over a group G asigns an R-module $M(H)$ to each subgroup H of G. These are linked by restriction maps, conjugation maps and transfers (or induction). The maps obey certain identities, in particular the Mackey double coset formula for decomposing transfer followed by restriction. The primordial example is the character ring of a group. Another example that will be important for us is $a(G)$, the Green ring of finite dimensional kG-modules under direct sum, with the usual restriction and induction maps. For more details see [6], [7] or [14].

Let C be a weak coefficient system on a finite group G (see [12]), that is, C is a contravariant functor from the category with objects the subgroups of G and morphisms the inclusion and conjugation maps to the category of abelian groups. C is called a coefficient system if H acts trivially on $C(H)$ by conjugation for all $H \leq G$. Define

$$\hat{C}(H) = \bigoplus_{J \leq H} C(J).$$

An element ϕ of $C(J) \subset \hat{C}(H)$ will be denoted by $(J,\phi)^H$. For $g \in G$ define ${}^g(J,\phi)^H = ({}^gJ, {}^g\phi)^{{}^gH}$, where ${}^g\phi({}^gj) = \phi(j)$. The conjugation maps in C combine to yield a map $\hat{c}_g : \hat{C}({}^gH) \to \hat{C}(H)$, where $\hat{c}_g(J,\phi)^H = {}^{g^{-1}}(j,\phi)^H$.

Whenever $K \leq L \leq G$ there are restriction morphisms between the values of $\hat{C}$ given by $r^L_K(J,\phi)^L = (J \cap K, \operatorname{res}^J_{J\cap K}\phi)^K$. These make $\hat{C}$ into a weak coefficient system. There are also inclusion morphisms given by $i^L_K(J,\phi)^K = (J,\phi)^L$, which give $\hat{C}$ the dual structure i.e. make it into a covariant functor on the category of conjugation and inclusion morphisms.

Define:

$$C^+(H) = H^0(H; \hat{C}(H))$$

and

$$C_+(H) = H_0(H; \hat{C}(H)).$$

We now define restriction and transfer maps, denoted by R and I, on these groups. We denote the restriction and transfer in cohomology by res and tr.

On C^+, $R^L_K = (r^L_K)_* \operatorname{res}^L_K$ and $I^L_K = \operatorname{tr}^L_K (i^L_K)_*$.

On C_+,

$$R^L_K(J,\phi)^L = (r^L_K)_* \sum_{g \in K\backslash L/J} {}^g(J,\phi)^L = \sum_{g\in K\backslash L/J} ({}^gJ \cap K, \operatorname{res}^J_{J\cap K^g} {}^g\phi)^K,$$

and $I^L_K(J,\phi)^K = (J,\phi)^L$.

The following is now straightforward to check:

Proposition 2.1. *With the morphisms mentioned above, both C^+ and C_+ are Mackey functors.*

If the coefficient system C was obtained by forgetting the transfers in a Mackey functor (also denoted C), then there is a morphism of Mackey functors $\alpha : C_+ \to C$, given by $\alpha_H(J,\phi)^H = \operatorname{tr}^H_J \phi$.

Proposition 2.2. *If C is a coefficient system then there is a morphism of Mackey functors $N : C_+ \to C^+$, given by*

$$N_H(J,\phi)^H = \sum_{g\in H/J} {}^g(J,\phi)^H.$$

It is an isomorphism if $|G|$ is invertible in R.

Proof. From the definitions we obtain for $K \leq L$:

$$R^L_K N_L(J,\phi)^L = \sum_{l \in L/J} (K \cap {}^l J, {}^\phi)^L,$$

and

$$N_K R^L_K (J,\phi)^L = \sum_{l \in K\backslash L/J} \sum_{k \in K \cap {}^l J} (K \cap {}^{kl} J, {}^{kl}\phi)^K,$$

which are both equal.

Also:

$$N_L I^L_K (J,\phi)^K = \sum_{l \in L/J} {}^l(J,\phi)^H,$$

and

$$I^L_K N_K (J,\phi)^K = \sum_{k \in K/J} \sum_{l \in L/K} {}^{hk}(J,\phi)^L,$$

are again equal.

That N commutes with conjugations is clear.

