DOUBLE COSET FORMULAS FOR PROFINITE GROUPS

PETER SYMONDS

Abstract. We show that in certain circumstances there is a sort of double coset formula for induction followed by restriction for representations of profinite groups.

1. Introduction

The double coset formula, which expresses the restriction of an induced module as a direct sum of induced modules, is a basic tool in the representation theory of groups, but it is not always valid as it stands for profinite groups. Based on our work on permutation modules in [8], we give some sufficient conditions for a strong form of the formula to hold. These conditions are not necessary, although they do often hold in interesting cases, and at the end we give some simple examples where the formula fails.

We also formulate a weaker version of the formula that does hold in general.

2. Results

We work with profinite groups G and their representations over a complete commutative noetherian local ring R with finite residue class field of characteristic p. There is no real loss of generality in taking $R = \hat{\mathbb{Z}}_p$. Our representations will be in one of two categories: the discrete p-torsion modules $\mathcal{D}_R(G)$ or the compact pro-p modules $\mathcal{C}_R(G)$. These are dual by the Pontryagin duality functor $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$, which we denote by $*$, so we will usually work in $\mathcal{D}_R(G)$. Our modules are normally left modules, so dual means contragredient. For more details see [7, 8].

For any profinite G-set X and $M \in \mathcal{D}_R(G)$ we let $F(X, M) \in \mathcal{D}_R(G)$ denote the module of continuous functions $X \to M$, with G acting according to $(gf)(x) = g(f(g^{-1}x))$, $g \in G$, $f \in F(X, M)$, $x \in X$. We set $F(X) = F(X, T)$, where $T = R^*$, the Pontryagin dual of the trivial module R.

When H, G are profinite groups with $H \leq G$ and $M \in \mathcal{D}_R(H)$, we define the coinduced module $\operatorname{Coind}_H^G M \in \mathcal{D}_R(G)$ by $\operatorname{Coind}_H^G M = F(G, M)^H \cong F(G)\check{\otimes}^H M$. Here H acts on the right of G, and G acts on the left, so we have $((g, h)f)(x) = hf(g^{-1}xh)$ for $g, x \in G$, $h \in H$, $f \in F(G, M)$. The operation $\check{\otimes}$ on discrete modules is Pontryagin dual to the more familar completed tensor product $\hat{\otimes}$ on compact modules ([8] 2.4); it commutes with direct limits.

The dual concept is induction: for $N \in \mathcal{C}_R(H)$, $\operatorname{Ind}_H^G N = R[[G]]\hat{\otimes}_H N$, where $R[[G]]$ is the complete group algebra. This is related to coinduction by Pontryagin duality: $\operatorname{Ind}_H^G N \cong (\operatorname{Coind}_H^G N^*)^*$.

These functors have most of the properties that one would expect by analogy with the discrete case ([8] 2.4).

Partially supported by a grant from the Leverhulme Trust.

Now if $K \le G$ then $\operatorname{Res}^G_K \operatorname{Coind}^G_H M = \operatorname{Res}^G_K (F(G)\check{\otimes}^H M)$ and the action of K is via its action on $F(G)$ only. What we want to do is to decompose $F(G)$, considered as a left $K \times H$-module. The action of $K \times H$ on G is given by $(k,h)x = kxh^{-1}$ and the action on $F(G)$ by $((k,h)f)(x) = F(k^{-1}xh)$, for $h \in H$, $k \in K$, $g, x \in G$.

The reader might prefer to think of the dual problem of decomposing $R[[G]]$, but by working in $\mathcal{D}_R(G)$ we avoid having to deal with the topology.

What we have is a permutation module in the sense of [8]. Note that $\operatorname{Stab}_{K\times H}(g) = \{(k,h) \in K \times H \mid kgh^{-1} = g\} = \{({}^g h, h) \mid h \in K^g \cap H\}$.

What we want to do is to decompose this into pieces of the form $F((K \times H)/S)$, where $S = S_g = \operatorname{Stab}_{K\times H}(g)$ for some $g \in G$.

