GROUP ACTIONS ON RINGS AND THE ČECH COMPLEX

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Abstract. We have previously shown that when a finite group acts on a polynomial ring over a finite field k then only finitely many isomorphism classes of indecomposable kG-modules occur as summands of S. We have also shown that the regularity of the invariant subring S^G is at most zero, which has various consequences, for example S^G is generated in degrees at most $n(|G|-1)$ (provided $n, |G| \geq 2$). Both of these theorems depend on the Structure Theorem of Karagueuzian and the author, which is proved by means of a long and complicated calculation. The aim of this paper is to prove these results using a more conceptual method.

1. Introduction

We have previously shown with Dikran Karagueuzian that, when a finite group G acts on a polynomial ring $S = k[x_1, \ldots, x_n]$ over a finite field k by homogeneous linear transformations, only finitely many isomorphism classes of indecomposable kG-modules occur as summands of S [16]. We have also shown that the regularity of the invariant subring S^G is at most zero, which has various important consequences: for example, S^G is generated in degrees at most $n(|G|-1)$ (provided $n, |G| \geq 2$); more generally, if S is finitely generated over a polynomial subring $k[d_1, \ldots, d_n] < S^G$ then S^G is generated in degrees at most $\sum_i(\deg(d_i)-1)$ (provided $\deg(d_i) > 1$ for at least two i) [20].

Both of these results depend on the Structure Theorem of Karagueuzian and the author [16], which is proved by means of a long and complicated calculation. The aim of this paper is to prove these results using a simpler, more conceptual method, based on considering the Čech complex associated to S and showing that it is split exact over kG in degrees greater than $-n$. This approach also has the advantage that it applies to a somewhat more general class of rings. Some other results along these lines have been obtained by Bleher and Chinburg [5] by considering Koszul resolutions. We wish to thank Burt Totaro for his helpful comments.

2. Background

Throughout this paper, k is a field. Our results are trivial if k has characteristic zero, so implicitly k has characteristic $p > 0$. Unless otherwise indicated, all rings will be $\mathbb{Z}$-graded noetherian k-algebras, and before any localization it will be assumed that they are 0 in negative degrees and are finite dimensional over k in each degree. For brevity, we will refer to such rings that are also commutative as rings of standard type. A ring such as kG is implicitly in degree 0. Modules will also be graded, and when a group acts on a ring or module, it must preserve the grading. For a module M and an integer N we will write $M_{\geq N} = \oplus_{i=N}^{\infty} M_i$, and similarly for other inequalities.

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Our basic tool is the Čech complex, strictly speaking the "extended Čech complex" or the "stable Koszul complex".

Given a ring R, a sequence of homogeneous elements $\mathbf{x} = x_1, \ldots, x_r$ from R and an R-module M, the Čech complex $\check{C}(\mathbf{x}; M)$ is a cochain complex

$$M \to \bigoplus_i M_{x_i} \to \bigoplus_{i<j} M_{x_i x_j} \to \cdots \to M_{x_1 \cdots x_r},$$

where M_x denotes the localization obtained by inverting x. It can be obtained as follows. $\check{C}(x_i; R)$ is the complex

$$R \to R_{x_i},$$

with R in degree 0 and R_{x_i} in degree 1 and

$$\check{C}(\mathbf{x}; M) = \left(\bigotimes_{i=1}^{r}{}_R\, \check{C}(x_i; R) \right) \otimes_R M.$$

Observe that if G is a finite group and M is an RG-module then $\check{C}(\mathbf{x}; M)$ is a complex of graded RG-modules; up to isomorphism it is independent of the ordering of the elements of $\mathbf{x}$.

The jth homology group of this complex is equal to the local cohomology group $H^j_{(\mathbf{x})}(M)$, where $(\mathbf{x})$ denotes the ideal in R generated by $\mathbf{x}$. It is known that if $\mathbf{y}$ is another sequence such that $\operatorname{rad}(\mathbf{y}) = \operatorname{rad}(\mathbf{x})$ then $H^*_{(\mathbf{y})}(M) = H^*_{(\mathbf{x})}(M)$. We are particularly interested in the case when $\operatorname{rad}(\mathbf{x}) = \mathfrak{m} = \mathfrak{m}_R := \operatorname{rad}(R_{>0})$, the radical of the ideal of elements in positive degrees (but this ideal is only maximal if $R_0/\operatorname{rad}(R_0)$ is a field). For more information, see [7, 9, 15].

One very useful technique is to change categories in such a way that all the kG-modules in question become projective. Before we do this we fix some notation. For any ring Λ, let Λ-Mod denote the category of left Λ-modules and Mod-Λ denote the category of right Λ-modules. The full subcategory of Mod-Λ on the projective modules is denoted by Proj-Λ. Let A be a finite dimensional left kG-module. Then $\operatorname{Add}(A)$ is the full subcategory of kG-Mod on the modules that can be expressed as a summand of a direct sum of copies of A. Set $E = \operatorname{End}_{kG}(A)$.

