THE MODULE STRUCTURE OF A GROUP ACTION ON A POLYNOMIAL RING: EXAMPLES, GENERALIZATIONS, AND APPLICATIONS

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Abstract. The proof of a module-structure result for group actions on polynomial rings, which the authors published in a previous paper, is discussed in some detail. Also examined are possible generalizations of the result and an application to invariant theory.

1. Introduction

This article is a commentary on the authors' paper [12]. We address three main themes. First, the difficulties in the proof of the main theorem (9.1) of [12]. Second, possible generalizations of this module-structure result. Third, applications of the result to invariant theory.

The main theorem of [12] can be stated as follows.

Let k be a finite field of characteristic p and order q and let $U_n = U_n(k)$ denote the group of $n \times n$ upper triangular matrices over k with 1s on the diagonal. This acts on the polynomial ring in n variables $S = k[x_1, \dots, x_n]$. The ring of invariants under this group, for which we write S^{U_n}, is known to be polynomial in generators $\{d_i : i = 1, \dots, n\}$, where the degree of d_i is q^{i-1}. Note that any p-group P acting faithfully on S may be considered to be a subgroup of U_n after a change of variables. Our structure theorem describes the kP-module structure of S for any such P:

Theorem 1.1. *There is an isomorphism of graded kP-modules:*

$$S \cong \bigoplus_{J \subseteq I} k[d_i : i \in I \cup \{n\} - J] \otimes_k \bar{X}_J(I),$$

where $\bar{X}_J(I)$ is a finite dimensional graded kP-module, P acts trivially on $k[d_i]$, and I is the set $\{1, 2, \dots, n-1\}$.

This should be read as saying that S contains one copy of $\bar{X}_J(I)$ for each monomial in the d_i with $i \in I \cup \{n\} - J$.

One consequence of this theorem is a finiteness result.

Corollary 1.2. *For any group G (not necessarily a p-group) acting on S, only a finite number of isomorphism classes of indecomposable kG-modules appear as a summand of S.*

Section 2 presents a worked example, illustrating the proof of 1.1 and Section 3 comments on the trace lemma ([12], 15.1), which is the most complicated part of the proof.

Next, in Section 4, we comment on several possible generalizations of the structure theorem or its finiteness corollary; in particular Section 5 calculates some examples of a structure theorem for $GL_n(k)$.

Some partial results in this direction are proved in Sections 6 and 7. Finally, in Section 8, we sketch how studying the ring S as a module for kG has been used in calculations in invariant theory.

2. Extended Example

In this section we present an extended example to help the reader sort through the details of the proof of the main theorem (9.1) of [12]. We take $n = 3$, $q = 3$, and $I = \langle 1, 2\rangle$; i.e. we consider the proof of the theorem for the group U_3 over the field k with three elements. For simplicity of notation we write $U_{1,2}$ instead of $U_{\{1,2\}} = U_3$, and similarly for $U_1 = U_{\{1\}}$ and $U_2 = U_{\{2\}}$ (even though this conflicts with the upper triangular matrices definition of these groups).

In this case the polynomial ring $S = k[x_1, x_2, x_3]$ contains the invariant ring $S^{U_{1,2}} = k[d_1, d_2, d_3]$, where the degrees of d_1, d_2, and d_3 are 1, 3, and 9 respectively. We study the $kU_{1,2}$-module structure of S by first reducing to the study of the $kU_{1,2}$-module structure of $T = S/(d_3)$, using Lemma 5.4 of [12] which tells us that, as $kU_{1,2}$-modules, $S = k[d_3] \otimes T$.

Now we examine the statement of the main theorem (9.1) of [12] in this case. Specializing the claims, the result tells us that there is a decomposition

$$T = X_{\langle 1,2\rangle}(\langle 1, 2\rangle) \oplus X_{\langle 1\rangle}(\langle 1, 2\rangle) \oplus X_{\langle 2\rangle}(\langle 1, 2\rangle) \oplus X_{\varnothing}(\langle 1, 2\rangle)$$

with certain properties. (In what follows we will eliminate the angle-brackets in the notation.)

Perhaps the simplest property to understand is the statement about Poincaré series. To unwind this statement ([12], 9.1,(4)) we note that the sets $S_J(I)$ here live in $P(I) = \mathbb{N}_0^I = \mathbb{N}_0^2$. These four sets $S_{1,2}(1, 2)$, $S_1(1, 2)$, $S_2(1, 2)$, and $S_{\varnothing}(1, 2)$ are defined by the inequalities in [12], 4.6. In the present case, with the natural coordinates a_1, a_2 on $\mathbb{N}_0^2$, we have

$$\begin{aligned} S_{1,2}(1, 2) &= \{(a_1, a_2) \mid a_2 < 2 \text{ and } a_1 + 2a_2 < 6\},\\ S_1(1, 2) &= \{(a_1, a_2) \mid a_2 \geq 2 \text{ and } a_1 < 4\},\\ S_2(1, 2) &= \{(a_1, a_2) \mid a_2 < 2 \text{ and } a_1 + 2a_2 \geq 6\}, \text{ and}\\ S_{\varnothing}(1, 2) &= \{(a_1, a_2) \mid a_2 \geq 2 \text{ and } a_1 \geq 4\}. \end{aligned}$$

Note that these sets form a partition of $\mathbb{N}_0^2$, as shown in Proposition 4.10 of [12].

It follows from the definition of the κs ([12], 7.5) that the sequence $\kappa_i^{(3-1)}$ is $1, 3, 6, 8, 9, 9, 9, \ldots$ and that the sequence $\kappa_j^{(3-2)}$ is $1, 2, 3, 3, 3, \ldots$. We are interested in the matrix of products $\kappa_{a_1}^{(3-1)} \kappa_{a_2}^{(3-2)}$, which looks like this:

$$\begin{matrix} 1 & 3 & 6 & 8 & 9 & 9 & 9 & 9 & 9\\ 2 & 6 & 12 & 16 & 18 & 18 & 18 & 18 & 18\\ 3 & 9 & 18 & 24 & 27 & 27 & 27 & 27 & 27\\ 3 & 9 & 18 & 24 & 27 & 27 & 27 & 27 & 27 \end{matrix}$$

By noting the regions corresponding to the $S_J(I)$, we can obtain explicitly the Poincaré Series for the $X_J(I)$ claimed in the result ([12], 9.1). For example, the Poincaré Series of $X_{1,2}(1, 2)$ is

$1+3t+6t^2+(8+2)t^3+(9+6)t^4+(9+12)t^5+16t^6$. This is the only Poincaré series which is a polynomial; the others are

$$\begin{aligned}
\mathrm{PS}(X_2(1,2)) &= \frac{9t^6+18t^7}{1-t},\\
\mathrm{PS}(X_1(1,2)) &= \frac{3t^6+9t^7+18t^8+24t^9}{1-t^3}, \text{ and}\\
\mathrm{PS}(X_\varnothing(1,2)) &= \frac{27t^{10}}{(1-t)(1-t^3)}.
\end{aligned}$$

It follows from Lemma 7.8 of [12] that these four Poincaré series add up to the Poincaré series of T; here this can be checked directly by adding the four rational functions.

