THE NON-PROJECTIVE PART OF THE TENSOR POWERS OF A MODULE

DAVE BENSON AND PETER SYMONDS

Abstract. Let M be a finite dimensional modular representation of a finite group G. We consider the dimensions of the non-projective part of the tensor powers $M^{\otimes n}$ of M, and we write $\gamma_G(M)$ for the lim sup of their nth roots. We investigate the properties of the invariant $\gamma_G(M)$, using tools from representation theory, and from the theory of commutative Banach algebras.

1. Introduction

This paper is exploratory in nature. We introduce a new invariant $\gamma_G(M)$ of a kG-module M (where, throughout the paper, G is a finite group, k is a field of characteristic p, and we only consider finitely generated kG-modules). This invariant is difficult to compute, but captures interesting asymptotic properties of tensor products. We have carried out a large number of computations, and we present a selection of them near the end of the paper.

The invariant $\gamma_G(M)$ is similar in nature to the *complexity* $c_G(M)$ [2, 9, 25], which describes the polynomial rate of growth of a minimal resolution of M, or equivalently the rate of growth of the non-projective part of $M \otimes \Omega^n(k)$ (here and for the remainder of the paper, $\otimes$ means $\otimes_k$ with diagonal group action). Instead, we examine the rate of growth of the dimension of the non-projective part of $M^{\otimes n}$. It turns out that in this case the growth is usually exponential, so the appropriate way to measure it is to consider lim sup of the nth root.

Definition 1.1. For a kG-module M, we write $M = M' \oplus (\mathsf{proj})$ where M' has no projective direct summands and (proj) denotes a projective module. Then M' is called the *core* of M and denoted $\mathsf{core}_G(M)$. We write $\mathsf{c}_n^G(M)$ for the dimension of $\mathsf{core}_G(M^{\otimes n})$; it is well defined by the Krull–Schmidt Theorem. We define

$$\gamma_G(M) = \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^G(M)}.$$

This is also equal to $1/r$, where r is the radius of convergence of the generating function

$$f_M(t) = \sum_{n=0}^{\infty} \mathsf{c}_n^G(M)t^n,$$

see Corollary 8.2.

In the case where M is algebraic (i.e., there are only finitely many isomorphism classes of indecomposable summands of tensor powers of M, see Alperin [1] or §II.5 of Feit [19])

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the invariant is a root of the minimal polynomial of M in the Green ring $a(G)$. Thus $\gamma_G(M)$ is closely related to the discussion in Chapter 3 of the book of Etingof, Gelaki, Nikshych and Ostrik [18]. Recall that $a(G)$ is the ring whose generators $[M]$ correspond to isomorphism classes of kG-modules, and whose relations are given by $[M \oplus N] = [M] + [N]$ and $[M \otimes N] = [M][N]$. Additively, the ring $a(G)$ is a free abelian group on the basis elements $[M]$ with M indecomposable. In general, the indecomposables are unclassifiable (wild representation type) and $a(G)$ is very large and hard to describe. It usually has nilpotent elements of the form $[M] - [N]$, but has a large semisimple quotient [5, 30, 31]. See Section 9 for more notations concerning Green rings.

All modules are algebraic if and only if the Sylow p-subgroups of G are cyclic. For an example, if G is a cyclic group of order five, k is a field of characteristic five, and M is the two dimensional indecomposable kG-module, then the non-projective summands of $M^{\otimes n}$ follow a Fibonacci pattern. We have $\mathsf{c}^G_{2n}(M) \approx \tau^{2n+1}$ and $\mathsf{c}^G_{2n+1}(M) \approx 2\tau^{2n+1}$, where

$$\tau = (1 + \sqrt{5})/2 = 2\cos(\pi/5) \approx 1.618034$$

is the golden ratio, and so $\gamma_G(M) = \tau$ in this case. We calculate the value of $\gamma_G(M)$ for all modules for a cyclic group of order p in Theorem 10.1.

Most modules are not algebraic. In the case where M is not algebraic, we interpret $\gamma_G(M)$ as the spectral radius of $[M]$ as an element of a suitable completion of the Green ring. This brings in the theory of commutative Banach algebras, playing the role of a sort of infinite dimensional Perron–Frobenius theory.

In common with the complexity, the value of $\gamma_G(M)$ is determined by the restrictions of M to elementary abelian p-subgroups of G. For this reason, most of our examples are modules for elementary abelian p-groups. The examples were worked out using the computer algebra system Magma [7]. Note that this paper is written using left modules, while Magma uses right modules, so the matrices have been transposed.

The following theorem summarises the results of this paper.