There are functors

$$U = \operatorname{Hom}_{kG}(A, -) : \operatorname{Add}(A) \to \text{Proj-E}$$

and

$$V = - \otimes_E A : \text{Proj-E} \to \operatorname{Add}(A),$$

which furnish an equivalence of categories. For more details, see [23, Ch. 10] or [19].

These functors take graded modules to graded modules. In addition, they preserve any structure as an R-module for commutative k-algebras R, meaning that if M is an RG-module then $U(M)$ is naturally a right $R \otimes_k E$-module that is projective on restriction to E and similarly for V. The functors also preserve the property of being finitely generated over R.

Lemma 2.1. *The functors U and V commute with localization. In other words, if M is an RG-module and $x \in R$ then $U(M_x) \cong U(M)_x$, and if N is a right $R \otimes_k E$-module then $V(N_x) \cong V(N)_x$.*

Proof. Because A is finite dimensional, $\operatorname{Hom}_{kG}(A, -)$ commutes with direct limits, and the localization can be constructed as a direct limit; this proves the lemma for U. Tensor products always commute with direct limits, which proves it for V. □

Definition 2.2. A kG-module is of *finite decomposition type* if it can be expressed as a sum of finite dimensional indecomposable kG-modules with only finitely many isomorphism classes of indecomposable modules appearing.

A kG-module is of finite decomposition type if and only if it is in $\operatorname{Add}(A)$ for some finite dimensional A. Such a module satisfies the Krull-Schmidt-Azumaya property that any direct sum decomposition can be refined into a sum of indecomposables and any two decompositions into indecomposables will involve the same isomorphism classes with the same multiplicities [22]. It follows that any summand of a module of finite decomposition type is also of finite decomposition type. Finite decomposition type is also preserved by induction, restriction and tensor product over k.

Lemma 2.3. *If M is an RG-module that is of finite decomposition type when considered as a kG-module, then so is M_x for any $x \in R$.*

Proof. As a kG-module, M is in $\operatorname{Add}(A)$ for some finite dimensional kG-module A. The right $R \otimes_k E$-module $U(M)$ is projective over E, hence flat over E. Its localization, $U(M)_x$, is a direct limit of flat modules and thus is also flat. But, over a finite dimensional k-algebra, a flat module is projective ([2, Theorem P] or [11, 22.29, 22.31A] or devise a proof by considering the projective cover of the module). Thus $U(M)_x \in$ Proj-E, and so $V(U(M)_x) \in \operatorname{Add}(A)$. But $V(U(M)_x) \cong V(U(M))_x \cong M_x$, by Lemma 2.1. □

Proposition 2.4. *Let R be a ring of standard type, $x_1, \ldots, x_r$ a sequence of elements such that* $\operatorname{rad}(\mathbf{x}) = \mathfrak{m}$ *and let M be a finitely generated RG-module. Then M is of finite decomposition type if and only if*

(1) $\check{C}(\mathbf{x}; M)$ is split exact over kG in sufficiently high degrees and
(2) each M_{x_i} is of finite decomposition type.

Note that when we say that a complex is exact we mean exact at every module, so the homology is 0 everywhere.

Proof. If the two listed conditions are satisfied, then there is a number D such that $M_{>D}$ is a summand of $(\oplus_{i=1}^r M_{x_i})_{>D}$, thus of finite decomposition type. But $M_{\leq D}$ is of finite decomposition type, because it is finite dimensional, and thus M is of finite decomposition type.

Conversely, if M is of finite decomposition type then M is in $\operatorname{Add}(A)$ for some finite dimensional kG-module A. The second condition holds, by Lemma 2.3. In order to see that the first condition is also satisfied, apply U to $\check{C}(\mathbf{x}; M)$ and note that $U(\check{C}(\mathbf{x}; M)) \cong \check{C}(\mathbf{x}; U(M))$, by Lemma 2.1. Since M is finitely generated, so is $U(M)$; thus we know that $H^*_{\mathfrak{m}}(U(M))$ is zero in sufficiently large degrees [7, 15.1.5], [9, 3.6.19]. It follows that $\check{C}(\mathbf{x}; U(M))$ is exact in high degrees. But this is a complex of projective E-modules, so it is split exact as a complex of E-modules in high degrees. Thus $V(\check{C}(\mathbf{x}; U(M))) \cong \check{C}(\mathbf{x}; M)$ is split exact as a complex of kG-modules in high degrees. □

Proposition 2.5. *Let M be an RG-module of finite decomposition type. If $\mathbf{x} = x_1, \ldots, x_r$ and $\mathbf{y} = y_1, \ldots, y_s$ are two sequences of elements of R such that* $\operatorname{rad}(\mathbf{x}) = \operatorname{rad}(\mathbf{y})$*, then there is a homotopy equivalence of complexes of graded kG-modules $\check{C}(\mathbf{x}; M) \simeq \check{C}(\mathbf{y}; M)$.*

Proof. Write $\mathbf{xy} = x_1, \ldots, x_r, y_1, \ldots, y_s$. We will show that $\check{C}(\mathbf{x}; M) \simeq \check{C}(\mathbf{xy}; M)$; together with the variation with $\mathbf{x}$ and $\mathbf{y}$ interchanged, this will prove the result. We can add the y_i to $\mathbf{x}$ one at a time, so the key case is to show that $\check{C}(\mathbf{x}; M) \simeq \check{C}(\mathbf{x}y; M)$ for any $y \in \mathrm{rad}(\mathbf{x})$. Just as in the previous proof, it is easier to apply the functor U and show that $\check{C}(\mathbf{x}; U(M)) \simeq \check{C}(\mathbf{x}y; U(M))$ as a complex of right E-modules, then apply V to obtain the original statement.