Properties (1), (2), (3), and (4) of the decomposition of [12], 9.1 will be treated as we examine the proof of this result in the case at hand. (Observe that thus far we have only written down the desired Poincaré series, rather than constructing modules which actually have the corresponding series, so further discussion of (4) is required.) We note that the proof of this decomposition depends through the induction on other decompositions (for smaller groups), and so we describe these now.

We consider S as a module for the group U_1 (of order 9); to understand its structure we study $T(1) = k[x_1,x_2,x_3]/(d_2,e_3)$, where $S^{U_1} = k[x_1,d_2,e_3]$. (In fact e_3 may be written explicitly as $x_3^3 - x_1^2x_3$.) As a kU_1-module, $T(1)$ splits into a projective part and a non-projective part $B_1(1)$ with Poincaré series $1+3t+6t^2+8t^3$. The module $B_1(1)$ is easy to describe: it is just the part of $T(1)$ with degree less than 4.

The $kU_{1,2}$ module $\bar{X}_1(1,2)$ is obtained by induction from $B_1(1)$: as noted in the proof of Proposition 8.4 of [12], there is a map $\mathrm{Ind}_{U_1}^{U_{1,2}} B_1(1) \to T$ defined by $g\otimes b \mapsto g\cdot(G(1,2\mid 1)\cdot b)$. The proof that this map is injective requires the Trace Lemma ([12], 15.1); in this special case that lemma states that

$$\mathrm{Tr}_{U_2}\frac{d_3}{d_2}\cdot x_1^i = -d_2^2\cdot d_1^i,$$

for $i=0,1,2,3$. We omit any discussion of filtration and terms lying in the "error ideal" since in this case, and in the other cases of the trace lemma being treated here, there are no such terms, and we may arrange the modules coming from lower stages of the induction to have monomial socles also. The proof of the injectivity gives an identification (generally in the sense of leading terms, but here exactly) of the socle of $\bar{X}_1(1,2)$; since the module $X_1(1,2)$ is obtained from $\bar{X}_1(1,2)$ by propagating by d_2, it follows that we have identified the socle of $X_1(1,2)$.

This means that $X_1(1,2)$ has the properties (1), (2), (3), and (4) required by the statement of [12], 9.1: (1) is automatic from the propagation construction, (2) follows from the construction of $\bar{X}_1(1,2)$ by mapping in an induced module, while (3) comes from the proof of injectivity in Proposition 8.4 of [12], which gives an identification of the socle of $\bar{X}_1(1,2)$. Finally, property (4) follows from our knowledge of the Poincaré series of $B_1(1)$.

The constructions of $X_2(1,2)$ and $X_\varnothing(1,2)$ are similar; we abbreviate the treatments because we have already discussed $X_1(1,2)$ in detail.

To construct $\bar{X}_2(1,2)$ we start with the preliminary module $B_2(2)$, which is the part of $T(2)$ with degree less than 2; this module has Poincaré series $1+2t$. Note that $T(2) = k[x_1, x_2, x_3]/(x_2, f_3)$, where $f_3 = x_3^3 - x_2^2 x_3$. $B_2(2)$ is a U_2-module, which we induce to $U_{1,2}$, and map to T by $g \otimes b \mapsto g \cdot (G(1,2 \mid 2) \cdot b)$. We prove that this map is injective and indentify (in the sense of leading terms) the socle using the following special case of the Trace Lemma:

$$\mathrm{Tr}_{U_1} \frac{d_3}{x_3^3 - x_2^2 x_3} \cdot x_2^i = -d_1^{6-2i} \cdot d_2^i,$$

where $i = 0, 1$. We then propagate by d_1; the resulting module $X_2(1,2)$ has the desired properties.

To construct $\bar{X}_\varnothing(1,2)$ we need only note that the following special case of the Trace Lemma implies the existence of a projective module in the degree 10 part of T:

$$\mathrm{Tr}_{U_{1,2}} G(1,2 \mid \varnothing) = \mathrm{Tr}_{U_{1,2}} \frac{d_3}{x_3} \frac{d_2}{x_2} = d_1^4 \cdot d_2^2.$$

We then propagate by d_1 and d_2; the resulting module $X_\varnothing(1,2)$ has the desired properties.

We turn to the construction of $X_{1,2}(1,2)$, the complement to the direct sum of the other three. The existence and properties of this module follow from Proposition 8.1 of [12]. We review the proof of this proposition to call attention to a subtle point. To show the existence of the complement $X_{1,2}(1,2)$, we note that the other modules we have constructed do not intersect (because it is clear from our bookkeeping that their socles do not), and in large degrees, their Poincaré series add up to the Poincaré series of T. It follows that the modules $X_1(1,2)$, $X_2(1,2)$, and $X_\varnothing(1,2)$ are summands of T in large degrees, and from this it easily follows that they are *individually* summands in small degrees ([12], 6.18). The subtle point is that our control of the socles is sufficient to show that these modules are *simultaneously* summands in small degrees, i.e. that the complement $X_{1,2}(1,2)$ exists. This is handled in Lemma 6.19 of [12].

From the decomposition of T we have studied, a similar decomposition for S follows, using [12], 5.4. This completes our study of the extended example.

3. Remarks on the Trace Lemma

The proof of the main result of [12] is a very detailed exercise in bookkeeping, and it is natural to wonder if the effort required could somehow be reduced by a slightly different approach. Many of the complications of the trace lemma ([12]) are connected with the "higher terms" which must be handled in the accounting.

The reader of Section 2 may wonder if such higher terms are ever nonzero. Below we give an explicit example in which these higher terms are not zero.