For an H-module M we write gM for the ${}^gH$-module which is isomorphic to M as an R-module but where ${}^g h$ acts as h.

Lemma 2.1. *$F((K\times H)/S)\check{\otimes}^H M \cong \operatorname{Coind}^K_{K\cap {}^g H} \operatorname{Res}^{H^g}_{K\cap {}^g H} gM$ in $\mathcal{D}_R(K)$, where $S = \operatorname{Stab}_{K\times H}(g)$.*

Proof. $F((K\times H)/S)\check{\otimes}^H M \cong F((K\times H)/S, M)^H \cong F(((K\times(K^g\cap H))/S)\times_{K^g\cap H} H, M)^H \cong F((K \times (K^g \cap H))/S, M)^{K^g\cap H}$.

Regard $g^{-1}K$ as a $(K \times (K^g \cap H))$-set by $(k_1, h_1)g^{-1}k = h_1 g^{-1} k k_1^{-1}$, $h_1 \in H$, $k, k_1 \in K$. This action is clearly transitive, and $\operatorname{Stab}_{K\times(K^g\cap H)}(g) = S$, so $g^{-1}K \cong (K \times (K^g \cap H)/S)$. Now the last module in the chain of isomorphisms is isomorphic to $F(g^{-1}K, M)^{K^g\cap H} \cong F(K, gM)^{K\cap {}^g H} \cong \operatorname{Coind}^K_{K\cap {}^g K} gM$. □

This shows that a double coset formula can be obtained as a consequence of a decomposition of a permutation module. For example, we obtain the following easy and well-known result.

Corollary 2.2. *If $H\backslash G/K$ is finite then for any $M \in \mathcal{D}_R(H)$ we have the usual double coset formula $\operatorname{Res}^G_K \operatorname{Coind}^G_H M \cong \bigoplus_{g\in K\backslash G/H} \operatorname{Coind}^K_{K\cap {}^g H} gM$ and dually for $N \in \mathcal{C}_R(H)$ we have $\operatorname{Res}^G_K \operatorname{Ind}^G_H N \cong \bigoplus_{g\in K\backslash G/H} \operatorname{Ind}^K_{K\cap {}^g H} gN$.*

Proof. Since each orbit is closed and there are only finitely many orbits, we see that $G \cong \coprod_{g\in K\backslash G/H} (K \times H)/S_g$ is a disjoint union of open and closed subspaces, hence $F(G) \cong \bigoplus_{g\in H\backslash G/K} F((K \times H)/S_g)$. Now use Lemma 2.1 and the discussion preceding it. □

The bewildered reader might wish to consult [1] 3.3.4, where a similar argument is used in the more familiar context of finite groups.

Our more general decomposition of $F(G)$ is based on [8] 3.21, but we will go over the proof again because there the stabilizers were assumed to be finite and we do not want this restriction.

All subgroups and sub-G-sets will be assumed to be closed.

If X is a G-set and $Y \le X$ is a sub-G-set then we let $F(X, Y)$ denote the functions on X that are zero on Y. Thus there is a short exact sequence $F(X, Y) \to F(X) \to F(Y)$. In fact $F(X, Y) \cong F(X/Y, *)$.

For the next four lemmas we will assume the following hypotheses: G is a profinite group, $H < G$, X is a G-set, $Y < X$ and if $x \in X - Y$ then $\operatorname{Stab}_G(x)$ is conjugate to H in G.

Lemma 2.3. *The G-module $F(X, Y)$ is a summand of a direct sum of terms $\operatorname{Coind}^G_H T \cong F(G/H)$.*

Proof. The case of a free action, that is when $H = 1$, is dealt with in [6] 5.6 when $Y = \emptyset$ and in [8] 3.11 otherwise. Basically, the idea is to let N be an open normal subgroup of H and prove the result for the G/N-set X/N; then use the fact that, in this context, "a direct limit of injectives is injective".