Theorem 1.2. *The invariant $\gamma_G(M)$ has the following properties:*

(i) *We have* $\gamma_G(M) = \lim_{n\to\infty} \sqrt[n]{\mathsf{c}^G_n(M)} = \inf_{n\geqslant 1} \sqrt[n]{\mathsf{c}^G_n(M)}$.
(ii) *We have* $0 \leqslant \gamma_G(M) \leqslant \dim M$.
(iii) *A kG-module M is p-faithful (Definition 6.1) if and only if* $\gamma_G(M) < \dim M$.
(iv) *A kG-module M is projective if and only if* $\gamma_G(M) = 0$. *Otherwise* $\gamma_G(M) \geqslant 1$.
(v) *If p divides $|G|$ then a kG-module M is endotrivial if and only if* $\gamma_G(M) = 1$.
(vi) *If a kG-module M is neither projective nor endotrivial then* $\gamma_G(M) \geqslant \sqrt{2}$.
(vii) *If* $\gamma_G(M) = \sqrt{2}$ *then* $M \otimes M^* \otimes M \cong M \oplus M \oplus (\mathsf{proj})$.
(viii) *We have* $\gamma_G(M^*) = \gamma_G(M)$.
(ix) *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a short exact sequence of kG-modules then each* $\gamma_G(M_i)$ *is at most the sum of the other two.*
(x) *We have* $\max\{\gamma_G(M), \gamma_G(N)\} \leqslant \gamma_G(M \oplus N) \leqslant \gamma_G(M) + \gamma_G(N)$.
(xi) *If N is isomorphic to a direct sum of m copies of M then* $\gamma_G(N) = m.\gamma_G(M)$.
(xii) *We have* $\gamma_G(k \oplus M) = 1 + \gamma_G(M)$.
(xiii) *We have* $\gamma_G(M \otimes N) \leqslant \gamma_G(M)\gamma_G(N)$.

(xiv) *We have* $\gamma_G(M^{\otimes m}) = \gamma_G(M)^m$.
(xv) *We have* $\gamma_G(\Omega M) = \gamma_G(M)$.
(xvi) *If* $H \leqslant G$ *then we have* $\gamma_H(M) \leqslant \gamma_G(M)$.
(xvii) *We have* $\gamma_G(M) = \max_{E \leqslant G} \gamma_E(M)$, *where the maximum is taken over the elementary abelian* p*-subgroups* $E \leqslant G$.
(xviii) *If* M *is a one dimensional module and* p *divides* $|G|$ *then for every* $n \in \mathbb{Z}$ *we have* $\gamma_G(\Omega^n(M)) = 1$.

The proofs of these can be found as follows: (i) in Theorem 4.4, (ii) and (iv) in Lemma 2.6, (iii) in Theorem 6.3, (v) in Theorem 7.5, (vi) in Theorem 5.8, (vii) in Theorem 12.6, (viii) in Lemma 2.7, (ix) in Corollary 5.6, (x) in Theorem 3.2, (xi) in Theorem 3.4, (xii) in Theorem 4.6, (xiii) in Theorem 5.1, (xiv) in Theorem 5.2, (xv) in Theorem 5.4, (xvi) in Lemma 2.10, and (xvii) in Theorem 7.2. Part (xviii) follows by combining part (xv) with Lemma 2.6 (iv).

We use the spectral theory of commutative Banach algebras to connect the invariant $\gamma_G(M)$ to the structure of the Green ring $a(G)$. Recall from Benson and Parker [6] that a *species* of $a(G)$ is a ring homomorphism $s\colon a(G) \to \mathbb{C}$ (we avoid the Banach algebra term "character" for obvious reasons). We say that a species s is *core-bounded* if for all kG-modules M we have $|s([M])| \leqslant \dim \mathsf{core}_G(M)$. The proof of the following theorem can be found in Section 9.

Theorem 1.3. *If* M *is a* kG*-module then* $\gamma_G(M)$ *is the supremum of* $|s([M])|$, *where* s *runs over the core-bounded species* $s\colon a(G) \to \mathbb{C}$. *Furthermore, there exists a core-bounded species* s *of* $a(G)$ *such that* $s([M]) = \gamma_G(M)$.

We formulate some conjectures about the behaviour of the invariant $\gamma_G(M)$, two of which we restate here:

Conjecture 1.4 (Conjecture 5.3)**.** *We have* $\gamma_G(M \otimes M^*) = \gamma_G(M)^2$.

Conjecture 1.5 (Conjecture 13.3)**.** *For all large enough values of* n, *the function* $\mathsf{c}_n^G(M)$ *satisfies a homogeneous linear recurrence relation with constant coefficients.*

The latter conjecture implies that the value of $\gamma_G(M)$ is always an algebraic integer. We do not know whether this is the case, but we at least show in Proposition 13.5 that $\gamma_G(M)$, for all primes, fields, finite groups and finitely generated modules, can only take countably many values.

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2. The Invariant $\gamma_G(M)$

We begin with some properties of tensor products.

Proposition 2.1. *Let M be a kG-module.*

(i) *If the dimension of M is not divisible by p then $M \otimes M^*$ has a direct summand isomorphic to k.*
(ii) *M is isomorphic to a direct summand of $M \otimes M^* \otimes M$.*
(iii) *If the dimension of M is divisible by p then $M \otimes M^* \otimes M$ has a direct summand isomorphic to $M \oplus M$.*

Proof. Let m_i be a basis for M and f_i the dual basis of M^*. Thus $\sum_i f_i(m_i) = \dim(M)$, and for $m \in M$ we have $m = \sum_i f_i(m)m_i$.

For (i) we have maps $k \to M \otimes M^*$ given by $1 \mapsto \sum_i m_i \otimes f_i$ and $M \otimes M^* \to M$ given by $m \otimes f \mapsto f(m)$, with composite multiplication by $\dim(M)$.

For (ii) we have maps $M \to M \otimes M^* \otimes M$ given by $m \mapsto \sum_i m \otimes f_i \otimes m_i$ and $M \otimes M^* \otimes M \to M$ given by $m \otimes f \otimes m' \mapsto f(m)m'$, with composite the identity on M.