Example 3.1. Let $n = 4$, $q = 3$. We write U for the upper-triangular group over k, which we take to act on $k[w, x, y, z]$ in such a way as to preserve the flag of subspaces $\langle w \rangle \subset \langle w, x \rangle \subset \langle w, x, y \rangle \subset \langle w, x, y, z \rangle$. We write d_w, d_x, d_y, and d_z for the orbit polynomials of w, x, y, and z. In the notation of [12], we will study $\mathrm{Tr}_{U_1} : k[w, x, y, z]^{U_{23}} \to k[w, x, y, z]^U$. We recall for the reader's convenience that U_1 is the "add w" group, isomorphic to $(k^+)^3$, and U_{23} is the "add x,y" group, isomorphic to the group of upper-triangular 3×3 matrices over k. This group acts on $k[x, y, z]$; its invariants are generated by the orbit polynomials (under U_{23}!) of x, y, and z, which we write

e_x, e_y, and e_z. In this notation we have $G(1,2,3 \mid 2,3) = d_z/e_z$. We now give a short table of traces:

	$i=0$	1	2	3	4	5
$j=0$	$-d_w^{18}$	$d_w^{16}d_x + d_w^{10}d_x^3$	$-d_w^{14}d_x^2 + d_w^8 d_x^4$	$d_w^{12}d_x^3$	$-d_w^{10}d_x^4$	$d_w^8 d_x^5$
$j=1$	$d_w^{12}d_y - d_w^6 d_x^2 d_y$	$-d_w^{10}d_x d_y$	$d_w^8 d_x^2 d_y$	$-d_w^6 d_x^3 d_y$	0	0

Table 3.2. $\mathrm{Tr}_{U_1} G(1,2,3 \mid 2,3) \cdot e_x^i e_y^j$

In this table of values there are three examples where the trace is *not* a monomial in the chosen generators of the invariant ring; i.e. where the "higher terms" are nonzero. The energetic reader may wish to verify that the leading terms are as specified in the trace lemma ([12], 15.1).

The proof of the trace lemma would be vastly simplified if one could produce a basis with the convenient property that the required traces took basis elements to basis elements (as opposed to linear combinations of these). Such a basis would eliminate all of the complications of "higher terms". Needless to say, the authors have not managed to produce such basis.

The authors noted in [11] that a very appealing proof of the main theorem of [12], based on a "factorization" of the module structure, was possible for $n = 3$, $q = 2$, but that this proof did not generalize to other values of n and q. (The proof was given in detail in [2].) One possible avenue to creating a generalization of the proof in [2] would be to take the components of the factorization, which are n submodules, each with Krull dimension 1, then invent a construction which takes these components as input and yields the full polynomial ring. However, it is not clear what sort of construction this might be. For example, when $n = 3$ and $q = 3$, one would expect to construct $k[x, y, z]$ out of three components, one having Poincaré series $(1 + 3t + 6t^2 + 8t^3 + 9t^4 + 9t^5 + \cdots)$, and corresponding to the successive radicals of the module $k[U_{1,2}/U_2]$ in the notation of Section 2. Thus, one would expect the Jacobson radical J_8 of $k[U_{1,2}/U_2]$ to figure in the construction of S^3. However, it can be verified with Magma that J_8 is neither a submodule nor a quotient of S^3.

4. Examples of Possible Generalizations

The reader of [12] will note that the proof of the main theorem in that paper seems to depend on a remarkable collection of coincidences; it is not easy to see how to generalize the module-structure finiteness result.

In this section we give some examples which suggest that the phenomenon of module-structure finiteness may hold in a more general setting than that of our theorem. There are limits to this generality, as seen in Examples 4.4 and 4.5.

Example 4.1 (Hopf Algebras)**.** Let $B = \Lambda(a, b)$, an exterior algebra on two generators over the field of two elements, which we write k in this example. We regard B as an algebra of cohomology operations by taking a to act as Sq^1 and b to act as $Sq^3 + Sq^2\, Sq^1$. We let this algebra act on the cohomology of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which we write $k[x, y]$. We have the following module-structure decomposition:

$$k[x,y] \cong k \oplus \left(k[x^2,y^2] \otimes F\right) \oplus (L_1 \oplus L_2 \oplus L_3)\,,$$

where $k = \langle 1 \rangle$ is the trivial B-module, $F = \langle xy, x^2y + xy^2, x^4y + xy^4, x^4y^2 + x^2y^4 \rangle$ is a free B-module of rank 1, $L_1 = \langle x, x^2, x^3, \ldots \rangle$, $L_2 = \langle y, y^2, y^3, \ldots \rangle$, and $L_3 = \langle x^2y, x^2y^2, x^2y^3, \ldots \rangle$. (We have given vector-space bases for each module appearing.)

We note that, as B-modules, $L_1 \cong L_2 \cong L_3$, thus only the three isomorphism classes k, F, and L_1 appear.

We sketch a proof of the above assertions. Note that the Poincaré series of all these modules add up to the Poincaré series of the polynomial ring. So, to get the decomposition result, we need only prove the surjectivity of the map from the decomposition to the polynomial ring.

Using the given basis of the module L_3, plus an induction, we can show that the free module $x^{2i}y^{2j}F$ lets us obtain the monomials $x^{2i+1}y^{2j+1}$, $x^{2i+1}y^{2j+2}$, $x^{2i+4}y^{2j+1}$, and $x^{2i+4}y^{2j+2}$. This gives us every monomial of $x^\alpha y^\beta$, where $\alpha > 0$ and $\beta > 0$. The rest of the monomials are covered by L_1, L_2, and k.

To see that all of the given modules are indecomposable is standard; The only tricky part is to show that the L_i are indecomposable. Perhaps the simplest method of doing this is to use the isomorphism $\Lambda(a, b) \cong \mathbb{F}_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$ and quote the results on infinitely generated modules for this group ring in [5].

Remark 4.2. The reader should be aware that there is some question of whether a module-structure decomposition is well-defined in a context involving modules which are not finite-dimensional. Since we have not chosen hypotheses or made specific conjectures, we will not address this issue.

The finiteness corollary still holds over the algebraic closure of a finite field, because any representation over such a field can be written inside a finite field. It is not clear that this remains the case for other infinite fields.

The next example, which was considered indepedently by Derksen and Kemper, shows that the finiteness result can continue to hold even when the field contains transcendentals.

Example 4.3 (Infinite Fields)**.** Let $K = k(s, t) \supset k$ be the extension of k generated by two algebraically independent elements s, t. Then $G = \{y \mapsto y + (\lambda s + \mu t)x,\ x \mapsto x \mid \lambda, \mu \in k\}$ is a group of automorphisms of $S = K[x, y]$. We note that $S^G = K[x, d_y]$, where $d_y = \prod_{(\lambda,\mu)\in k^2}(y + (\lambda s + \mu t)x)$.

We will show, by the method of [12], that

$$S \cong \left\{ K[x, d_y] \otimes S^{q^2-1} \right\} \oplus \left\{ K[d_y] \otimes (\oplus_0^{q^2-2} S^i) \right\},$$

i.e. we will prove that the main theorem of [12] holds in this context.