By construction, $\check{C}(\mathbf{x}y; U(M)) \cong \check{C}(y; R) \otimes_R \check{C}(\mathbf{x}; U(M))$. This leads to a short exact sequence of complexes

$$0 \to \check{C}(\mathbf{x}; U(M_y))(-1) \to \check{C}(\mathbf{x}y; U(M)) \to \check{C}(\mathbf{x}; U(M)) \to 0,$$

where the (-1) indicates a shift in complex degree of 1.

The homology of $\check{C}(\mathbf{x}; U(M_y))$ is $H^*_{\mathbf{x}}(U(M_y))$, which is $(\mathbf{x})$-torsion, yet the action of y is invertible and $y \in \mathrm{rad}(\mathbf{x})$, so this homology is 0. It follows that $\check{C}(\mathbf{x}; U(M_y))$ is an exact complex of projective right E-modules; hence it is split exact, which means that it is 0 in the homotopy category. The other two complexes in the short exact sequence are thus homotopy equivalent. □

3. Preliminary Results

Given a graded R-module M, let $a_i(M)$ denote the highest degree in which $H^i_{\mathfrak{m}}(M)$ is non-zero (possibly ∞ or $-\infty$). The Castelnuovo-Mumford regularity of M is $\mathrm{reg}(M) := \max\{a_i(M) + i\}$. In [20] we also considered $\mathrm{hreg} := \max\{a_i(M)\}$. Because $H^i_{\mathfrak{m}}(M)$ vanishes for $i > \dim(M)$, we have $\mathrm{reg}(M) \leq \mathrm{hreg}(M) + \dim(M)$. If M is a ring and R is not specified then we take $R = M$, although any subring of M over which it is still finite will yield the same answer.

Let R be a ring of standard type, $x_1, \ldots, x_r$ a sequence of homogeneous elements such that $\mathrm{rad}(\mathbf{x}) = \mathfrak{m}$ and M a finitely generated RG-module. If we know that $\mathrm{hreg}(M) \leq N$ for some integer N then $\check{C}(\mathbf{x}; M)_{>N}$ is exact. Notice that $(M_x)^G = (M^G)_x$, because $M^G \cong \mathrm{Hom}_{kG}(k, M)$, which commutes with direct limits; hence $\check{C}(\mathbf{x}; M)^G \cong \check{C}(\mathbf{x}; M^G)$. Thus if we happen to know that $\check{C}(\mathbf{x}; M)_{>N}$ is split exact as a complex of kG-modules, then $\check{C}(\mathbf{x}; M^G)_{>N}$ is exact, and so $\mathrm{hreg}(M^G) \leq N$.

We are interested in the case when M is actually a ring of standard type, which we call S, G acts on S by grading-preserving ring automorphisms, and R is S^G or some subring of S^G over which it is still finite. The aim is to find a bound on $\mathrm{hreg}(S^G)$. It is this that leads to bounds on the degrees of the generators and relations: see [20] for details.

The fixed point subscheme of $\mathrm{Spec}(S)$ under the action of G is the closed subscheme, denoted by $\mathrm{Spec}(S)^G$, defined by the ideal $I_{G,S} < S$ generated by all the elements of the form $(g-1)s$ for $g \in G$ and $s \in S$.

We will need the following geometric result due to Fleischmann [12, 5.9] (see also [17]). Recall that for a kG-module M and a subgroup $H \leq G$ the relative trace is the map $\mathrm{tr}^G_H : M^H \to M^G$ defined by $\mathrm{tr}^G_H(m) = \sum_{g \in G/H} gm$.

Proposition 3.1. *Let S be a commutative ring (*not *assumed graded or noetherian here) on which a finite group G acts as ring automorphisms. If $(\mathrm{Spec}(S))^G = \emptyset$, then $S^G = \sum_{H \lneqq G} \mathrm{tr}^G_H(S^H)$.*

The formulation in [12] is more general in that it also describes the case when the fixed point set is not empty, but it is stated only for polynomial rings. For the convenience of the reader we sketch a proof.

Proof. Notice that $T := \sum_{H \lneq G} \mathrm{tr}_H^G(S^H)$ is an ideal in S^G; we need to show that it is not contained in any maximal ideal.