For (iii) (cf. Proposition 4.9 in Auslander and Carlson [3], where this is proved with the further hypothesis that M is indecomposable, but this hypothesis is not used in the proof), we have maps $M \oplus M \to M \otimes M^* \otimes M$ given by

$$(m, m') \mapsto \sum_i (m \otimes f_i \otimes m_i + m_i \otimes f_i \otimes m')$$

and $M \otimes M^* \otimes M \to M \oplus M$ given by $m \otimes f \otimes m' \mapsto (f(m)m', f(m')m)$. If M has dimension divisible by p then the composite is the identity on $M \oplus M$. □

Lemma 2.2. *If M is a kG-module, then $M \otimes M^*$ is projective if and only if M is projective.*

Proof. The tensor product of any module with a projective module is projective. So by Proposition 2.1, M projective implies $M \otimes M^*$ projective implies $M \otimes M^* \otimes M$ projective implies M projective. □

Lemma 2.3. *For any kG-module M, if $M^{\otimes n}$ is projective for some $n \geqslant 1$, then so is M.*

Proof. It follows from Proposition 2.1 that $M^{\otimes(n-1)}$ is isomorphic to a direct summand of $M^{\otimes n} \otimes M^*$. So $M^{\otimes n}$ is projective if and only if $M^{\otimes(n-1)}$ is projective. The result now follows by induction on n. □

For a kG-module M, recall that we write $\mathsf{c}_n^G(M)$ for the dimension of $\mathsf{core}_G(M^{\otimes n})$.

Definition 2.4. We define

$$\gamma_G(M) = \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^G(M)}.$$

Remarks 2.5. (i) The invariant $\gamma_G(M)$ is robust, in that the dimension of $\mathsf{core}_G(M)$ may be replaced by the number of composition factors of $\mathsf{core}_G(M)$ or the number of composition factors of the socle of $\mathsf{core}_G(M)$, and so on.

(ii) An interesting invariant is $\gamma_G(M)/\dim M$, which we think of as the "non-projective proportion of M in the limit." In the example of the introduction, we have $\gamma_G(M) = \tau$. Thus $\gamma_G(M)/\dim M \approx 0.809$, and so we think of M as "about 19.1% projective in the limit."

(iii) We shall see in Section 4, using the theory of submultiplicative functions, that in fact $\lim_{n\to\infty} \sqrt[n]{\mathsf{c}_n^G(M)}$ exists and is equal to $\inf_{n\geqslant 1} \sqrt[n]{\mathsf{c}_n^G(M)}$.

We begin with some obvious properties of the invariant $\gamma_G(M)$.

Lemma 2.6. *For any kG-module we have:*

(i) $0 \leqslant \gamma_G(M) \leqslant \dim M$,

(ii) $\gamma_G(M) = 0$ *if and only if* M *is projective,*

(iii) *If* M *is not projective then* $\gamma_G(M) \geqslant 1$.

(iv) *If* M *is a one dimensional module and* $|G|$ *is divisible by* p *then* $\gamma_G(M) = 1$.

Proof. Part (i) is because

$$\mathsf{c}_n^G(M) = \dim \mathsf{core}_G(M^{\otimes n}) \leqslant \dim M^{\otimes n} = (\dim M)^n.$$

If M is projective then clearly $\gamma_G(M) = 0$. Conversely, if M is not projective then, by Lemma 2.3, no $\mathsf{c}_n^G(M)$ is 0. Since $\mathsf{c}_n^G(M)$ is a non-negative integer we have $\mathsf{c}_n^G(M) \geqslant 1$, proving parts (ii) and (iii). Part (iv) now follows, since M is not projective in this case. □

Lemma 2.7. *We have* $\gamma_G(M^*) = \gamma_G(M)$.

Proof. We have $\mathsf{core}_G(M^*) \cong \mathsf{core}_G(M)^*$ and so $\mathsf{c}_n^G(M^*) = \mathsf{c}_n^G(M)$. □

Recall that we have the syzygy operator Ω, where ΩM is defined to be the kernel of a projective cover $P \to M$. Similarly, $\Omega^{-1}M$ is defined to be the cokernel of an injective hull $M \to I$. Since projective kG-modules are the same as injective modules, we have $\Omega(\Omega^{-1}M) \cong \mathsf{core}_G(M) \cong \Omega^{-1}(\Omega M)$.

For $n > 0$, $\Omega^n M$ denotes $\Omega(\Omega^{n-1}M)$, $\Omega^{-n}M$ denotes $\Omega^{-1}(\Omega^{-n+1}M)$, and $\Omega^0 M$ denotes $\mathsf{core}_G(M)$. For $n \in \mathbb{Z}$ we have $\mathsf{core}_G(\Omega^n k \otimes M) \cong \Omega^n M$.

Lemma 2.8. *We have* $\gamma_G(\Omega k) = \gamma_G(\Omega^{-1}k) = 1$*, provided that* p *divides* $|G|$.

Proof. We have $\mathsf{core}_G((\Omega k)^{\otimes n}) \cong \Omega^n k$, and $\dim \Omega^n k$ grows polynomially in n (see for example [4] §5.3). Therefore $\gamma_G(\Omega k) = 1$. Since $(\Omega k)^* \cong \Omega^{-1}k$, Lemma 2.7 shows that $\gamma_G(\Omega^{-1}k) = 1$. □

Lemma 2.9. *If* $\mathsf{core}_G(M) \cong \mathsf{core}_G(N)$ *then* $\gamma_G(M) = \gamma_G(N)$.