Note that $C^+(H)$ has a basis consisting of elements of the form

$$S(J,\phi)^H = \sum_{h \in H/N_H(J,\phi)} {}^h(J,\phi)^H.$$

If $|G|$ is invertible then N has an inverse M given by

$$M_H S(J,\phi)^H = |N_H(J,\phi)^H/J|^{-1}(J,\phi)^H.$$

□

Remark. The definition of C_+ goes back to Deligne [5]. The definition of C^+ is based on Thévenaz's twin functor [13]. It appears not to be quite the same as this or the one in [3], because of the definition of the restriction maps, but in fact the two are equivalent (see [12]). The present version corresponds better with the geometry. The functors $C \mapsto C_+$ and $C \mapsto C^+$ have various important properties (loc. cit.).

3. Vector Bundles

Let ξ be a G-vector bundle over a field k with base a G-space X. This means that ξ is a vector bundle and G is a finite group of automorphisms. For any $J \leq H \leq G$ and any finite-dimensional representation ϕ of J over k, let $X_{(J,\phi)^H}$ denote the subspace of points $x \in X$ such that $\mathrm{Stab}_H\, x = J$ and the representation of J on the fibre of ξ is isomorphic to ϕ. We call this a stratum.

We write $(J,\phi) \leq (K,\theta)$ to mean that $J \leq K$ and $\mathrm{Res}^K_J\, \theta \cong \phi$. Let χ be a conjugation invariant function on the $(J,\phi)^H$ satisfying the addition property

$$\chi_{(J,\phi)^H} = \sum_{\substack{(K,\theta),\, K \leq H \\ (J,\phi) \leq (K,\theta)}} \chi_{(K,\theta)^G}.$$

In practice, χ will always be the Euler characteristic in compactly supported cohomology of some $X_{(J,\phi)^H}$.

We define a characteristic class $c_H(\chi) \in a^+(H)$ by

$$c_H(\chi) = \sum_{(J,\phi)/H} \chi_{(J,\phi)^H}(J,\phi)^H.$$

(The summation is over the $(J,\phi)^H$ with $J \leq H$ up to conjugation.)

The additivity property of χ ensures that this construction is natural in H in the sense that for $K \leq H \leq G$ we have $c_K(\chi) = R^H_K c_H(\chi)$.

If we define $\chi_{(J,\phi)^H} = \chi_c(X_{(J,\phi)^H})$ as above,then we write $c_H(\xi)$ instead of $c_H(\chi)$. If we consider ξ as belonging to a suitable category of equivariant bundles, we can form the associated Grothendieck group M under disjoint union. This is even a Mackey functor if we define the transfer of the K vector bundle ξ over X by $\operatorname{tr}^L_K \xi = L \times_K \xi$, a vector bundle over $L \times_K X$. Then c gives a morphism of Mackey functors over G from M to a^+.

Usually it is more convenient to consider instead $L_H(\xi) = N^{-1}c_H(\xi) \in \mathbb{Q} \otimes a_+(H)$. In this case we have:

$$L_H(\xi) = \sum_{(J,\phi)/H} \frac{\chi_c(X_{(J,\phi)})}{|N_H(J,\phi)^H/J|}(J,\phi)^H.$$

Notice that $(\mathbb{Q} \otimes a)_+ = \mathbb{Q} \otimes (a_+)$, so there is no ambiguity.

Proposition 3.1. *a) If X is a G-CW-complex and χ is the usual Euler characteristic in compactly supported cohomology, then $L_H(\xi)$ is integral, i.e. it has image in $a_+(H)$.*

b) If X is a quasi-projective variety over an algebraically closed field of characteristic p and χ is the Euler characteristic in compactly supported étale cohomology, then $L_H(\xi) \in \mathbb{Z}[\frac{1}{p}] \otimes a_+(H)$.

Proof. a) Since $N_H(J,\phi)^H/J$ acts freely on $X_{(J,\phi)^H}$, its order divides the Euler characteristic.

b) In [8] it is shown that if a group K acts freely then the p' part of its order divides the Euler characteristic. Now we proceed as in part (a). ◻

Given a p-subgroup P of G, let $N = N_G(P)$ and $\overline{N} = N/P$. The Brauer map is defined on a kG-module M as $\operatorname{Br}_P M = H^0(P; M)/\sum_{Q \lneqq P} \operatorname{Im} \operatorname{tr}^P_Q$, and this leads to a map $\operatorname{Br}_P : a(G) \to a(\overline{N})$ (see [1]). We can also define it on bundles by first restricting to the part of the base fixed by P, and then using the previous construction on the fibres. A Brauer map can also be defined on both a^+ and a_+ by restricting to N, projecting onto the $(J,\phi)^N$ with $P \leq J$ and then replacing $(J,\phi)^N$ by $(J/P, \operatorname{Br}_P \phi)^{\overline{N}}$.