Let I be a maximal ideal in S^G. Clearly S is integral over S^G, so there is a maximal ideal $J < S$ such that $J \cap S^G = I$ [1, 5.8, 5.10]. Let H be its stabilizer, and consider the surjection $\pi : S \twoheadrightarrow \prod_{g \in G/H} S/{}^gJ$. If J is not fixed under the action of G, let e be the element of the product that is 1 at S/J and 0 at the other coordinates, so $e^2 = e$ and $he = e$ for $h \in H$; then $\mathrm{tr}_H^G(e) = \pi(1)$. Let $e' \in S$ be such that $\pi(e') = e$ and let $\tilde{e} = \prod_{h \in H} he'$. Then $\tilde{e} \in S^H$ and $\pi(\tilde{e}) = \prod_{h \in H} \pi(he') = \prod_{h \in H} h\pi(e') = \prod_{h \in H} he = e$. Thus $\pi(\mathrm{tr}_H^G(\tilde{e})) = \mathrm{tr}_H^G(\pi(\tilde{e})) = 1$. It follows that $T \nleq I$ in this case.

If G fixes J, consider the action of G on S/J. The condition on the fixed point subscheme shows that S/J is generated by elements of the form $(g-1)s$ for $g \in G$ and $s \in S$, so the action is not trivial; let H be its kernel. By the surjectivity of the trace in Galois theory, there is an element $f \in S/J$ such that $\mathrm{tr}_H^G(f) = 1$. Let p be the characteristic of S/J and let P be the Sylow p-subgroup of H ($P = 1$ if $p = 0$). Notice that for any $x \in S/J$, $\mathrm{tr}_H^G(x^p) = (\mathrm{tr}_H^G(x))^p$ (provided $p \neq 0$). Let $f' \in S$ be such that $\pi(f') = f$ and let $\tilde{f} = \mathrm{tr}_P^H(\prod_{g \in P} gf')$. Then $f' \in S^H$ and $\pi(\mathrm{tr}_H^G(\tilde{f})) = \mathrm{tr}_H^G \mathrm{tr}_P^H(f^{|P|}) = \mathrm{tr}_H^G(|H : P| f^{|P|}) = |H : P| (\mathrm{tr}_H^G f)^{|P|} = |H : P| \neq 0$. Again, it follows that $T \nleq I$. □

Remark. If S is a k-algebra, G acts by k-algebra automorphisms and $\bar{k}$ denotes the algebraic closure of k then the fixed point subscheme is empty if and only the action of G on the set of closed points of $\mathrm{Spec}(\bar{k} \otimes_k S)$ has no fixed point.

Remark. If G is a p-group and $p \in \mathrm{rad}(S)$ then the converse statement to that in the proposition is also true. When S is the ring of k-valued functions on a G-set this is a well-known consequence of the properties of the Brauer construction [8].

Definition 3.2. If a group G acts on a commutative k-algebra S, then by an SG-module M we mean an S-module that is also a kG-module in such a way that $g(sm) = (gs)(gm)$ for $g \in G$, $s \in S$ and $m \in M$. In other words, M is a module for the twisted group algebra.

When M is a kG-module we can regard $\mathrm{End}_k(M)$ as a kG-module by setting $(gf)(m) = g(f(g^{-1}m))$ for $g \in G, f \in \mathrm{End}_k(M), m \in M$.

Let $\mathcal{C}$ be a set of subgroups of G. Recall that a kG-module M is said to be projective relative to $\mathcal{C}$ if the following equivalent conditions hold.

(1) The module M is a summand of a sum of induced modules of the form $N{\uparrow}_H^G$, where $H \in \mathcal{C}$ and N is a kH-module.
(2) The module M is a summand of $\oplus_{H \in \mathcal{C}} M{\downarrow}_H^G{\uparrow}_H^G$.
(3) Any surjection of kG-modules $L \twoheadrightarrow M$ that splits on restriction to any $H \in \mathcal{C}$ is also split over kG.
(4) $\mathrm{Id}_M \in \sum_{H \in \mathcal{C}} \mathrm{tr}_H^G(\mathrm{End}_{kH}(M))$.

The equivalence of these conditions is known as Higman's criterion and there are proofs in many places for the case when $\mathcal{C}$ consists of just one group [14] [3, 3.6.13]. The general case is not much harder and can be found in [18, 3.5.8] or [6, 2.2.3]. In characteristic p, a module is projective relative to its Sylow p-subgroups.

Lemma 3.3. *Let S be a commutative k-algebra on which a p-group P acts by algebra automorphisms in such a way that $1 = \sum_{Q \lneqq P} \mathrm{tr}_Q^P(s_Q)$ for some $s_Q \in S^Q$. Then any SP-module M is projective relative to proper subgroups of P as a kP-module.*

Proof. By characterization (4) of relative projectivity, it suffices to show that $\mathrm{Id}_M = \sum_{Q \lneqq P} \mathrm{tr}_Q^P(m_Q)$ for some $m_Q \in \mathrm{End}_{kQ}(M)$. Let $m_Q \in \mathrm{End}_{kQ}(M)$ be multiplication by s_Q. □

Lemma 3.4. *Let M be an R-module, $\mathbf{x}$ a sequence of homogeneous elements of $R_{>0}$ and N an integer. Then $(M_{>N})_{x_{i_1}\cdots x_{i_\ell}} \cong M_{x_{i_1}\cdots x_{i_\ell}}$ if $\mathbf{x}$ is non-empty, and $\check{C}_{\mathbf{x}}(M_{>N})_{>N} = \check{C}_{\mathbf{x}}(M)_{>N}$.*