Proof. If $\mathsf{core}_G(M) \cong \mathsf{core}_G(N)$ then, since the tensor product of a projective module with any module is projective, we have

$$\mathsf{core}_G(M^{\otimes n}) \cong \mathsf{core}_G(\mathsf{core}_G(M)^{\otimes n}) \cong \mathsf{core}_G(\mathsf{core}_G(N)^{\otimes n}) \cong \mathsf{core}_G(N^{\otimes n}).$$

Thus $\mathsf{c}_n^G(M) = \mathsf{c}_n^G(N)$, and so $\gamma_G(M) = \gamma_G(N)$. □

Lemma 2.10. *If* H *is a subgroup of* G *and* M *is a* kG*-module then* $\gamma_H(M) \leqslant \gamma_G(M)$.

Proof. We have $\mathsf{c}_n^H(M) \leqslant \mathsf{c}_n^G(M)$ for all n. □

Lemma 2.11. *If* K *is an extension field of* k *then* $\gamma_G(K \otimes_k M) = \gamma_G(M)$.

Proof. This will follow immediately if we can show that for any kG-module N, we have $K \otimes_k \mathsf{core}_G(N) \cong \mathsf{core}_G(K \otimes_k N)$. For this, we need to show that if $K \otimes_k N$ has a projective summand P then N also has a projective summand. Consider the restrictions of $K \otimes_k N$ and P from KG to kG. The restriction $P{\downarrow_{kG}}$ must be a sum of finite dimensional indecomposable projective kG-modules; let P' be one of them. It is a summand of $K \otimes_k N{\downarrow_{kG}}$, which is a sum of copies of N. Because it is finite dimensional, P' is a summand of a finite sum of copies of N, hence is a summand of N, by the Krull–Schmidt Theorem. $\square$

Example 2.12. Let M be the 3 dimensional faithful uniserial module for $G = \mathbb{Z}/3 \times \mathbb{Z}/3 = \langle g, h \rangle$ over $\mathbb{F}_3$ given by

$$g \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\Omega^2 M \cong M$, M is algebraic (Craven [13], Section 3.3.2), and ΩM has dimension 6. The modules $M' = M \otimes M^*$ and $\Omega M' \cong \Omega M \otimes M^* \cong M \otimes \Omega(M^*)$ are indecomposable, of dimensions 9 and 18 respectively. The indecomposable summands of tensor powers of M are determined by the equations

$$\begin{aligned} M \otimes M &\cong M^* \oplus \Omega(M^*), & M \otimes M' &\cong 2M \oplus 2\Omega M \oplus P, \\ M \otimes \Omega M &\cong M^* \oplus \Omega(M^*) \oplus P, & M \otimes \Omega M' &\cong 2M \oplus 2\Omega M \oplus 4P, \end{aligned}$$

where P is the 9 dimensional projective module. These equations imply that

$$M^{\otimes 5} \cong 8M^{\otimes 2} \oplus 19P.$$

It follows that for $n \geqslant 5$ we have $\mathsf{c}_n(M) = 8\mathsf{c}_{n-3}(M)$, and so $\gamma_G(M) = 2$. Similarly we have $\gamma_G(M') = 4$.

3. Short Exact Sequences and Direct Sums

Lemma 3.1. *Let a_n, b_n and c_n be sequences of non-negative real numbers, satisfying*

$$c_n \leqslant \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leqslant \limsup_{n\to\infty} \sqrt[n]{a_n} + \limsup_{n\to\infty} \sqrt[n]{b_n}.$$

Proof. The statement that $\limsup_{n\to\infty} \sqrt[n]{a_n} = \alpha$ implies that for all $\varepsilon > 0$, there exists m such that for all $n \geqslant m$ we have $a_n \leqslant (\alpha+\varepsilon)^n$. Introducing a positive constant A, we can assume that $a_n \leqslant A(\alpha+\varepsilon)^n$ for all $n \geqslant 0$. Similarly, if $\limsup_{n\to\infty} \sqrt[n]{b_n} = \beta$ then for all $\varepsilon > 0$ there exists a positive constant B such that for all $n \geqslant 0$ we have $b_n \leqslant B(\beta+\varepsilon)^n$. Thus for all $\varepsilon > 0$ there is a positive constant $C = AB$ such that for all $n \geqslant 0$ we have

$$c_n \leqslant \sum_{i=0}^{n} \binom{n}{i} A(\alpha+\varepsilon)^i B(\beta+\varepsilon)^{n-i} = C(\alpha+\beta+2\varepsilon)^n,$$

and so $\limsup_{n\to\infty} \sqrt[n]{c_n} \leqslant \alpha + \beta$. $\square$

Theorem 3.2. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of kG-modules then*

$$\gamma_G(M_2) \leqslant \gamma_G(M_1) + \gamma_G(M_3).$$

If the sequence splits then we also have

$$\max\{\gamma_G(M_1), \gamma_G(M_3)\} \leqslant \gamma_G(M_2).$$

Proof. The module $M_2^{\otimes n}$ has a filtration of length 2^n where the filtered quotients are $\binom{n}{i}$ copies of $M_1^{\otimes i} \otimes M_3^{\otimes(n-i)}$ $(0 \leqslant i \leqslant n)$. Projective summands of a filtered quotient split off the entire module, since they are also injective. So

$$\begin{aligned}\mathsf{c}_n^G(M_2) = \dim \mathsf{core}_G(M_2^{\otimes n}) &\leqslant \sum_{i=0}^{n} \binom{n}{i} \dim \mathsf{core}_G(M_1^{\otimes i} \otimes M_3^{\otimes(n-i)}) \\ &\leqslant \sum_{i=0}^{n} \binom{n}{i} \mathsf{c}_i^G(M_1)\mathsf{c}_{n-i}^G(M_3)\end{aligned}$$