Recall the basic property that $\operatorname{Br}_P k(G/J) \cong k({}^P(G/J))$ as $k\overline{N}$-modules, where we write ${}^P(G/J)$ for the fixed points of P acting on G/J on the left.

Remark. What we call $\operatorname{Br}_P M$ is by many authors denoted $M[P]$.

Proposition 3.2. *The Brauer map Br_P commutes with both $c : M \to a^+$, with $N : a_+ \to a^+$ and with $\alpha : a_+ \to a$.*

Proof. We will omit to indicate certain obvious restrictions of ϕ.

If we consider an equivariant stratum

$$X_{S(J,\phi)^G} = \coprod_{g \in G/N_G(J,\phi)} X_{{}^g(J,\phi)^G},$$

then

$$\operatorname{Br}_P X_{S(J,\phi)^G} = \coprod_{\substack{g \in G/N_G(J,\phi) \\ P \leq {}^gJ}} X_{((N \cap {}^gJ)/P, \operatorname{Br}_P {}^g\phi)^{\overline{N}}}.$$

On $a^+(G)$ we have

$$\operatorname{Br}_P S(J,\phi)^G = \sum_{\substack{g \in G/N_G(J,\phi) \\ P \leq {}^gJ}} ((N \cap {}^gJ)/P, \operatorname{Br}_P {}^g\phi)^{\overline{N}},$$

and on $a_+(G)$

$$\mathrm{Br}_P(J,\phi)^G = \sum_{g\in N\backslash^P(G/J)} ((N\cap {}^gJ)/P, \mathrm{Br}_P\, {}^g\phi)^{\overline{N}}.$$

Finally, in $a(G)$ (cf. [1])

$$\mathrm{Br}_P \operatorname{Ind}_J^G \phi = \sum_{g\in n\backslash^P(G/J)} \operatorname{Ind}_{(N\cap {}^gJ)/P}^{\overline{N}} \mathrm{Br}_P\, {}^g\phi.$$

We leave to the reader the easy task of calculating the six compositions and verifying the three identities claimed. □

As a consequence we have the following result.

Proposition 3.3. $\mathrm{Br}_P\, L_G(\xi) = L_{N_G(P)/P}(\mathrm{Br}_P\,\xi)$.

4. Projective spaces

From now on we consider only line bundles over an algebraically closed field k of characteristic p. We will mostly only be using the part of $a_+(G)$ generated by 1-dimensional representations, which we denote by $a^1_+(G)$.

Remark. The map L identifies the Grothendieck group of equivariant line bundles over finite G-sets (an extended Burnside ring) with $a^1_+(G)$.

Given a representation V of G, there are two spaces that we shall consider: the usual projective space, $\mathbb{P}(V)$, and also $\mathbb{P}_\oplus(V) = \{(L,M)\,\text{subspaces of}\,V; \dim L = 1, L\oplus M = V\}$. Thus a point consists of a line through the origin in V and a complement to it. Notice that $\mathbb{P}_\oplus(V)$ is a quasi-projective variety since $\mathbb{P}_\oplus(V)\cong \{(u,v)\in\mathbb{P}(V)\times\mathbb{P}(V^*); \langle u,v\rangle\neq 0\}$.

Both $\mathbb{P}(V)$ and $\mathbb{P}_\oplus(V)$ carry canonical line bundles ξ and $\xi_\oplus$ respectively, in which the fibre above the point associated to the line L in V is the line L itself.

Remark. In characteristic zero, $\mathbb{P}_\oplus(V)$ is equivariantly homotopy equivalent to $\mathbb{P}(V)$ and there is little point in considering it.

We write $L(V)$ (resp. $L_\oplus(V)$) for $L(\xi)$ (resp. $L(\xi_\oplus)$). Note that both lie in $\mathbb{Q}\otimes a^1_+$.