To see this, note that by [7], Proposition 9.5, we have that $\mathrm{Tr}_G\, y^{q^2-1} = \alpha x^{q^2-1}$, where α is a nonzero element of K. Thus, the submodule of S^{q^2-1} generated by y^{q^2-1} is free. By dimension-counting, this module is S^{q^2-1} itself. In this setting, projective and injective are the same, so we may propagate S^{q^2-1} by any monomial in x and d_y to obtain new summands, which do not intersect as their socles are different.

The other summand in the decomposition, $K[d_y] \otimes (\oplus_0^{q^2-2} S^i)$, can be mapped to the polynomial ring by multiplication, and does not intersect the first summand. (Again, consider the socles.) Thus, the map we have constructed from the decomposition to the polynomial ring is injective. By counting graded dimensions using Poincaré series, we see that the map is surjective and therefore an isomorphism.

The point of this example is not so much that the finiteness result continues to hold, but that the method of proof of [12] still works. This is because (as noted in [11]) cases where there are three or fewer variables in the polynomial ring have special characteristics which may make alternate proofs of the finiteness result possible. In fact, in Section 6 we will prove that the finiteness result holds for the action of a finite group on a polynomial ring in three variables over *any* field.

The next example shows that the finiteness result fails for free modules over a polynomial ring.

Example 4.4 (Free Modules over a Polynomial Ring)**.** Let k be the field of two elements and let $V = \langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ act on $k[x, y]\langle 1, z\rangle$ by $\alpha\colon z \mapsto z + x$ and $\beta\colon z \mapsto z + y$. We can regard $k[x, y]\langle 1, z\rangle$ as a subset of $k[x, y, z]$ or as a free $k[x, y]$-module of rank two.

If $n \in \mathbb{N}_0$, we write $\Omega^n k$ for the n-th Heller translate of the trivial kV-module k. These modules are pairwise nonisomorphic, since their ranks over k are all different. $\Omega^n k$ can be pictured using the diagrams popularized by Benson and Carlson [4]. Writing $a = 1+\alpha \in kV$, $b = 1+\beta$ we have:

$$\Omega^1 k = \bullet[dl]_a[dr]^b \bullet \bullet \qquad \Omega^2 k = \bullet[dl]_a[dr]^b \bullet [dl]_a[dr]^b \bullet \bullet \bullet$$

$$\Omega^3 k = \bullet[dl]_a[dr]^b \bullet [dl]_a[dr]^b \bullet [dl]_a[dr]^b \bullet \bullet \bullet \bullet \qquad \text{etc.}$$

The dots represent basis vectors and the arrows the action of the group ring; see [4].

Now we note that if we attach a grading to our module by assigning x, y, and z grading 1, then $M = k[x, y]\langle 1, z\rangle$ is the direct sum $\oplus_{i=0}^{\infty} M_i$. Further, the basis of M_i consisting of the monomials of total degree i has exactly the same diagram as $\Omega^i k$:

$$M_1 = z[dl]_a[dr]^b xy \qquad M_2 = zx[dl]_a[dr]^b zy[dl]_a[dr]^b x^2 xy y^2 \qquad \text{etc.}$$

This gives an explicit isomorphism $M_i \cong \Omega^i k$, and therefore

$$M \cong \oplus_{i=0}^{\infty} M_i \cong \oplus_{i=0}^{\infty} \Omega^i k.$$

Thus, the module-structure finiteness property does not hold for M.

Example 4.5 (Tensor Algebras)**.** In this example once again k is the field of two elements and V the elementary abelian group of order 4. We write T for the tensor algebra of $\Omega^1 k$, so that $T_n = (\Omega^1 k)^{\otimes n}$. Then $T_n \cong \Omega^n k \oplus F$, where F is a free kV-module. As noted above, the modules $\Omega^n k$ are pairwise nonisomorphic, so that the finiteness result does not hold in this context either.

Various results about the modules for which a finiteness result does hold for the tensor algebra are given by Feit in [10], II.5. The obvious generalization of the example above to any prime and the results of Feit show that the only groups for which there is a finiteness result for all modules are those of finite representation type.

5. The General Linear Group

In this section we indicate the form a module-structure theorem for the general linear group should take by giving two examples. We will study the cases of two and three variables over the field of two elements. Thus, throughout this section, we take $k = \mathbb{F}_2$. The reader should note that in our examples, we will be proving "best possible" module-structure theorems in the sense that all the summands exhibited are indecomposable.

Our first goal is to handle the case of two variables. We want to describe the module structure that results when $G = GL_2(k)$ acts on $k[y,z]$ by homogeneous linear substitutions. We begin by recalling the following standard result:

Lemma 5.1. *There are exactly three indecomposable kG-modules: the Steinberg module St, the trivial module k and the free module for the quotient C_2, M.*

Now we identify instances of these modules in $k[y,z]$. Recall that $k[y,z]^G = k[c_2, c_3]$, where $c_2 = y^2 + yz + z^2$, $c_3 = y^2z + yz^2$.

Lemma 5.2. $\langle y, z\rangle \cong \langle y^2, z^2\rangle \cong St$, *$\langle y^2 + yz + z^2\rangle \cong k$, and $\langle y^3 + y^2z + z^3\rangle \cong M$.*

Writing $St^{(1)}$, $St^{(2)}$ and $M^{(3)}$ for the copies of these modules we have identified in $k[y,z]$, we have

Proposition 5.3. $k[y,z] \cong k[c_2] \oplus k[c_2, c_3] \otimes (St^{(1)} \oplus St^{(2)}) \oplus k[c_2, c_3] \otimes M^{(3)}$.

Proof. We construct a map from the decomposition on the right-hand-side to the polynomial ring using the inclusion of the submodules exhibited in Lemma 5.2 and multiplication by the invariants. We note that the Poincaré Series of the right-hand-side is

$$\frac{2t + 2t^2}{(1-t^2)(1-t^3)} + \frac{2t^3}{(1-t^2)(1-t^3)} + \frac{1}{1-t^2} = \frac{1}{(1-t)^2},$$

i.e. the same as the Poincaré Series of the left-hand side. So, to prove that the map is an isomorphism, it suffices to show that it is injective.

To show this, we show that there is no linear relation between the socles of the three parts of our decomposition. Such a relation would have the form

$$f(c_2) + c_3 \cdot g(c_2, c_3) + (ax + by + cx^2 + dy^2) \cdot h(c_2, c_3) = 0,$$

where f, g, and h are polynomials and a, b, c, and d are elements of k. If we rearrange the equation so that all the invariants are on one side, we see that $(ax + by + cx^2 + dy^2)$ must be invariant, and it follows that a, b, c, and d are all zero. Then we have reduced to $f(c_2) + c_3 \cdot g(c_2, c_3) = 0$, and since the Dickson invariants are algebraically independent, it follows that f and g are zero.