Applying Lemma 3.1, we deduce that

$$\gamma_G(M_2) \leqslant \gamma_G(M_1) + \gamma_G(M_3).$$

If the sequence splits, then each $\mathsf{c}_n^G(M_2)$ is at least as big as $\mathsf{c}_n^G(M_1)$ and also at least as big as $\mathsf{c}_n^G(M_3)$. $\square$

Example 3.3. Let M be the two dimensional natural module for $SL(2, \mathbb{F}_4)$ and let N be its Frobenius twist. Then $M \otimes N$ is the four dimensional Steinberg module St, which is projective. Furthermore, $M^{\otimes 3} \cong M \oplus M \oplus \mathsf{St}$ and $N^{\otimes 3} \cong N \oplus N \oplus \mathsf{St}$. Since $M \otimes N$ is projective, for all $n \geqslant 1$ we have

$$\mathsf{core}((M \oplus N)^{\otimes n}) \cong \mathsf{core}(M^{\otimes n}) \oplus \mathsf{core}(N^{\otimes n}),$$

and so

$$\gamma_G(M) = \gamma_G(N) = \gamma_G(M \oplus N) = \sqrt{2}.$$

This shows that the first inequality in the theorem is not always an equality, even for direct sums.

We shall make further use of this example in Remark 5.9.

On the other hand, for sums of isomorphic modules, we have the following.

Theorem 3.4. *If N is isomorphic to a direct sum of m copies of a kG-module M then $\gamma_G(N) = m\gamma_G(M)$.*

Proof. The module $N^{\otimes n}$ is isomorphic to a direct sum of m^n copies of $M^{\otimes n}$, so we have $\mathsf{c}_n^G(N) = m^n \mathsf{c}_n^G(M)$ and $\sqrt[n]{\mathsf{c}_n^G(M)} = m(\sqrt[n]{\mathsf{c}_n^G(M)})$. Now take $\limsup_{n\to\infty}$. $\square$

Here is another useful bound.

Theorem 3.5. *If $M_1 \otimes \cdots \otimes M_m$ is not projective then $\gamma_G(M_1 \oplus \cdots \oplus M_m) \geqslant m$.*

Proof. By Lemma 2.3, no power of $M_1 \otimes \cdots \otimes M_m$ is projective, so neither is any module of the form $M_1^{i_1} \otimes \cdots \otimes M_m^{i_m}$. The module $(M_1 \oplus \cdots \oplus M_m)^{\otimes n}$ thus has at least m^n non-projective summands, so we have

$$\mathsf{c}_n^G(M_1 \oplus \cdots \oplus M_m) \geqslant m^n$$

and

$$\sqrt[n]{\mathsf{c}_n^G(M_1 \oplus \cdots \oplus M_m)} \geqslant m.$$

Now take $\limsup_{n\to\infty}$. $\square$

Corollary 3.6. *If M is not projective then $\gamma_G(k \oplus M) \geqslant 2$.*

Proof. This follows by taking $m = 2$, $M_1 = k$ and $M_2 = M$ in Theorem 3.5. $\square$

Remark 3.7. We shall prove in the next section, using the theory of submultiplicative sequences, that $\gamma_G(k \oplus M)$ is always equal to $1 + \gamma_G(M)$.

4. Submultiplicative Sequences

In this section, we investigate the submultiplicative properties of γ_G, and deduce Theorem 4.6. We shall revisit this from the point of view of Banach algebras and Gelfand's spectral radius theorem later on, but for the moment we shall try to stay elementary.

Definition 4.1. We say that a sequence $c_0, c_1, c_2, \ldots$ of non-negative real numbers is *submultiplicative* if $c_0 = 1$, and for all $m, n \geqslant 0$ we have $c_{m+n} \leqslant c_m.c_n$.

Lemma 4.2. *If M is a kG-module then $\mathsf{c}_n(M)$ is a submultiplicative sequence, provided that p divides $|G|$.*

Proof. This follows from the fact that

$$\mathsf{core}_G(M^{\otimes m} \otimes M^{\otimes n}) \cong \mathsf{core}_G(\mathsf{core}_G(M^{\otimes m}) \otimes \mathsf{core}_G(M^{\otimes n})). \qquad \square$$

Lemma 4.3 (Fekete [20])**.** *If c_n is a submultiplicative sequence then*

$$\limsup_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \sqrt[n]{c_n} = \inf_{n\geqslant 1} \sqrt[n]{c_n}.$$

Proof. It suffices to show that $\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \inf_{n\geqslant 1} \sqrt[n]{c_n}$. If some c_n is equal to zero, then so are all subsequent ones. So we assume that all $c_n > 0$. Suppose that L is a number such that

$$\inf_{n\to\infty} \sqrt[n]{c_n} < L.$$

Then there is an $m \geqslant 1$ with $\sqrt[m]{c_m} < L$. For $n > m$ we use division with remainder to write $n = mq_m + r_m$ with $0 \leqslant r_m < m$. By the definition of submultiplicativity, we have

$$c_n = c_{mq_m + r_m} \leqslant c_{mq_m} c_{r_m} \leqslant (c_m)^{q_m} c_{r_m}.$$