As a consequence of the naturality of c_H we have the important property:

Proposition 4.1. *Both L and $L_\oplus$ are natural in H, i.e. $L_H(\operatorname{Res}^G_H V) = R^G_H L_G(V)$ and $L_{\oplus,H}(\operatorname{Res}^G_H V) = R^G_H L_{\oplus,G}(V)$.*

Proposition 4.2. *Both L and $L_\oplus$ are additive in V.*

Proof. Just as in the complex case ther is an action of k^* on $\mathbb{P}(V\oplus W)$ by multiplication on the first factor only: $\lambda(u,v) = (\lambda u, v)$. This preserves the strata, has fixed point set $\mathbb{P}(V)\cup\mathbb{P}(W)$ and acts freely elsewhere. By [8], the order of each element of finite order in k^* must divide the Euler characteristic of a non-fixed stratum, so the latter must be 0.

The same argument applies with $\mathbb{P}$ replaced by $\mathbb{P}_\oplus$. □

Remark. For any equivariant bundle over a space X,

$$\overline{X_{(J,\phi)^G}} = X_{(J,\phi)^J}$$

and also

$$\overline{X_{(J,\phi)^G}} = \coprod_{(J,\phi)\leq(K,\theta)} X_{(K,\theta)^G}.$$

It follows that the coefficients of L and $L_\oplus$ can be calculated algebraically as follows: It suffices to know the Euler characteristics of the closures of the strata, $\chi(\overline{\mathbb{P}(V)_{(J,\phi)}})$, since the numbers $\chi(\mathbb{P}(V)_{(J,\phi)})$ can be obtained from them formally by using a Moebius function. (And similarly with the subscript $\oplus$.)

Proposition 4.3. *a) $\chi(\overline{\mathbb{P}(V)_{(J,\phi)^G}})$ is the multiplicity of ϕ as a submodule of $\operatorname{Res}^G_J V$.*

b) $\chi(\overline{\mathbb{P}_\oplus(V)_{(J,\phi)^G}})$ is the multiplicity of ϕ as a summand of $\operatorname{Res}^G_J V$.

Proof. Recall that for any vector space U, $\chi(\mathbb{P}(U)) = \dim U$.

a) $\overline{\mathbb{P}(V)_{(J,\phi)^G}}$ is the projective space on the subspace of $\operatorname{Res}^G_J V$ generated by submodules isomorphic to ϕ, and this has the dimension stated.

b) $\overline{\mathbb{P}_\oplus(V)_{(J,\phi)}}$ is the coefficient of $(J,\phi)^J$ in $L_{\oplus,J}(\operatorname{Res}^G_J V)$. But if $\operatorname{Res}^G_J V \cong \phi^r \oplus W$ (where ϕ is not a summand of W), then $L_{\oplus,J}(\operatorname{Res}^G_J V) = r(J,\phi)^J + L_{\oplus,J}(W)$, and $L_{\oplus,J}(W)$ has no $(J,\phi)^J$ term. □

Corollary 4.4. *If V is indecomposable with vertex P, then $L_{\oplus,G}(V)$ only involves $(J,\phi)^G$ for which the Sylow p-subgroup of J is conjugate to a subgroup of P.*

Proof. The vertex of ϕ is the Sylow p-subgroup of J. Now $\operatorname{Res}^G_J V$ can not have a summand with vertex not subconjugate to P. □

Theorem 4.5. *As kG-modules, V, $\alpha L(V)$ and $\alpha L_\oplus(V)$ all have the same simple factors.*

Proof. It is sufficient to check the Brauer characters so, by naturality, we can restrict to the cyclic p' groups. But now V is a sum of 1-dimensional representations, and by additivity it is enough to consider the case $\dim V = 1$. But now the result is clear. □

Thus αL and $\alpha L_\oplus$ are our two explicit Brauer induction formulas.

Example. Let C be the cyclic group of order p, and V_r the non-trivial indecomposable kC-module of dimension r, $1 \le r \le p$.