Proof. The first part follows from the short exact sequence $M_{>N} \to M \to \bar{M}_{\le N}$, where $\bar{M}_{\le N}$ denotes $M/M_{>N}$, and the fact that $(\bar{M}_{\le N})_{x_i} = 0$. For the second part, consider the short exact sequence $\check{C}_{\mathbf{x}}(M_{>N}) \to \check{C}_{\mathbf{x}}(M) \to \check{C}_{\mathbf{x}}(\bar{M}_{\le N})$. From the first part we see that $\check{C}_{\mathbf{x}}(\bar{M}_{\le N}) = \bar{M}_{\le N}$. □

Lemma 3.5. *Let C and D be complexes of graded kG-modules and let M and N be integers such that $C_{>M}$ and $D_{>N}$ are split exact. Then $(C \otimes_k D)_{>M+N}$ is split exact.*

Proof. We have $(C \otimes_k D)_i = \sum_{u+v=i} C_u \otimes_k C_v$. If $i > M + N$, then either $u > M$ and so C_u is split exact, or $v > N$ and so D_v is split exact. A split exact complex tensored with any other complex is split exact. □

Lemma 3.6. *Let G be a finite group and let M be a kG-module. Let $\mathcal{C}$ be a set of subgroups of G.*

(1) If M is projective relative to $\mathcal{C}$ and M is of finite decomposition type after restriction to any subgroup in $\mathcal{C}$, then M is of finite decomposition type.
(2) If $0 \to C_0 \to C_1 \to \cdots \to C_n \to 0$ is a complex of kG-modules that is split exact after restriction to any subgroup in $\mathcal{C}$ and $C_1, \ldots, C_n$ are projective relative to $\mathcal{C}$, then the complex is split exact over kG.

Proof. For (1), use characterization (2) of relative projectivity. For (2), use induction on n and characterization (3) to split the map $C_{n-1} \twoheadrightarrow C_n$. □

Definition 3.7. A kG-module X is semi-invertible if there exist kG-modules A and B such that $A \otimes_k X \cong k \oplus B$. Similarly, a complex of kG-modules X is semi-invertible if there exist complexes A and B such that $A \otimes_k X \simeq k \oplus B$.

The module A in this definition can be assumed to be finite dimensional. For if c is a generator of the copy of k on the right hand side then we have $c = \sum_{i=1}^n x_i \otimes a_i$ with $x_i \in X, a_i \in A$, and the a_i generate a finite dimensional kG-submodule of A that we can use instead of A.

4. Main Theorem

Theorem 4.1. *Let S be a graded k-algebra of standard type on which a finite group G acts and let M be an SG-module. Let N be an integer such that $\mathrm{hreg}(M) \le N$. Suppose that for each p-subgroup P of G there exist homogeneous elements $y_1, \ldots, y_r$ and $z_1, \ldots, z_s$ of $S_{>0}^P$ such that:*

(a) y_i vanishes on $(\mathrm{Spec}(S))^P$,
(b) $\mathrm{rad}_S(\mathbf{yz}) = \mathfrak{m}_S$,

(c) there is a $k[\mathbf{y}]P$-module T and a $k[\mathbf{z}]P$-module U such that $M_{>N} \cong (T \otimes_k U)_{>N}$ as $k[\mathbf{yz}]P$-modules,

(d) (i) either the action of P on U is trivial or

(ii) U is semi-invertible and of finite decomposition type and for some integer d the complex $\check{C}_{\mathbf{z}}(U)_{>d}$ is split exact over kP and $\check{C}_{\mathbf{z}}(U)_d$ is semi-invertible, and

(e) (i) either M is finitely generated over S or

(ii) $M_{\le N}$ is of finite decomposition type and the $T_{\le N-d}$ for each p-subgroup $P < G$ are also of finite decomposition type.

Then:

(1) for any sequence of homogeneous elements $\mathbf{x}$ in $S^G_{>0}$ such that $\mathrm{rad}_S(\mathbf{x}) = \mathfrak{m}_S$, the complex of kG-modules $\check{C}_{\mathbf{x}}(M)_{>N}$ is split exact;

(2) $\mathrm{hreg}(M^G) \le N$;

(3) M is of finite decomposition type.

Proof. Part (2) follows from part (1) by the discussion at the beginning of Section 3. Parts (1) and (3) hold over kG if they hold on restriction to a Sylow p-subgroup, by Lemma 3.6 with $\mathcal{C}$ the class of Sylow p-subgroups. We may therefore assume that G is a p-group.

We will prove (1) and (3) together, but with G replaced by one of its p-subgroups, P, using induction on the order of P. The conclusions are clearly valid when $|P| = 1$, because $\mathrm{hreg}(M) \le N$, so assume that $P \neq 1$ and the conclusions hold over all proper subgroups of P. Notice that (d.i) implies (d.ii), so we will only consider (d.ii). In fact, (e.i) also implies (e.ii) as follows. If M is finitely generated over S, then $M_{>N}$ is finitely generated over $k[\mathbf{yz}]$; under the isomorphism in (b), there is a finite set of generators $m_i = \sum_j t_{ij} \otimes u_{ij}$ with $t_{ij} \in T$ and $u_{ij} \in U$, all homogeneous. We can replace T by its $k[\mathbf{yz}]P$-submodule generated by the t_{ij}; then $T_{\le N-d}$ is finitely dimensional, so certainly of finite decomposition type.