Now $q_m \leqslant n/m$, so $q_m/n \leqslant 1/m$. So we have

$$\sqrt[n]{c_n} \leqslant \sqrt[m]{c_m}\sqrt[n]{c_{r_m}} < L.\sqrt[n]{c_{r_m}}.$$

As n tends to infinity, the numbers $\sqrt[n]{c_0}, \ldots, \sqrt[n]{c_{m-1}}$ all tend to one, and so

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leqslant L. \qquad \square$$

Theorem 4.4. *If M is a kG-module then $\gamma_G(M) = \lim_{n\to\infty} \sqrt[n]{\mathsf{c}_n^G(M)} = \inf_{n\geqslant 1} \sqrt[n]{\mathsf{c}_n^G(M)}$.*

Proof. This follows from Lemmas 4.2 and 4.3. $\square$

Proposition 4.5. *Suppose that a_n and b_n are submultiplicative sequences. Define a sequence c_n by*

$$c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

Then c_n is also a submultiplicative sequence, and we have

$$\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \sqrt[n]{a_n} + \lim_{n\to\infty} \sqrt[n]{b_n}.$$

Proof. Using the fact that

$$\binom{m+n}{\ell} = \sum_{i+j=\ell} \binom{m}{i}\binom{n}{j}$$

and the submultiplicativity of the sequences a_n and b_n, we have

$$\sum_{\ell=0}^{m+n} \binom{m+n}{\ell} a_\ell b_{m+n-\ell} \leqslant \left(\sum_{i=0}^{m} \binom{m}{i} a_i b_{m-i}\right) \cdot \left(\sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}\right)$$

and so the sequence c_n is submultiplicative.

By Lemma 3.1 we have

$$\lim_{n\to\infty} \sqrt[n]{c_n} \leqslant \lim_{n\to\infty} \sqrt[n]{a_n} + \lim_{n\to\infty} \sqrt[n]{b_n}.$$

The reverse inequality is proved similarly. If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \alpha$ and $\lim\limits_{n\to\infty} \sqrt[n]{b_n} = \beta$ then given $\varepsilon > 0$ there exist positive constants A and B such that for all $n \geqslant 0$ we have $a_n \geqslant A(\alpha-\varepsilon)^n$ and $b_n \geqslant B(\beta-\varepsilon)^n$. So for all $\varepsilon > 0$ there is a positive constant $C = AB$ such that for all $n \geqslant 0$ we have

$$c_n \geqslant \sum_{i=0}^{n} \binom{n}{i} A(\alpha-\varepsilon)^i B(\beta-\varepsilon)^{n-i} = C(\alpha+\beta-2\varepsilon)^n,$$

and so

$$\lim_{n\to\infty} \sqrt[n]{c_n} \geqslant \alpha+\beta-2\varepsilon = \lim_{n\to\infty} \sqrt[n]{a_n} + \lim_{n\to\infty} \sqrt[n]{b_n} - 2\varepsilon. \qquad \square$$

Theorem 4.6. *If p divides $|G|$ and M is a kG-module then we have $\gamma_G(k \oplus M) = 1 + \gamma_G(M)$.*

Proof. We have

$$\mathsf{c}_n^G(k \oplus M) = \sum_{i=0}^{n} \binom{n}{i} \mathsf{c}_i^G(M).$$

So we can apply Proposition 4.5 with $a_n = \mathsf{c}_n^G(M)$, $b_n = 1$, and $c_n = \mathsf{c}_n^G(k \oplus M)$. □

Corollary 4.7. *If If p divides $|G|$, M is a kG-module, and N is isomorphic to a direct sum of a copies of k and b copies of M then we have $\gamma_G(N) = a + b\gamma_G(M)$.*

Proof. This follows inductively from Theorems 3.4 and 4.6. □

5. Tensor Products

Theorem 5.1. *We have* $\gamma_G(M \otimes N) \leqslant \gamma_G(M)\gamma_G(N)$.

Proof. We have

$$\mathsf{core}_G(M \otimes N) = \mathsf{core}_G(\mathsf{core}_G(M) \otimes \mathsf{core}_G(N)).$$

Therefore

$$\mathsf{c}_n^G(M \otimes N) \leqslant \mathsf{c}_n^G(M)\mathsf{c}_n^G(N)$$

and

$$\sqrt[n]{\mathsf{c}_n^G(M \otimes N)} \leqslant \sqrt[n]{\mathsf{c}_n^G(M)}\,\sqrt[n]{\mathsf{c}_n^G(N)}.$$

Now apply $\limsup_{n\to\infty}$ to both sides. □

This inequality may be strict. For example, it is possible for $M \otimes N$ to be projective with neither M nor N projective. However for tensor powers of a single module, we have the following.

Theorem 5.2. *We have* $\gamma_G(M^{\otimes m}) = \gamma_G(M)^m$.