Then

$$L_C(V_r) = (C,k)^C + \frac{r-1}{p}(1,k)^C,$$

and

$$L_{\oplus,C}(V) = \begin{cases} (C,k)^C & \text{if } r = 1, \\ \frac{r}{p}(1,k)^C & \text{if } r \ge 2. \end{cases}$$

Proposition 4.6. *If V is a trivial source module (i.e. a summand of a permutation module) then, as virtual $kN_G(P)/P$-modules, $\operatorname{Br}_P \alpha_G L_{\oplus,G}(V)$ and $\alpha_{N_G(P)/P} L_{\oplus,N_G(P)/P}(\operatorname{Br}_P V)$ have the same simple factors.*

Proof. For a line bundle, Br_P is just restriction to the part of the base space fixed by P.Thus in view of 3.3, we need $\chi(\overline{\mathbb{P}_\oplus(V)^P}_{(J,\phi)^{N_G(P)/P}}) = \chi(\overline{\mathbb{P}_\oplus(V[P])}_{(J,\phi)^{N_G(P)/P}})$ for each $J \le N_G(P)/P$. We check this by restricting V to J', the inverse image of J in $N_G(P)$. The result then follows from the following lemma. □

Lemma 4.7. *ϕ^r is a summand of $\operatorname{Res}^G_{N_G(P)} V$ if and only if it is a summand of $\operatorname{Br}_P V$.*

Proof. One implication is clear. For the other, use [1] theorem 3.2, or the method of lifting idempotents in the endomorphism rings. □

Corollary 4.8. *If V is a trivial source module then $V = \alpha L_\oplus(V)$ in $\mathbb{Q} \otimes a(G)$.*

Proof. Both sides have the same simple components after the Brauer map at any p-subgroup P by 4.6, so they are equal, by a version of Conlon's Induction Theorem (see [4]). □

Although in characteristic zero L does not commute with induction, in the modular case it does behave well in certain cases.

Lemma 4.9. *If $H \trianglelefteq G$ is of p-power index, then $L_{\oplus,G}(\operatorname{Ind}_H^G V) = I_H^G L_{\oplus,H}(V)$.*

Proof. It is sufficient to prove the case of index p, by induction on a chain of subgroups. We can suppose that V is indecomposable. If $U \not\leq H$, $\operatorname{Res}_U^G \operatorname{Ind}_H^G V = \operatorname{Ind}_{U\cap H}^U V$, which is indecomposable. Thus both sides of the equation lie in the part of $\mathbb{Q} \otimes a(G)$ in which the coefficient of $(J,\phi)^G$ is 0 when $J \not\leq H$. But R_H^G is injective on this, and

$$R_H^G L_{\oplus,G}(\operatorname{Ind}_H^G V) = L_{\oplus,H}(\operatorname{Res}_H^G \operatorname{Ind}_H^G V) = \sum_{g\in G/H} L_{\oplus,H}({}^gV) = R_H^G I_H^G L_{\oplus,H}(V).$$

□

5. Integrality

Theorem 5.1. *If V is a trivial source module then $L_{\oplus,G}(V) \in a_+^1(G)$, i.e. the coefficients are integers.*

The proof will take up the whole of this section.

Let us make the coefficients in $L_\oplus(V)$ explicit by writing

$$L_{\oplus,G}(V) = \sum_{(J,\phi)/G} l_V(J,\phi)^G (J,\phi)^G.$$

Suppose that G and V form a counterexample to the theorem with minimal possible $|G|$, and that V is indecomposable. Let $(J,\phi)^G$ have maximum $|J|$ among the $(J,\phi)^G$ such that $l_V(J,\phi)^G$ is not integral.

Lemma 5.2. *In the circumstances above, (J,ϕ) is normal in G and J is of p-power index in G.*

Proof. For $H \leq G$,

$$\begin{aligned} L_{\oplus,H}(\operatorname{Res}_H^G V) = R_H^G L_{\oplus,G}(V) &= R_H^G \sum_{(K,\theta)/G} l_V(K,\theta)^G (K,\theta)^G \\ &= \sum_{(K,\theta)/G} \sum_{g\in H\backslash G/K} l_V(K,\theta)^G (H \cap {}^g K, {}^g\theta)^H. \end{aligned}$$

If $H = N_G(J,\phi)$, then the coefficient of $(J,\phi)^H$ is $l_V(J,\phi)^G + \sum l_V(I,\eta)$, where the summation is over terms with $|I| \gneqq |J|$ so, by hypothesis, is a sum of integers. Thus $l_V(J,\phi)^H$ is not integral and so we must have $H = G$.