This proves that the map is injective and therefore an isomorphism. □

Now we turn to the result for three variables, that is, we study the module structure resulting from the action of $G = GL_3(k)$ on $k[x,y,z]$ by homogeneous linear transformations. Recall that the Dickson invariants $k[x,y,z]^G$ are a polynomial algebra $k[c_4, c_6, c_7]$ on homogeneous generators of degree 4, 6, and 7.

The result is quite complex, and to state it, we introduce a notational convention: $M_r^{(d)}$ denotes a module of rank r (over k) which is homogeneous of degree d. We write St for the Steinberg module, which has dimension 8 over k. The proof, which we omit, is handled using MAGMA.

Our decomposition is:

Proposition 5.4. *There is an isomorphism of $GL_3(k)$-modules:*

$$\begin{aligned} k[x,y,z] \cong k[c_4] \otimes (k \oplus M_3) \oplus k[c_4,c_6] \otimes (N_6^{(2)} \oplus N_6^{(4)} \oplus X_{10}^{(3)} \oplus X_{10}^{(5)} \oplus Z_{14}^{(6)} \oplus Y_{10}^{(7)}) \\ \oplus k[c_4,c_6,c_7] \otimes (St^{(4)} \oplus St^{(5)} \oplus St^{(6)} \oplus St^{(7)\oplus 2} \oplus St^{(8)} \oplus St^{(9)} \oplus St^{(10)} \\ \oplus P_{16}^{(8)} \oplus P_{16}^{(9)} \oplus P_{16}^{(11)} \oplus P_{16}^{*(10)} \oplus P_{16}^{*(12)} \oplus P_{16}^{*(13)} \oplus W_8^{(14)}). \end{aligned}$$

The modules P_{16}, P_{16}^*, St, and W_8 are projective.

These two examples suggest that the result for $GL_n(k)$, where $k = \mathbb{F}_q$, should have n parts, propagated by the subsets of the Dickson invariants

$$\{c_{q^n-q^{n-1}}\}, \{c_{q^n-q^{n-1}}, c_{q^n-q^{n-2}}\}, \dots, \{c_{q^n-q^{n-1}}, c_{q^n-q^{n-2}}, \dots, c_{q^n-1}\}.$$

One can make more specific conjectures, but since we do not have a completely precise conjecture on the "best possible" decomposition of $k[x_1, \dots, x_n]$ as a $GL_n(k)$-module, we will stop here.

6. A General Proof for Three Variables

Here we present a version of the structure theorem for three variables that applies to any finite group and any field. In particular, this result (6.1) gives a decomposition for the action of $GL_3(k)$ on $k[x_1, \dots, x_n]$, where k is a finite field (see Proposition 5.4 and the remarks following). Further, Theorem 6.1 applies to a field that contains transcendentals (see also Example 4.3).

Theorem 6.1. *If a finite group G acts on $S = k[x_1, x_2, x_3]$ by homogeneous transformations then there are homogeneous invariant elements $u, v \in S^G$ and graded kG-submodules P, U, B of S such that P is projective (but infinite dimensional) and U, B are finite dimensional and there is an isomorphism of kG-modules*

$$S \cong P \oplus (U \otimes k[u,v]) \oplus (B \otimes k[u]).$$

The proof depends on several lemmas, which hold for any number of variables. There is no content when k has characteristic 0, so we assume that the characteristic is p.

Lemma 6.2. *Let R be a commutative graded ring, finitely-generated as an algebra over $R_0 = k$ by homogeneous elements of positive degree. Let M be a finitely generated graded RG-module for which there is a number N such that $\dim_k M_r \le N$ for all $r \in \mathbb{Z}$. Then there are finite-dimensional graded kG-submodules B, U of M and a homogeneous element $v \in R$ such that $M = B \oplus (U \otimes k[v])$.*

Proof. By Noether normalization there is a homogeneous element $v \in R$ of degree s, say, such that M is finitely-generated over $k[v]$. It follows that there is some N such that if $r \ge N$ then $M_{r+s} = vM_r$. Let $B = \bigoplus_{r<n} M_r$ and $U = \bigoplus_{r=N}^{N+s-1} M_r$. □

Lemma 6.3. *If G acts on $S = k[x_1, \dots, x_n]$ then there is an invariant homogeneous element $u \in S^G$ such that, as kG-modules, $S \cong (S/uS) \otimes k[u]$.*

Proof. Let Q be the Sylow p-subgroup of G. We may assume that the matrices have triangular form with respect to the basis $x_1, \dots, x_n$ (such that x_1 is fixed). Let $R = S^G$. Since S is finitely generated over R and S contains x_n^r for all $r \in \mathbb{N}$ there must be a homogeneous element $u \in R$ that contains the monomial x_n^r for some $r \in \mathbb{N}$.

Let T denote the k-subspace of S spanned by all monomials not divisible by x_n^r. Then $S = T \oplus uS$ as kQ-modules. Thus the inclusion of uS in S is split over Q, so it is split over G: let T' be a complement as a kG-module. Then $S = T' \oplus uS = T' \otimes k[u]$. □

Remark 6.4. If G is a p-group in triangular form then we may take u to be the orbit product of x_n. If k is finite we may take u to be the bottom Dickson invariant for $GL_n(k)$.

Lemma 6.5. *Returning to G acting on $S = k[x_1, x_2, x_3]$, with u as in lemma 6.3, the non-projective part of S/uS, which we denote by $(S/uS)/(proj)$, has dimension bounded by some N in each degree.*

Proof. Since $S/(\mathrm{proj}) \cong ((S/uS)/(\mathrm{proj})) \otimes k[u]$ and $\dim_k S/(\mathrm{proj})_r$ is bounded by a linear function in r by [16], the result follows. □

Proof. of 6.1. By lemma 6.3 we can write $S = (S/uS) \otimes k[u]$ and, by lemma 6.5, we can apply lemma 6.2 to obtain $(S/uS)/(\mathrm{proj}) = B \oplus (U \otimes k[v])$. □

7. Conlon's Induction Theorem

One approach to obtaining a structure theorem for general G is to use Conlon's Induction Theorem to reduce it to the case of subgroups that are cyclic modulo p.

Let $T_n = T_n(k)$ be the subgroup of $GL_n(k)$ consisting of upper triangular matrices where any non-zero entries are allowed on the diagonal. The generators d_i of the invariants under U_n are eigenvectors for T_n.