The case of general H is almost the same as in [8] 3.17. If $Y \neq \emptyset$ then $F(X,Y) \cong F(X/Y,*)$ so we can assume that $Y = *$. The case when $Y = \emptyset$ is simpler and is left to the reader.

Now apply the case of a free action to the action of $N_G(H)/H$ on X^H to see that $F(X^H,*)$ is a summand of a sum of $\mathrm{Coind}_1^{N_G(H)/H} T \cong \mathrm{Coind}_H^{N_G(H)} T$'s. Thus $F(G\times_{N_G(H)}X^H, G\times_{N_G(H)} *) = \mathrm{Coind}_{N_G(H)}^G F(X^H,*)$ is a summand of a sum of $\mathrm{Coind}_H^G T$'s. But the multiplication map $(G\times_{N_G(H)} X^H)/(G\times_{N_G(H)} *) \to X$ is a continuous map of compact Hausdorff sets, hence a homeomorphism, so $F(X,*)$ also has this form. □

Lemma 2.4. *The short exact sequence $F(X^H,Y^H) \to F(X^H) \to F(Y^H)$ is split over $N_G(H)/H$.*

Proof. $F(X^H,Y^H)$ is an injective module over $N_G(H)/H$, by 2.3. □

Lemma 2.5. *The short exact sequence $F(X^H\cup Y,Y) \to F(X^H\cup Y) \to F(Y)$ is split over $N_G(H)$.*

Proof. If $Y^H = \emptyset$ then $X^H \cup Y$ is a disjoint union and the result is clear. Otherwise, the natural map $X^H/Y^H \to (X^H\cup Y)/Y$ is a homeomorphism, so $F(X^H\cup Y,Y) \cong F(X^H,Y^H)$.

Consider the diagram

$$\begin{array}{ccccc} F(X^H\cup Y,Y) & \longrightarrow & F(X^H\cup Y) & \longrightarrow & F(Y) \\ \cong\downarrow & & \downarrow & & \downarrow \\ F(X^H,Y^H) & \longrightarrow & F(X^H) & \longrightarrow & F(Y^H). \end{array}$$

The bottom short exact sequence is split, by 2.4, hence the top one is also split. □

Lemma 2.6. *The short exact sequence $F(X,Y) \to F(X) \to F(Y)$ is split over G.*

Proof. Let $s : F(X^H\cup Y) \to F(X^H\cup Y,Y)$ be the splitting of the previous lemma. Define $s' : F(X) \to F(X,Y)$ by $(s'(f))(x) = (s((gf)|_{X^H\cup Y}))(gx)$, where $f \in F(X)$, $x \in X$ and $g \in G$ is such that $gx \in X^H$.

It is easy to check that s' is well defined, G-equivariant and a splitting. □

For any two subgroups A, B of G, we write $A \geq_G B$ if some G-conjugate of B is a subgroup of A. We write $A >_G B$ if B is conjugate to a proper subgroup of A. Since G is compact we can not have $A >_G A$.

Given any G-set X, define:

$$X^H = \{x\in X \,|\, \mathrm{Stab}_G(x) \geq H\}, \qquad X^{>H} = \{x\in X \,|\, \mathrm{Stab}_G(x) > H\},$$
$$X^{(H)} = \{x\in X \,|\, \mathrm{Stab}_G(x) \geq_G H\}, \quad X^{(>H)} = \{x\in X \,|\, \mathrm{Stab}_G(x) >_G H\}.$$

Clearly X^H is closed; so is $X^{(H)}$, because it is the image of the map $G\times X^H \to X$, $(g,x) \mapsto gx$. If there are only finitely many conjugacy classes of stabilizers then it is easy to check that $X^{>H}$ and $X^{(>H)}$ are also closed in X.