On restriction to any proper subgroup Q of P, the complex $\check{C}_{\mathbf{yz}}(M)_{>N}$ is split exact, by induction; the same must be true for $\check{C}_{\mathbf{yz}}(T \otimes_k U)_{>N} = (\check{C}_{\mathbf{y}}(T) \otimes_k \check{C}_{\mathbf{z}}(U))_{>N}$, by Lemma 3.4.

The complex $\check{C}_{\mathbf{z}}(U)_d$ is semi-invertible, so there is a complex A in degree $-d$ such that $A \otimes_k \check{C}_{\mathbf{z}}(U)_d \simeq k \oplus B$, with k in degree 0. Hence $\check{C}_{\mathbf{y}}(T)$ is homotopy equivalent over kP to a summand of $A \otimes_k (\check{C}_{\mathbf{y}}(T) \otimes_k \check{C}_{\mathbf{z}}(U))$; the latter, on restriction to Q, is split exact in degrees greater than $N-d$, by the previous paragraph, (d.ii) and Lemma 3.5. Thus $\check{C}_{\mathbf{y}}(T)_{>N-d}$ is split exact on restriction to any proper subgroup of P.

By condition (a), $y_i \in I_{P,S}$, the ideal defining the fixed point subscheme. But $I_{P,S_{y_i}} = (I_{P,S})_{y_i}$, so $I_{P,S_{y_i}} = S_{y_i}$; this means that $(\mathrm{Spec}(S_{y_i}))^P = \emptyset$. Proposition 3.1 and Lemma 3.3 now imply that M_{y_i}, or indeed any $M_{y_{i_1}\cdots y_{i_\ell}}$ with $\ell \ge 1$, is projective relative to proper subgroups. But $M_{y_{i_1}\cdots y_{i_\ell}} \cong (M_{>N})_{y_{i_1}\cdots y_{i_\ell}} \cong ((T \otimes_k U)_{>N})_{y_{i_1}\cdots y_{i_\ell}} \cong (T \otimes_k U))_{y_{i_1}\cdots y_{i_\ell}} \cong T_{y_{i_1}\cdots y_{i_\ell}} \otimes_k U$, by Lemma 3.4. Since U is semi-invertible by condition (d.ii), $T_{y_{i_1}\cdots y_{i_\ell}}$ must be projective relative to proper subgroups for $\ell \ge 1$.

We can now use Lemma 3.6 with $\mathcal{C}$ equal to the class of proper subgroups of P to deduce that $\check{C}_{\mathbf{y}}(T)_{>N-d}$ is split exact over kP. From the assumption that $\check{C}_{\mathbf{z}}(U)_{>d}$ is split exact over kP and Lemma 3.5, it follows that $\check{C}_{\mathbf{yz}}(U \otimes_k T)_{>N}$ is split exact; hence so is $\check{C}_{\mathbf{yz}}(M)_{>N}$, by Lemma 3.4. This proves (1) in the case $(\mathbf{x}) = (\mathbf{yz})$.

Since U is semi-invertible, there exist kP-modules A and B, with A finite dimensional, such that $U \otimes_k A \cong k \oplus B$. Using Lemma 3.4 again, we obtain $M_{y_i} \otimes_k A \cong T_{y_i} \otimes_k U \otimes_k A \cong T_{y_i} \oplus (T_{y_i} \otimes_k B)$. By induction, M is of finite decomposition type on restriction to any proper subgroup; hence so is M_{y_i}, by Lemma 2.3, and also $M_{y_i} \otimes_k A$, since A is finite dimensional. It follows that T_{y_i} is of finite decomposition type after restriction to any proper subgroup.

Because we have seen that T_{y_i} is projective relative to proper subgroups, it follows from characterization (2) of relative projectivity that each T_{y_i} is of finite decomposition type over kG; hence so is $T_{>N-d}$, by the splitting of $\check{C}_{\mathbf{y}}(T)_{>N-d}$. Since $T_{\leq N-d}$ is of finite decomposition type by (e.ii), it follows that T is of finite decomposition type. Finally, U is assumed to be of finite decomposition type, hence so is $T \otimes U$ and thus $M_{>N}$. But $M_{\leq N}$ is assumed to be of finite decomposition type; hence so is M, proving (3).

By Proposition 2.5, we can now change $\mathbf{yz}$ to $\mathbf{x}$, which proves (1). □

Remark. Condition (d.ii) is rather artificial, but it serves to clarify the proof. Other formulations for non-trivial action on U are possible. The key point arises when $Q < P$ and (writing T_Q for the T associated to Q etc.) we need to be able to deduce that $\check{C}_{\mathbf{y}_P}(T_P)$ is split exact over kQ given that $\check{C}_{\mathbf{y}_Q}(T_Q)$ is.