Proposition 7.1. *The main structure formula 1.1 remains valid over T_n, with T_n acting on the d_i as above.*

Proof. Examine the proof in [12] to verify that the pieces $\bar{X}_I(J)$ can be defined over T_n. □

Rather than a structure theorem we will produce one of a result that would be one of its main consequences: a formula for the multiplicity as a summand of any indecomposable kG-module.

For any indecomposable kG-module M let $\mathrm{PS}_M(S,t)$ be the power series in which the coefficient of t^r is the multiplicity of M as a summand of S_r.

By Conlon's Induction Theorem ([8]; [9], 80.61), there are cyclic modulo p subgroups C_i of G and rational numbers a_i such that

$$S = \sum_i a_i \operatorname{Ind}_{C_i}^G \operatorname{Res}_{C_i}^G S$$

in the Green ring.

Let ℓ be an extension of k such that all the p' elements of G become diagonalizable in ℓ. We write $\ell S = \ell \otimes_k S$. The image of each C_i in $GL_n(\ell)$ can be conjugated in to $T_n(\ell)$, so we can apply 7.1 to $\operatorname{Res}_{C_i}^G \ell S$ to obtain

$$\ell S = \sum_j \mathrm{PS}_{N_{ij}}(\ell S, t) N_{ij},$$

where N_{ij} runs through the finite set of indecomposable summands of ℓS. Putting this together we obtain

$$\ell S = \sum_{ij} a_i \operatorname{PS}_{N_{ij}}(\ell S, t) \operatorname{Ind}_{C_i}^{G} N_{ij}.$$

Now we restrict scalars and, to simplify the notation, we let $b_{ij}(M)$ denote the multiplicity of the indecomposable kG-module M as a summand of $\operatorname{Ind}_{C_i}^{G} \operatorname{Res}_{C_i}^{G} \operatorname{Res}_{k}^{\ell} N_{ij}$. We obtain:

Theorem 7.2.

$$\operatorname{PS}_M(S,t) = \frac{1}{[\ell : k]} \sum_{ij} a_i b_{ij}(M) \operatorname{PS}_{N_{ij}}(\ell S, t),$$

where $\operatorname{PS}_{N_{ij}}(\ell S, t)$ *encodes the multiplicity of* N_{ij} *as a summand of* ℓS *as an* ℓC_i*-module.*

To use this formula involves calculating the C_i and the a_i, which is routine, and also the $\operatorname{PS}_{N_{ij}}(\ell S, t)$. The latter can all be calculated by restricting the formula for the decomposition of ℓS as an $\ell T_n(\ell)$-module. According to [12] this involved calculating ℓS up to degree $r^{n-1} - n$, where r is the order of ℓ.

Such a procedure is probably uneconomical, but it does yield an explicit algorithm.

8. Applications to Vector Invariants

In this section we present an application of module-structure theory to invariant theory. We show how to obtain the Poincaré Series of the vector invariants $k[V \oplus V \oplus \cdots \oplus V]^G$ using the G-module structure of $k[V]$. We also show how to take this technique further and in some cases obtain a Poincaré series for what we call *tensor* invariants, that is, $k[V \otimes W]^{G \times H}$, where W carries a H-module structure. Most of these techniques were shown to the authors by Milgram [14], who states that their use was standard in certain types of homotopy-theoretic calculations about thirty years ago, but offers no specific attribution.

The basic idea for vector invariants is simple: the G-module structure of $k[V]$ determines the Poincaré Series of $k[V]^G$. Since the G-module structure of $k[V]$ determines the G-module structure of $k[V^{\oplus n}] \cong k[V]^{\otimes n}$, it also determines the Poincaré series of $k[V^{\oplus n}]^G$. Similarly, if W is a monomial representation of H, we will be able to determine the $G \times H$-module structure of $k[V \otimes W]$ in terms of the G-module structure of $k[V]$ and the H-module structure of $k[W]$; the vector invariants are simply the special case where W is the trivial H-module of dimension n.

Because the idea is not deep, and because the appropriate implementation will vary with the particular application, we will confine ourselves to illustrating the method by the study of certain special cases. We begin with the case of vector invariants of the regular representation F of the cyclic group of order 2, C_2, over the field of two elements k.

Example 8.1 (Vector invariants: the regular representation of $G = C_2$, $n = 2$.)**.** The C_2-module structure of the polynomial ring $k[F]$ is $(k[d_2] \otimes k) \oplus (k[d_1, d_2] \otimes F)$. From this fact we have

$$k[F \oplus F] \cong k[F] \otimes k[F] \cong \{(k[d_2] \otimes k) \oplus (k[d_1, d_2] \otimes F)\} \otimes \{(k[e_2] \otimes k) \oplus (k[e_1, e_2] \otimes F)\}.$$

Noting that $F \otimes F \cong F \oplus F$, we may expand this decomposition to yield

$$k[F \oplus F] \cong (k[d_2, e_2] \otimes k) \oplus (k[d_1, d_2, e_2] \otimes F) \oplus (k[d_2, e_1, e_2] \otimes F) \oplus (k[d_1, d_2, e_1, e_2] \otimes F \oplus F) \tag{8.2}$$

Now we note that $\dim_k F^{C_2} = \dim_k k^{C_2} = 1$, so that 8.2 gives us an expression for the Poincaré Series of $k[F \oplus F]^{C_2}$:

$$\frac{1}{(1-t^2)^2} + \frac{t}{(1-t)(1-t^2)^2} + \frac{t}{(1-t)(1-t^2)^2} + \frac{2t^2}{(1-t)^2(1-t^2)^2} = \frac{1+t^2}{(1-t)^2(1-t^2)^2}. \tag{8.3}$$

The skeptical reader will be pleased to note that our series in Equation 8.3 can be, and in fact has been, checked using Kemper's software [13], which has been incorporated into MAGMA [6].

Now that we have illustrated the basic idea, we proceed to some refinements. Our next example studies tensor invariants; in the notation of the opening paragraph of this section we take $G \cong H \cong C_2$ and V and W to be the regular representations of G and H.

Example 8.4 (Tensor invariants: $G = C_2$, $H = C_2'$, $V = F$, and $W = F'$.)**.** We regard $F \otimes F' \cong F \tilde{\oplus} F$ as a $C_2 \times C_2'$ module, where the C_2' exchanges the Fs. This is the same as the $C_2 \times C_2'$-module obtained by taking the tensor product of the regular representations F, F' of C_2, C_2'. We study $k[F \otimes F']$ as a $C_2 \times C_2'$-module by writing $k[F \otimes F'] \cong k[F \tilde{\oplus} F] \cong k[F] \tilde{\otimes} k[F]$, where the tilde indicates that the action of C_2' switches the factors in the direct sum or tensor product. Of course underneath all the notation we are just studying the regular representation of $C_2 \times C_2'$; the point is to illustrate the method.