Now take a subgroup S with $J \leq S \leq G$ and such that S/J is a Sylow p-subgroup of G/J. Setting $H = S$ in the previous formula we see that the coefficient of $(J,\phi)^S$ in $L_{\oplus,S}(\operatorname{Res}_S^G V)$ is

$$\begin{aligned} \sum_{(K,\theta)/G} \sum_{\substack{g\in S\backslash G/K \\ (S\cap {}^gK, {}^g\theta) =_S (J,\phi)}} l_V(K,\theta)^G &\equiv \sum_{(K,\theta)/G} \sum_{\substack{g\in S\backslash G/K \\ ({}^gK, {}^g\theta) =_S (J,\phi)}} l_V(K,\theta)^G \pmod{\mathbb{Z}} \\ &\equiv \sum_{(K,\theta)/G} \sum_{\substack{g\in S\backslash G/K \\ (K,\theta) = (J,\phi)}} l_V(K,\theta)^G \pmod{\mathbb{Z}} \end{aligned}$$

since (J,ϕ) is normal

$$\equiv \sum_{g\in S\backslash G/J} l_V(J,\phi)^G \pmod{\mathbb{Z}}$$
$$\equiv |G/S| l_V(J,\phi)^S \pmod{\mathbb{Z}},$$

since $S\backslash G/J = S\backslash G$. This must equal $l_V(J,\phi)^S$, so if $S \neq G$ it is an integer, by minimality of G.

Since $|G/S|$ is coprime to p and the only possible denominator in $l_V(J,\phi)^S$ is a power of p, we must have $S = G$. □

Lemma 5.3. *Let P be the vertex of V. Then P is a Sylow p-subgroup of G.*

Proof. By 4.4 we may assume that $J \cap P$ is a Sylow p-subgroup of J. Since V is a summand of $\operatorname{Ind}_P^G k \cong \operatorname{Ind}_{PJ}^G \operatorname{Ind}_P^{PJ} k$, it must also be a summand of $\operatorname{Ind}_{PJ}^G U$ for some indecomposable trivial source kPJ-module U.

But PJ is of p-power index in G, so by Green's Indecomposability Theorem, $\operatorname{Ind}_{PJ}^G U$ is indecomposable and thus isomorphic to V. By 4.9, the coefficient of $(J,\phi)^G$ in $L_{\oplus,G}(V)$ is the same as that of $(J,\phi)^{PJ}$ in $L_{\oplus,PJ}(U)$, so by minimality, $PJ = G$. But $|PJ/P| = |J/J\cap P|$, which is coprime to p. □

Lemma 5.4. *If k is a summand of $\operatorname{Res}_J^G V$, then $V \cong k$.*

Proof. V is a summand of $\operatorname{Ind}_P^G k$, and notice that $\operatorname{Ind}_P^G k$ has exactly one summand isomorphic to k.

$\operatorname{Res}_J^G V$ is a summand of $\operatorname{Res}_J^G \operatorname{Ind}_P^G k \cong \operatorname{Ind}_{J\cap P}^J k$, by the previous lemma. Again this has only one trivial summand, so it must be the restriction of the one over G. □

The image of ϕ must lie in a finite subfield of k so there is some power of p, say p^r, such that $\phi(j)^{p^r} = \phi(j)$ for all $j \in J$. We may assume that r is big enough that $|J|$ divides p^r. Now define an extension Φ of ϕ to G by $\Phi(g) = \phi(g^{p^r})$ for $g \in G$.

Let Φ^* denote the dual of Φ. If V is a counterexample, then so is $\Phi^* \otimes V$, since $l_{\Phi\otimes V}(K,\theta)^H = l_V(K,\Phi\otimes\theta)^H$. But $l_V(J,\Phi)^G = l_{\Phi^*\otimes V}(J,k)^G$, so we can assume that $\Phi = k$.

But $l_V(J,k)^G$ can be expressed as a linear combination of the multiplicities of θ as a summand of $\operatorname{Res}_K^G V$ for pairs $(K,\theta) \geq (J,k)$. If k is not a summand of $\operatorname{Res}_J^G V$ then clearly these are all zero, and $l_V(J,k)^G = 0$, contradicting the assumption that it is not an integer. Thus k is a summand of Res_J^G, and by the previous lemma $V = k$. But clearly k is not a counterexample. This contradiction completes the proof of 5.1.

Example. From 4.4 and 5.1 we can obtain the known fact that every projective module is virtually a sum with integer coefficients of modules induced up from p'-subgroups.

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