Theorem 2.7. *Let X be a profinite G-set such that there are only finitely many conjugacy classes of stabilizers $S_1, \dots, S_n$. Then $F(X) \cong \oplus_{i=1}^n F(X^{(S_i)}, X^{(>S_i)})$. In particular, $F(X)$ is a summand of a direct sum of the $F(X/S_i)$'s.*

Proof. We use induction on n; the case $n = 0$ is trivial. Let S_1, say, be minimal among the S_i and their conjugates and let $Y = \cup_{i=2}^n X^{(S_i)}$.

From 2.6 we have $F(X) \cong F(X, Y) \oplus F(Y)$. But $F(Y)$ is of the right form by induction and so is $F(X, Y)$ by 2.3. □

Corollary 2.8. *Let $H, K \leq G$ be profinite groups and let $K \times H$ act on G by $(k, h)g = kgh^{-1}$. Suppose that the number of conjugacy classes of stabilizers of the action of $K \times H$ on G is finite and let $M \in \mathcal{D}_R(H)$. Then $\operatorname{Res}_K^G \operatorname{Coind}_H^G M$ is a summand of a direct sum of modules of the form $\operatorname{Coind}_{K \cap {}^g H}^K gM$.*

Dually, if $N \in \mathcal{C}_R(G)$ then $\operatorname{Res}_K^G \operatorname{Ind}_H^G N$ is a summand of a direct product of modules of the form $\operatorname{Ind}_{K \cap {}^g H}^K gN$.

Proof. This follows immediately from 2.7 and 2.1, using $K \times H$ as the group and G as X. □

Remark. If $F \trianglelefteq H$ acts trivially on M then we may replace $X = G$ by $X = G/F$ in the above argument. For example, if M is trivial, so we can take $F = H$, then we are just considering G/H as a left K-set.

Remark. There are various general conditions that will ensure that the hypotheses of the theorem hold. For example if each $K \cap {}^g H$ is finite then it certainly suffices to know that $K \times H$ has only finitely many conjugacy classes of finite subgroups. This is the case if H and K are p-adic analytic by [2]. More generally it holds if H and K are pro-p and virtually of type FP, by [8] 6.15.

For an interesting example of a calculation that uses these ideas see [3, 9].

The statement of the theorem says only *a summand of a direct sum of* because of the nature of the proof of 2.3. If we know that this module is in fact free then the statement can be made to look more like the usual double coset formula.

Corollary 2.9. *In the circumstances of 2.8, let $J = K \times H$ and write X for G considered as a J-set as in 2.8. If either both H and K are pro-p groups or if G is countably based then*

$$\operatorname{Res}_K^G \operatorname{Coind}_H^G M \cong \bigoplus_S F(X^S/N_J(S), X^{>S}/N_J(S)) \check{\otimes} \operatorname{Coind}_{K \cap {}^g H}^K gM,$$

where S runs through representatives of the conjugacy classes of stabilizers and S is the stabilizer of $g \in X$. The action of K is on the second factor only.

Dually,

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G N \cong \prod_S R[[X^S/N_J(S), X^{>S}/N_J(S)]] \hat{\otimes} \operatorname{Ind}_{K \cap {}^g H}^K gN.$$

Proof. The proof is the same as that of 2.8 except that we need to identify each term $F(X^{(S_i)}, X^{(>S_i)})$ in the statement of 2.7 as an explicit sum of $F(X/S_i)$'s. Let $J = K \times H$ and write S for some S_i.

Recall that, for a pro-p group P, every projective module is a product of $R[[P]]$'s by [11] 7.5.4, 7.4.1 or [8] 2.5, so every injective module is a sum of $F(P)$'s. We can tell how many by considering the fixed point module; if we denote the injective module by I then $I \cong I^P \check{\otimes} F(P)$.

But $F(X^{(S_i)}, X^{(>S_i)})^J \cong (\mathrm{Coind}^J_{N_J(S)} F(X^S, X^{>S}))^J \cong F(X^S, X^{>S})^{N_J(S)}$ and $F(X^S, X^{>S})$ is an injective module over $N_J(S)/S$ by 2.3, so $F(X^S, X^{>S}) \cong F(X^S, X^{>S})^{N_J(S)} \check{\otimes} F(N_J(S)/S) \in \mathcal{D}_R(N_J(S))$. Now observe that $F(X^S, X^{>S})^{N_J(S)} \cong F(X^S/N_J(S), X^{>S}/N_J(S))$.