Corollary 4.2. *If a finite group G acts by linear substitutions on a polynomial ring $S = k[x_1, \dots, x_n]$ with the x_i in degree 1, then S is of finite decomposition type and* $\operatorname{hreg}(S^G) \leq -n$ *and* $\operatorname{reg}(S^G) \leq 0$.

Proof. The condition on reg is weaker than that on hreg, so it suffices to verify the hypotheses of Theorem 4.1 with $N = -n$. The method is a variant of that in [16, §6]. Clearly $\operatorname{hreg}(S) \leq -n$.

Let P be a p-subgroup of G. Let V be the dual space to S_1, regarded as a left module in the usual way, and let $\{x_i^*\}$ be the dual basis; we regard S as $k[V]$. Letting $s = \dim V^P$ and $r = n - s$, we may change the basis of V (and of S_1) so that the matrices for the action on V are lower triangular and $x_{r+1}^*, \dots, x_n^*$ span V^P. For $g \in P$ we have

$$gx_i^* = \begin{cases} x_i^* + \sum_{i<j} \lambda_{i,j}(g) x_j^*, & \text{if } i \leq r \\ x_i^* & \text{if } i > r, \end{cases}$$

and thus

$$gx_i = x_i + \sum_{j<i,\, j\leq r} \lambda_{j,i}(g^{-1}) x_j.$$

Let $d_{x_i} = \prod_{g \in P/\operatorname{Stab} x_i} gx_i$ denote the orbit product of x_i. When expressed as a sum of monomials in the x_i, d_{x_i} involves $x_i^{\deg d_{x_i}}$ and does not contain any x_j with $j > i$.

Set $y_i = d_{x_i}$ for $1 \leq i \leq r$ and $z_i = d_{x_{i+r}}$ for $1 \leq i \leq s$. Let $\alpha_i = \deg z_i$.

Let T be the k-subspace of S spanned by the monomials with x_{i+r}-degree strictly less than α_i for $1 \leq i \leq s$; it is, in fact a kP-submodule. Let $U = k[z_1, \dots, z_s]$, with trivial action of P.

We claim that $S \cong T \otimes_k U$ by the natural map arising from the inclusion of T in S and multiplication by the z_i (cf. [16, 6.4]). The two sides have the same graded dimensions, so it suffices to check surjectivity; this can be done by a standard argument used in Gröbner basis theory. We order the basis elements by $x_n > x_{n-1} > \cdots > x_1$ and give the monomials the corresponding graded lexicographic order. Thus, if we write $z_i = x_{i+r}^{\alpha_i} + X_i$ for $1 \leq i \leq s$, each monomial appearing in X_i is strictly smaller than $x_{i+r}^{\alpha_i}$.

Suppose that a be the smallest monomial that is not in the image of the map above. If each x_{i+r} for $1 \leq i \leq s$ appears in a to a power strictly less than α_i, then $a \in T$ so is in the image. Otherwise, let j be the largest value of i with $1 \leq i \leq s$ such that $x_{i+r}^{\alpha_i}$ divides a; then $a = x_{j+r}^{\alpha_j} b = (z_j - X_j)b$. Since $b < a$, b is in the image, so is $z_j b$. Also, each monomial in $X_j b$ is strictly smaller than a, so must be in the image. Thus a is in the image, a contradiction.

This proves the claim, and the lettered hypotheses of Theorem 4.1 are satisfied for $M = S$ and any N. □

Remark. The finite decomposition type part of Corollary 4.2 was shown in [16] in the case when k is finite. The regularity part was the main result of [20].

Remark. Given a kG-module M, let $T(M) = \sum_{H<G,\, p \mid |G:H|} \mathrm{tr}_H^G(M^H)$ and set $F(M) = M^G/T(M)$, obtaining a functor with values in vector spaces. If M is also an RG-module and $x \in R$ then this preserves the R-module structure and it is easy to check that $T(M_x) \cong T(M)_x$ and hence $F(M_x) \cong F(M)_x$. It follows that $\check{C}_{\mathbf{x}}(F(M)) \cong F(\check{C}_{\mathbf{x}}(M))$.

In particular, take M to be the polynomial ring $S = k[V]$ of Corollary 4.2, where $\dim V = n$. The complex $\check{C}_{\mathbf{x}}(S)$ is split exact over kG in degrees greater than $-n$, hence so is $\check{C}_{\mathbf{x}}(F(M))$. Thus $\mathrm{hreg}(F(S)) \leq -n$.

It was shown by Fleischmann that $\dim F(k[V]) = \dim_k(V^P)$, where P is a Sylow p-subgroup of G. Thus $\mathrm{reg}(F(S)) \leq s - n$, where $s = \dim V^P$. Fleischmann's proof shows that a set of elements of $k[V]^G$ that restricts to a system of parameters on $k[V^P]$ also forms a system of parameters on $F(k[V])$. This allows us to find a bound on the degrees of the generators of $F(k[V])$ along the lines of [20, 2.1].