Proof. By Theorem 5.1 we have $\gamma_G(M^{\otimes m}) \leqslant \gamma_G(M)^m$. Conversely, if $n = ms + i$ with $0 \leqslant i < m$ then

$$M^{\otimes n} = M^{\otimes i} \otimes (M^{\otimes m})^{\otimes s}$$

and so

$$\mathsf{c}_n^G(M) \leqslant (\dim M)^m \mathsf{c}_s^G(M^{\otimes m}).$$

Thus

$$\sqrt[n]{\mathsf{c}_n^G(M)} \leqslant \sqrt[ms]{\mathsf{c}_n^G(M)} \leqslant \sqrt[s]{\dim M}\,\sqrt[m]{\sqrt[s]{\mathsf{c}_s^G(M^{\otimes m})}}.$$

Applying $\limsup_{n\to\infty}$, the factor $\sqrt[s]{\dim M}$ tends to 1. It follows that

$$\gamma_G(M) \leqslant \sqrt[m]{\gamma_G(M^{\otimes m})}.$$ □

The following conjecture is based on extensive computations, but we have failed to find a proof. See Remark 9.19 for an interpretation in terms of Banach algebas.

Conjecture 5.3. *We have* $\gamma_G(M \otimes M^*) = \gamma_G(M)^2$.

Theorem 5.4. *We have* $\gamma_G(\Omega M) = \gamma_G(M)$.

Proof. We have $\mathsf{core}_G(\Omega k \otimes M) \cong \mathsf{core}(\Omega M) = \Omega M$. So by Lemma 2.9, Theorem 5.1 and Lemma 2.8 we have

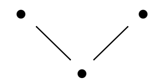
$$\gamma_G(\Omega M) = \gamma_G(\Omega k \otimes M) \leqslant \gamma_G(\Omega k)\gamma_G(M) = \gamma_G(M).$$

The reverse inequality follows in the same way from the fact that

$$\mathsf{core}_G(\Omega^{-1}k \otimes \Omega M) \cong \mathsf{core}_G(M).$$ □

Example 5.5. Let M be the three dimensional module $\mathsf{Soc}^2(kG)$ for $G = \mathbb{Z}/3\times\mathbb{Z}/3 = \langle g, h\rangle$ over $\mathbb{F}_3$, given by the following matrices, which has the diagram shown:

$$g \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



In this diagram and those in Section 15, the vertices represent basis vectors. The actions of $g-1$ and $h-1$ are represented by the lines going down to the left, respectively down to the right from the vertex, or zero if there is no such line. Then M is non-periodic and non-algebraic, and (Craven [13], Section 3.3.2) we have

$$M \otimes M \cong M^* \oplus \Omega(M^*).$$

Here, $M^* \cong kG/\mathsf{Rad}^2(kG)$ has dimension three, and $\Omega(M^*) \cong \mathsf{Soc}^3(kG)$ has dimension six. Using this, and the fact that $M' = M \otimes M^*$ is a non-projective indecomposable module, it is easy to compute that $\mathsf{core}_G(M^{\otimes n})$ has 2^{n-2} non-projective summands if n is divisible by three, and 2^{n-1} non-projective summands otherwise. So using Theorem 3.5 we have $\mathsf{c}^G_n(M) \geqslant 2^{n-2}$ and $\gamma_G(M) \geqslant 2$. On the other hand, using Theorem 5.2, Theorem 3.2, Lemma 2.7 and Theorem 5.4 we have

$$\gamma_G(M)^2 = \gamma_G(M \otimes M) = \gamma_G(M^* \oplus \Omega(M^*)) \leqslant \gamma_G(M^*) + \gamma_G(\Omega(M^*)) = 2\gamma_G(M)$$

and so $\gamma_G(M) \leqslant 2$. Combining these, we have $\gamma_G(M) = 2$. Similarly, we have $\gamma_G(M') = 4$.

Corollary 5.6. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of kG-modules. Then $\gamma_G(M_i) \leqslant \gamma_G(M_j) + \gamma_G(M_\ell)$, for any $\{i, j, \ell\} = \{1, 2, 3\}$.*

Proof. This follows from Theorems 3.2 and 5.4, together with the observation that there are short exact sequences

$$\begin{gathered} 0 \to M_2 \to M_3 \oplus (\mathsf{proj}) \to \Omega^{-1}(M_1) \oplus (\mathsf{proj}) \to 0 \\ 0 \to \Omega(M_3) \oplus (\mathsf{proj}) \to M_1 \oplus (\mathsf{proj}) \to M_2 \to 0. \end{gathered}$$

□

Definition 5.7. Recall that a kG-module M is *endotrivial* if $M \otimes M^* \cong k \oplus (\mathsf{proj})$.

Theorem 5.8. *If M is neither projective nor endotrivial then $\gamma_G(M) \geqslant \sqrt{2}$.*

Proof. Suppose that M is neither projective nor endotrivial. We divide into two cases according to whether the dimension of M is divisible by p.

If the dimension of M is divisible by p, then by Proposition 2.1, $(M \otimes M^*)^{\otimes 2}$ has a direct summand isomorphic to a direct sum of two copies of $M \otimes M^*$. Thus, $\gamma_G(M \otimes M^*)^2 = \gamma_G((M \otimes M^*)^{\otimes 2}) \geqslant 2\gamma_G(M \otimes M^*)$. As M is not projective, neither is $M \otimes M^*$ by Lemma 2.2 and we have $\gamma_G(M \otimes M^*) \neq 0$, so $\gamma_G(M \otimes M^*) \geqslant 2$.

On the other hand, if the dimension of M is not divisible by p and M is not endotrivial then, by Proposition 2.1, $M \otimes M^* \cong k \oplus X$ with X non-projective. Then $\gamma_G(X) \geqslant 1$, so $\gamma_G(M \otimes M^*) \geqslant 2$, by Corollary 3.6.