If G is countably based then so is X^S, so the quotient map $X^S \to X^S/N_J(S)$ has a continuous section s by [6] 5.6.7. But then the map $\mathrm{Im}(s) \times N_J(S)/S \to X^S$ induces an isomorphism $F(X^S, X^{>S}) \cong F((X^S/N_J(S)) \times N_J(S)/S, (X^{>S}/N_J(S)) \times N_J(S)/S) \cong F(X^S/N_J(S), X^{>S}/N_J(S)) \check{\otimes} F(N_J(S)/S)$.

In either case $F(X) \cong \oplus_S F(X^S/N_J(S), X^{>S}/N_J(S)) \check{\otimes} F(J/S)$, and hence the claim. □

3. The General Case

A completely different approach to these formulas is given by Mel'nikov [4, 5]. He succeeds in expressing $R[[X]] \cong \bigoplus_{x \in G\backslash X} R[[G/\mathrm{Stab}_G(x)]]$, where the $\oplus$ is not normally the usual direct sum, but some sort of completion of it. For example, if G is the trivial group then we have $R[[X]] \cong \bigoplus_{x \in X} R$.

This $\oplus$ does, however, have the good property of commuting with Tor.

By the discussion in section 2, we can now write $\mathrm{Res}^G_K \mathrm{Ind}^G_H M \cong \bigoplus_{g \in H\backslash G/K} \mathrm{Ind}^K_{H^g \cap K} g^{-1}M$, suitably interpreted, for any $H, K \leq G$.

We now present a result that is essentially dual to this.

Recall that for any $N \in \mathcal{C}_R(G), M \in \mathcal{D}_R(G)$ there are well-behaved Galois Ext-groups $\mathrm{Ext}^n_G(N, M)$ (see [6] 6.1, [8] 2.6).

For any left G-set X let $O_x(G, X) = Gx$ denote the orbit of $x \in X$. By abuse of notation we will often allow $x \in X/G$ and write just O_x. The inclusion of O_x in X induces a restriction map $r_x : \mathrm{Ext}^n_G(N, F(X)) \to \mathrm{Ext}^n_G(N, F(O_x))$ for any $N \in \mathcal{C}_R(G)$.

We give all the groups $\mathrm{Ext}^n_G(N, F(O_x))$ the discrete topology and form their direct product, giving it the product topology. The r_x combine to induce a map $\theta : \mathrm{Ext}^n_G(N, F(X)) \to \prod_{x \in X/G} \mathrm{Ext}^n_G(N, F(O_x))$.

When we take $n = 0$ and $N = R[[G]]$ this reduces to the more basic homomorphism $\theta : F(X) \to \prod_{x \in X/G} F(O_x)$.

Theorem 3.1. *The map θ is injective with dense image.*

Proof. The map θ is certainly an isomorphism when X is finite.

In order to simplify the notation we write $E(-)$ for $\mathrm{Ext}^n_G(N, F(-))$. Write $X = \varprojlim X_i$, an inverse limit of finite G-sets indexed by some set I (this is always possible by [6] 5.6.4). Thus there are G-equivariant morphisms $p_i : X \to X_i$ and we say that X_j covers X_i if there is a morphism $X_j \to X_i$ in the system. We know that $E(X) \cong \varinjlim E(X_i)$.

For any $a \in E(X)$ there is an $s \in I$ and an $a_s \in E(X_s)$ such that $a = p_s^* a_s$. Suppose that $\theta(a) = 0$ and consider $\theta(a_s)$.