Using Dade's Lemma we can find a system of parameters of degree at most $|G|$; this leads to the bound $\max\{(s|G| - n, |G|\}$. If k is the finite field $\mathbb{F}_q$ and G is a p-group, then we can use invariants for the full group of lower triangular matrices and obtain the bound $\max\{q^{n-s}(q^s-1)/(q-1)-n, q^{n-1}\}$. If G is not a p-group, we can use the Dickson invariants; it is those of degrees $q^n - q^{n-1}, \ldots, q^n - q^{n-s}$ that restrict correctly (see e.g. [4, 8.1.2]). This yields the bound $\max\{sq^n - q^{n-s}(q^s - 1)/(q - 1) - n, q^n - q^{n-1}\}$.

Fleischmann [13, 14.1, 14.2] showed that $F(k[V])/\operatorname{rad}(F(k[V]))$ is Cohen-Macaulay. Totaro [21] has shown that $F(k[V])$ is Cohen-Macaulay and also suggested this remark.

5. Examples

There are several observations that can help in checking the hypotheses of Theorem 4.1. First of all, we only need to check representatives of the p-subgroups up to conjugacy. Also, if $Q < P$ and $(\mathrm{Spec}(S))^Q = (\mathrm{Spec}(S))^P$ then we do not need to check for Q. If we happen to know for some subgroup P and some sequence $x_1, \ldots, x_n$ in S^P that $\check{C}_{\mathbf{x}}(S)_{>N}$ is split over kP, then this can replace conditions (a)-(e) for P.

The last of these is particularly useful if we only want to show that S is of finite decomposition type. For then the conditions at each P only have to hold for N large enough, and this is ensured by Proposition 2.4 if S is known to be of finite decomposition type over P, for example if P is cyclic.

Example. There are many interesting examples when $S = k[x_1, \ldots, x_n]$, with the x_i in positive degrees but not all the same, and G acts by k-algebra automorphisms that respect the grading.

For a simple case, let us assume that each element $g \in G$ acts with x_1 fixed and $gx_i = x_i + \lambda_i x_1^{r_i}$ for $i \geq 2$, $\lambda_i \in k$ and $r_i \deg x_1 = \deg x_i$ if $\lambda_i \neq 0$. Assuming that the action is faithful, for any non-trivial subgroup the fixed point subscheme of the action on the spectrum is given by $x_1 = 0$. By the discussion above, we only have to verify the hypotheses for the whole group.

Let $y_1 = x_1$ and let $z_i = d_{x_{i+1}}$ for $1 \le i \le n-1$ (this is the orbit product again). Let T be the part of S spanned by monomials with x_i-degree strictly less that $\deg d_{x_i}$ for $2 \le i \le n$. Let $U = k[z_1, \dots, z_{n-1}]$. This satisfies the hypotheses of Theorem 4.1 with $N = \operatorname{hreg}(S) = -\sum \deg x_i$. Thus S has finite decomposition type and $\operatorname{hreg}(S^G) \le -\sum \deg x_i$.

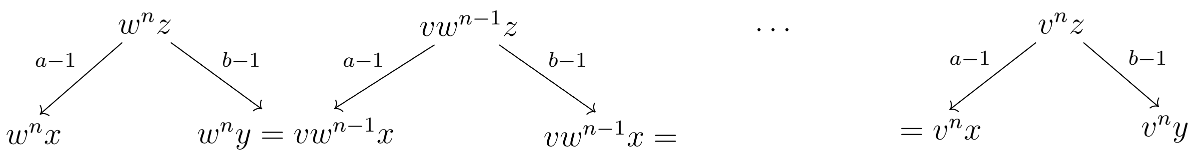
In the next two examples it is possible to calculate the invariants and their regularity directly, so we concentrate on finite decomposition type.

Example. Suppose that k has characteristic 2 and let $S = k[v, w, x, y, z]/(vx + wy)$, with all the generators in degree 1. Let G be the Klein four-group with generators a and b and let it act on S fixing v, w, x, y and with $a(z) = z + x$ and $b(z) = z + v$.

The fixed point subscheme of the action is where v and x vanish. Using the fact that $k[w, y]$ is free over $k[w + y, wy]$ on the basis $\{1, y\}$, we obtain $k[v, w, x, y, z] \cong (k[v, x] \oplus yk[v, x]) \otimes_k k[w + y, vx + wy, z]$. Set $y_1 = v$, $y_2 = w$, $z_1 = w + y$ and $z_2 = d_z$. Let T be the free $k[v, x]$-submodule of S spanned by $\{1, z, z^2, z^3, y, yz, yz^2, yz^3\}$ and $U = k[z_1, z_2]$. Then $S \cong T \otimes_k U$, verifying the conditions for G. By the discussion above, this suffices to show that S is of finite decomposition type.

Example. This example is the same as the previous one except that $b(z) = z + y$. The ring S is graded by total degree and also by v-degree+w-degree. Let A_n be the part of S with total degree $n+1$ and v-degree+w-degree=n. Then A_n is a kG-summand of S of dimension $2n+3$. We will show that A_n is indecomposable, which proves that S is not of finite decomposition type.

The module A_n has a monomial basis, after the obvious identifications. The group action can be described by a diagram, as in [10]:



.

This is known to be the diagram of an indecomposable module (in fact $\Omega^{-n-1}k$). Another approach is to consider matrices, as in [3, 4.3.3].

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