If we write down the structure of $k[F] \tilde{\otimes} k[F]$ as a C_2-module in the first few degrees, we obtain

	Degree	0	1	2	2
Degree	Module	k	F	F	k
0	k	$k \tilde{\otimes} k$	$\boxed{k \otimes F}$	$k \otimes F$	$\underline{k \otimes k}$
1	F	$\boxed{F \otimes k}$	$F \tilde{\otimes} F$	$F \otimes F$	$F \otimes k$
2	F	$F \otimes k$	$F \otimes F$	$F \tilde{\otimes} F$	$F \otimes k$
2	k	$\underline{k \otimes k}$	$k \otimes F$	$k \otimes F$	$k \tilde{\otimes} k$

TABLE 8.5. C_2-module structure of $k[F] \tilde{\otimes} k[F]$

We note that the action of C_2' swaps the entries in the table across the main diagonal. Thus, as representations of $C_2 \times C_2'$, the boxed modules combine to form $\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(F \otimes k)$, while the underlined modules combine to form $\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(k \otimes k)$. Similar remarks apply to the other off-diagonal entries in the table. The diagonal entries are marked with tildes to indicate that the action of C_2' switches the factors in the tensor product.

These regularities allow us to determine the $C_2 \times C_2'$-module structure of $k[F] \tilde{\otimes} k[F]$ in the following way. We first write down for each C_2-isomorphism type $[M]$ of summands of $k[F]$ a Poincaré series $P_M(t)$ for its multiplicity. This collection of isomorphism types $[M_i]$ gives rise to three classes of $C_2 \times C_2'$-isomorphism types of summands of $k[F] \tilde{\otimes} k[F]$, whose forms are listed below:

(1) (boxed type): $\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(M_i \otimes M_j)$
(2) (underlined type): $\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(M_i \otimes M_i)$

(3) (diagonal type): $M_i \tilde{\otimes} M_i$

The Poincaré series for the multiplicities of each of these types of modules as summands $k[F]\tilde{\otimes}k[F]$ of can be obtained easily in terms of the $P_{M_i}(t)$ as follows:

(1) (boxed type): $P_{M_i}(t) \cdot P_{M_j}(t)$
(2) (underlined type): $(1/2)[P_{M_i}(t)^2 - P_{M_i}(t^2)]$
(3) (diagonal type): $P_{M_i}(t^2)$

In the case at hand we see from our decomposition of $k[F]$ that $P_k[t] = (1-t^2)^{-1}$ and that $P_F(t) = t \cdot (1-t)^{-1} \cdot (1-t^2)^{-1}$. We then obtain Poincaré series for the multiplicities of the summands of $k[F]\tilde{\otimes}k[F]$:

Module	Poincaré series
$\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(k \otimes F)$	$\frac{t}{(1-t)(1-t^2)^2}$
$\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(F \otimes F)$	$\frac{t^3}{(1-t)^2(1-t^2)(1-t^4)}$
$F\tilde{\otimes}F$	$\frac{t^2}{(1-t^2)(1-t^4)}$
$\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(k \otimes k)$	$\frac{t^2}{(1-t^2)(1-t^4)}$
$k\tilde{\otimes}k$	$\frac{1}{(1-t^4)}$

TABLE 8.6. Poincaré series for the multiplicities of summands in $k[F]\tilde{\otimes}k[F]$

Finally, as in Example 8.2 we need to know the dimension of the invariant subspaces.

Module	$\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(k \otimes F)$	$\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(F \otimes F)$	$F\tilde{\otimes}F$	$\mathrm{Ind}_{C_2}^{C_2 \times C_2'}(k \otimes k)$	$k\tilde{\otimes}k$
$\dim_k(M^{C_2 \times C_2'})$	1	2	2	1	1

TABLE 8.7. Dimensions of invariant subspaces

This allows us to add rational functions to obtain the Poincaré series of $k[F \otimes F']^{C_2 \times C_2'}$.

$$\begin{aligned} &\frac{t}{(1-t)(1-t^2)^2} + \frac{2t^3}{(1-t)^2(1-t^2)(1-t^4)} + \frac{2t^2}{(1-t^2)(1-t^4)} + \\ &\qquad \frac{t^2}{(1-t^2)(1-t^4)} + \frac{1}{(1-t^4)} = \frac{t^2-t+1}{(1-t)^2(1-t^2)^2} \end{aligned} \tag{8.8}$$

This example, like the previous one, has been checked against the result obtained by Kemper's software.

Although the techniques of Example 8.4 may seem specialized to the case of tensor products with the regular representation of C_2, they can be adapted to arbitrary permutation representations by a more sophisticated version of the analysis following Table 8.5. Next we will see that how to handle monomial representations. In the next example we will consider the case of a tensor product with the natural representation St of $GL_2(k)$, i.e. in the setting of the opening paragraph of this section we take $W = St$ and $H = GL_2(k)$. Note that $GL_2(k) \cong S_3 = \langle a, b \mid a^3 = 1, b^2 = 1, bab = a^2\rangle$. We write C_2 for the subgroup $\langle b\rangle \subset S_3$.

Let $K = \{0, 1, \omega, \omega^2\}$ be the field of four elements. If V is a representation of G, we can describe the $G \times S_3$-module structure of $K[V \otimes St]$ by regarding $V \otimes St$ as $V \tilde{\oplus} V$, where b switches the two copies of V and a acts on the first as multiplication by ω and on the second by multiplication by ω^2. Notice that St can be defined over k (it is the natural representation of $GL_2(k)$) and that if we are only interested in the Poincaré series of the invariant ring, it does not matter whether we work over k or K.

We can adapt the results of Example 8.4 to the C_2-module structure of $S^*(V\tilde{\oplus}V)$. This gives us three types of submodules of $S^*(V\tilde{\oplus}V)$. The first type is $V \otimes W\tilde{\oplus}W \otimes V$, the second is $V \otimes V\tilde{\oplus}V \otimes V$, and the third is $V\tilde{\otimes}V$. Here we have used the tilde to indicate the action of C_2, as in Example 8.4.