Given $x \in X$, the restriction of p_s to $O_x(G, X)$ yields a map $E(O_{p_s(x)}(G, X_s)) \to E(O_x(G, X))$, under which the image of $r_{p_s(x)} a_s$ is zero. Thus for each $x \in X$ there is some X_i, say $X_{i(x)}$, that covers X_s and such that the image of $r_{p_s(x)} a_s$ under the natural map $E(O_{p_s(x)}(G, X_s)) \to E(O_{p_{i(x)}(x)}(G, X_{i(x)}))$ is zero.

Let $Y_x = p^{-1}_{i(x)}(O_{p_{i(x)}(x)}(G, X_{i(x)}))$. The Y_x, $x \in X$ provide an open cover of X so, by compactness, there is an open subcover $Y_{i_1}, \ldots, Y_{i_u}$. Let X_t be a set that covers all of $X_{i_1}, \ldots X_{i_u}$.

By construction, the image of $\theta(a_s)$ in $\prod_{x\in X_t/G} E(O_x(G,X_t))$ is zero. It follows that the image of a_s in $E(X_t)$ must be zero, so $a = 0$, proving injectivity.

For the density statement, notice that it just means that, given a finite set of disjoint orbits $O_{x_1},\dots,O_{x_v}$ and elements $a_t \in E(O_{x_t})$, we can find an $a \in E(X)$ such that $r_{x_t}(a) = a_t$ for $t = 1,\dots,v$.

There must be some X_i, say X_k, such that the orbits $O_{x_t}(G,X_k)$ for $t = 1,\dots,v$ are distinct. Also, for each x_t there is an X_i, say $X_{i(t)}$, such that a_t is the image of some $\bar{a}_t \in E(O_{p_{i(t)}(x_t)}(G,X_{i(t)}))$. Let X_w, say, cover X_k and all the $X_{i(t)}$ and let $\tilde{a}_t$ be the image of $\bar{a}_t$ in $E(O_{p_w(x_t)}(G,X_w))$.

Certainly there is an element $\tilde{a} \in E(X_w)$ such that $r_{p_w(x_t)}(\tilde{a}) = \tilde{a}_t$ for each t, so if a denotes the image of $\tilde{a}$ in $E(X)$ then $r_{x_t}(a) = a_t$ for $t = 1,\dots,v$, as required. □

Corollary 3.2. *Whenever we have subgroups $H,K \subseteq G$, the K-equivariant map*

$$F(G/H) \to \prod_{g\in K\backslash G/H} F(O_g(K,G/H)) \cong \prod_{x\in K\backslash G/H} F(K/(K\cap {}^gH))$$

and, for $N \in \mathcal{C}_R(K), M \in \mathcal{D}_R(H)$, the map

$$\mathrm{Ext}^n_K(N,\mathrm{Coind}_H^G M) \to \prod_{g\in K\backslash G/H} \mathrm{Ext}^n_K(N, F(O_g(K,G/H))\check{\otimes}^H M) \cong \prod_{g\in K\backslash G/H} \mathrm{Ext}^n_{K\cap {}^gH}(N, gM).$$

are injective with dense image.

Proof. The first part follows immediately from 3.1 and 2.1 in the same way as 2.8.

For the second part we also consider G as a $K\times H$-set, but first let X be an arbitrary $K\times H$-set on which H acts freely and consider $\mathrm{Ext}^n_{K\times H}(N\hat{\otimes}M^*, F(X))$.

Since H acts freely, $F(X)$ is injective over H. So a standard spectral sequence argument shows that $\mathrm{Ext}^n_{K\times H}(N\hat{\otimes}M^*, F(X)) \cong H^n(K, \mathrm{Hom}_H(N\hat{\otimes}M^*, F(X)))$ and in turn this is isomorphic to $H^n(K, \mathrm{Hom}(N, \mathrm{Hom}_H(M^*, F(X)))) \cong \mathrm{Ext}^n_K(N, \mathrm{Hom}_H(M^*, F(X))) \cong \mathrm{Ext}^n_K(N, F(X)\check{\otimes}^H M)$.

If we do take $X = G$ then this becomes $\mathrm{Ext}^n_K(N, \mathrm{Coind}_H^G M)$. But we can also decompose $F(X)$ into orbits first and we then obtain the terms in the middle direct product. These are isomorphic to those in the right hand product by 2.1 and Shapiro's Lemma. Now apply 3.1. □

Remark. A theorem for which this result plays a key role in the proof can be found in [10].

4. Example

Let G be the group of matrices of the form $\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}$ with $a \in \mathbb{F}_p$ and $b,c \in \mathbb{F}_p[[T]]$. This acts on $V = \mathbb{F}_p[[T]]^2$, considered as column vectors $\begin{bmatrix}x\\ y\\ 1\end{bmatrix}$. This action is transitive and the stabilizer of $\begin{bmatrix}x\\ y\\ 1\end{bmatrix}$ consists of those matrices with $c = 0$ and $b = -ay$, so is cyclic of order p.

Let H consist of those matrices with $b = c = 0$: then $V \cong G/H$. Let K consist of those matrices with $c = 0$ (an infinite elementary abelian p-group) and consider V as a K-set.

Let F denote the continuous functions with values in $\mathbb{F}_p$. Suppose that some $F(K/S)$ is a summand of $F(V)$ as a K-module, where S is a stabilizer of some point of V. Then because of the action of G, which conjugates the S's, every $F(K/S)$ is a summand. These are pairwise non-isomorphic and their endomorphism rings are local, by [9]. Thus, by the proof of the Krull-Schmidt property, any finite direct sum of distinct $F(K/S)$'s is a summand of $F(V)$, hence the full direct sum $\oplus_S F(K/S)$ is a submodule of $F(V)$. But $F(V)$ is countable and the indexing set of the S's is not: a contradiction.

Thus there can be no formula of the type discussed above for $\operatorname{Res}^G_K \operatorname{Coind}^G_H \mathbb{F}_p$.

There is a similar example with $a, b, c \in \hat{\mathbb{Z}}_p$. In this case $K \cong \hat{\mathbb{Z}}_p^2$ and each $S \cong \hat{\mathbb{Z}}_p$.

References

[1] Benson, D. J., *Representations and Cohomology I*, Cambridge Univ. Press. 1992.
[2] J.D. Dixon, M.P.F. du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups (2nd edition)*, London Math. Soc. Lecture Notes **157** (1999), Cambridge Univ. Press, Cambridge.
[3] P. Goerss, H.-W. Henn, M. Mahowald and C. Rezk, *A resolution of the $K(2)$-local sphere at the prime* 3, Annals Math. **162** (2005) 777-822.
[4] O.V. Mel'nikov, *Subgroups and homology of free products of profinite groups*, Math. USSR Izvestiya **34** (1990) 97-119.
[5] O.V. Mel'nikov, *Aspherical pro-p-groups*, Mat. Sbornik **193** (2002) 71-104.
[6] L. Ribes and P. Zalesskii, *Profinite Groups*, Springer-Verlag, Berlin 2000.
[7] P. Symonds and T. Weigel, *Cohomology of p-adic analytic groups*, in *New Horizons in pro-p groups*, du Sautoy, Segal, Shalev, eds., Progress in Math. **184** (2000), Birkhäuser, Boston.
[8] P. Symonds, *Permutation complexes for profinite groups*, Comment. Mathematici Helvetici **82** (2007) 1-37.
[9] P. Symonds, *On the construction of permutation complexes for profinite groups*, Geometry and Topology Monographs **11** (2007) 369-378.
[10] P. Symonds, *On cohomology isomorphisms of groups*, Jour. Algebra **313** (2007) 802-810.
[11] J.S. Wilson, *Profinite Groups*, Oxford Univ. Press, 1998.

School of Mathematics, University of Manchester, P.O. Box 88, Manchester M60 1QD, England

E-mail address: Peter.Symonds@manchester.ac.uk