We observe that all of these module types are actually S_3-invariant; i.e. working over K, the decomposition of the C_2-module structure is also a decomposition of the S_3-module structure. In the application of this result we used a compact description of the multiplicities of these three types of modules in $S^*(V\tilde{\oplus}V)$ in terms of their multiplicities in $S^*(V)$. We can give a similar description of the multiplicities of the different S_3-module types. To do this, we first refine our description of the decomposition. Modules of the first type are either $St \otimes V \otimes W$ or $M \otimes V \otimes W$, regarded as a $S_3 \times G$-module. (Here M is the free module for the quotient C_2 of S_3.) We have the second case if $\deg W + 2 \deg V \equiv 0 \mod 3$, and the first case if $\deg W + 2 \deg V \not\equiv 0 \mod 3$. The same holds for modules of the second type. Modules of the third type always have trivial C_3-action. Note that, for modules of the first and second type, if we are in the first case (i.e. a tensor product with St) there are no invariants, since $St^{S_3} = 0$.)

Observe that if $f(t) = \sum_i a_i t^i$, $g(t) = \sum_j b_j t^j$, then $\sum_{i+2j\equiv 0 \mod 3} a_i b_j t^{i+j}$ can be expressed in terms of f and g as

$$\frac{1}{3}\left[f(t)g(t) + f(\zeta t)g(\zeta^2 t) + f(\zeta^2 t)g(\zeta t)\right], \tag{8.9}$$

where $\zeta \in \mathbb{C}$ is a primitive cube root of 1. This discussion can be generalized to other roots of 1, and by combining this with arbitrary permutation representations, we can get results for monomial representations. However, the calculations required for such examples quickly become tedious.

We now proceed to an application of this general discussion.

Example 8.10 (Tensor invariants: $G = C_2$, $V = F$, $H = S_3$, and $W = St$)**.** Take $G = C_2$ and $V = F$, the regular representation of G. In the decomposition of $K[V \otimes St]$ we obtain modules of all five types mentioned above.

We recall from our discussion of the decomposition for $S^*(F)$ that $P_k(t) = 1/(1 - t^2)$ and $P_F(t) = t/(1 - t)(1 - t^2)$. Thus in $S^*(F\tilde{\oplus}F)$, regarded as an $C_2 \times S_3$-module, we have that the modules of type $M \otimes F \otimes k$ have the following multiplicity:

$$\frac{1}{3}\left[P_F(t)P_k(t) + P_F(\omega t)P_k(\omega^2 t) + P_F(\omega^2 t)P_k(\omega t)\right]$$

These modules have $C_2 \times S_3$-fixed set of dimension 1, so the function above gives exactly their contribution to the Poincare series of $S^*(F\tilde{\oplus}F)^{C_2\times S_3}$

Next we have to consider modules of type $M \otimes F \otimes F$ and $M \otimes k \otimes k$. In the first case we get the multiplicity

$$\frac{1}{2} \cdot \left\{ \frac{1}{3} \left[P_F(t)^2 + 2P_F(\omega t) P_F(\omega^2 t) \right] - P_F(t^2) \right\}.$$

These modules have $C_2 \times S_3$-fixed set of dimension 2, so we need to multiply their contribution to the Poincare series by 2. In the second case we have

$$\frac{1}{2} \cdot \left\{ \frac{1}{3} \left[P_k(t)^2 + 2P_k(\omega t) P_k(\omega^2 t) \right] - P_k(t^2) \right\},$$

and the modules have $C_2 \times S_3$-fixed set of dimension 1.

Finally there are two cases of modules of the third type. The $F \tilde{\otimes} F$-modules have multiplicity function $P_F(t^2)$, and $C_2 \times S_3$-fixed set of dimension 2. The $k \tilde{\otimes} k$-modules have multiplicity function $P_k(t^2)$, and $C_2 \times S_3$-fixed set of dimension 1.

Now we add all of this up and get

$$\frac{(t^8 - t^7 + t^6 + t^4 + t^2 - t + 1)(t+1)(t^2+1)}{(1-t^2)(1-t^3)(1-t^4)(1-t^6)},$$

which agrees with the result obtained from Kemper's program.

Now we apply these ideas to an example which is of interest in its own right and not just as an application of these techniques.

Example 8.11 ("Double Dickson" invariants)**.** Take $G = GL_3(k)$ and let V be the natural G-module. We compute the Poincaré series of the tensor invariants $k[V \otimes St]^{GL_3(k) \times GL_2(k)}$. Using the techniques described above, the decomposition of $k[V]$ as a $GL_3(k)$-module (5.4), and Magma we find that our series is

$$\begin{gathered}
(t^{46} + t^{43} + t^{42} + 2t^{41} + t^{40} + 3t^{38} + 2t^{36} + 3t^{35} + 5t^{34} + t^{33} + 4t^{32} + 3t^{31} + 3t^{30} + \\
t^{29} + 5t^{28} + 4t^{27} + 4t^{26} + 4t^{25} + 5t^{24} + 3t^{23} + 2t^{22} + 3t^{21} + 3t^{20} + t^{19} + 5t^{18} + 2t^{17} + \\
3t^{16} + t^{15} + 2t^{14} + t^{12} + t^{11} + 2t^{10} - t^9 + 2t^8 + t^6 - t + 1) \cdot \\
(t^{58} - t^{57} - t^{54} + t^{53} - t^{52} + t^{51} + t^{48} - t^{47} - t^{46} + t^{45} - t^{44} + t^{43} + t^{42} - t^{41} + 2t^{40} - \\
2t^{39} + t^{38} - 2t^{37} + t^{35} - t^{34} + 2t^{33} - t^{30} - t^{28} + 2t^{25} - t^{24} + t^{23} - 2t^{21} + t^{20} - \\
2t^{19} + 2t^{18} - t^{17} + t^{16} + t^{15} - t^{14} + t^{13} - t^{12} - t^{11} + t^{10} + t^7 - t^6 + t^5 - t^4 - t + 1)^{-1}
\end{gathered}$$

Admittedly this is not very appealing. One recalls the remark of Neusel and Smith ([15], p. 25) that "[Vector invariants] seem to have fallen out of favor, except as a means of providing nasty examples."

Part of the interest of this example is that it is very close to the limit of what can be done with current invariant-theory software. The authors were unable to verify the result using Kemper's software, though it may be possible to do this with better computing resources or algorithms adapted to specific features of this example. However, the series has been expanded and checked for correctness through degree 12.

This example was originally studied in the context of cohomology of groups. One of the maximal elementary abelian subgroups of M_{24}, E, is of rank 6, and the action of its normalizer realizes this representation. Further, if S is the Sylow 2-subgroup of M_{24}, then $E \subset S \subset M_{24}$ is a weakly closed system [3], and so the restriction map from $H^*(M_{24}, k)$ to $H^*(E, k)^{N_{M_{24}}(E)}$ is surjective, i.e. the cohomology of M_{24} maps onto our invariant ring.

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