In both cases, using Theorem 5.1 and Lemma 2.7, we have

$$2 \leqslant \gamma_G(M \otimes M^*) \leqslant \gamma_G(M)\gamma_G(M^*) = \gamma_G(M)^2$$

and so $\gamma_G(M) \geqslant \sqrt{2}$. □

Remark 5.9. Example 3.3 shows that equality can occur in Theorem 5.8. See Section 12 for more about what happens when $\gamma_G(M) \geqslant \sqrt{2}$.

We can deduce a theorem of Carlson on finite dimensional idempotent kG-modules (see Theorem 3.5 of [11]) as a corollary.

Corollary 5.10. *If $M \otimes M \cong M \oplus (\mathsf{proj})$ and M is not projective then $M \cong k \oplus (\mathsf{proj})$.*

Proof. The hypothesis implies that $\gamma_G(M) = 1$, so by Theorem 5.8, M is endotrivial. The endotrivial modules (modulo projective summands) form a group under tensor product, so the only idempotent element is the identity. □

6. Faithful Modules

Definition 6.1. We say that a kG-module M is *p-faithful* if it is not 0 and no element of order p in G acts trivially on M. So faithful implies p-faithful, and p-faithful is equivalent to being faithful on restriction to a Sylow p-subgroup of G.

Lemma 6.2. *Let M be a kG-module. Then some tensor power $M^{\otimes n}$ with $n \geqslant 1$ has a non-zero projective summand if and only if M is p-faithful.*

Proof. The lemma is clearly true if $M = 0$, so assume that $M \neq 0$. If M is not p-faithful, then there is an element $g \in G$ of order p acting trivially on M. It therefore acts trivially on $M^{\otimes n}$, so this module has no projective summands.

On the other hand, if M is p-faithful then the kernel of the action on M is a normal p'-subgroup $H \leqslant G$. Projective kG/H-modules are projective kG-modules, so we may assume that $H = 1$. This case follows from the Corollary to Theorem 1 of Bryant and Kovács [8]. □

Theorem 6.3. *We have $\gamma_G(M) < \dim M$ if and only if M is p-faithful.*

Proof. Again, we may assume that $M \neq 0$. We use Lemma 6.2. If M is not p-faithful then for all n we have $\mathsf{core}_G(M^{\otimes n}) = M^{\otimes n}$ and so $\gamma_G(M) = \dim M$. Conversely, if M is p-faithful, then some tensor power has a projective summand, say $M^{\otimes m} = P \oplus N$ with P a non-zero projective module. Thus using Theorem 5.2 we have

$$\gamma_G(M)^m = \gamma_G(M^{\otimes m}) \leqslant \dim N < (\dim M)^m$$

and so $\gamma_G(M) < \dim M$. □

7. Restriction to Elementary Abelian Subgroups

Theorem 7.1. *There exists a constant B, which depends only on p and G, such that if M is a kG-module then*

$$\dim \mathsf{core}_G(M) \leqslant B \max_{E \leqslant G} \dim \mathsf{core}_E(M)$$

where the maximum is taken over the set of elementary abelian p-subgroups E of G.

Proof. See Theorem 3.7 of Carlson [10]. □

Theorem 7.2. *Let M be a kG-module. Then $\gamma_G(M) = \max_{E \leqslant G} \gamma_E(M)$.*

Proof. By Theorem 7.1 and Lemma 2.10 we have

$$\max_{E\leqslant G} \sqrt[n]{\mathsf{c}_n^E(M)} \leqslant \sqrt[n]{\mathsf{c}_n^G(M)} \leqslant \sqrt[n]{B} \max_{E\leqslant G} \sqrt[n]{\mathsf{c}_n^E(M)}.$$

Taking $\limsup_{n\to\infty}$, the factor of $\sqrt[n]{B}$ tends to 1. □

Example 7.3. Let G be a generalised quaternion group and let k be a field of characteristic two. Then G has only one elementary abelian 2-subgroup $E = \langle z\rangle$, where z is the central element of order two. Let $X = 1 + z$, an element of kG satisfying $X^2 = 0$. If M is a kG-module then the restriction to kE is a direct sum of $\dim(\mathsf{Ker}(X, M)/\mathsf{Im}(X, M))$ copies of the trivial module plus a free module. It follows that

$$\gamma_G(M) = \dim(\mathsf{Ker}(X, M)/\mathsf{Im}(X, M)).$$

In particular, this is an integer.

Proposition 7.4 (Dade [16, 17])**.** *If E is an elementary abelian p-group, then the only indecomposable endotrivial kE-modules are the syzygies $\Omega^n(k)$ ($n \in \mathbb{Z}$) of the trivial module.* □

Theorem 7.5. *If p divides $|G|$ then a kG-module M is endotrivial if and only if $\gamma_G(M) = 1$.*

Proof. If M is projective then $\gamma_G(M) = 0$. If M is neither projective nor endotrivial then by Theorem 5.8 we have $\gamma_G(M) \geqslant \sqrt{2}$.

Conversely if M is endotrivial then its restriction to every elementary abelian p-subgroup of G is endotrivial. So by Theorem 7.2, we may assume that $G = E$ is an elementary abelian p-group. Since an endotrivial module is a direct sum of an indecomposable endotrivial module and a projective module, we may assume that M is indecomposable. By Proposition 7.4, M is a syzygy of the trivial module, so by Theorem 5.4 we have $\gamma_E(M) = 1$. □

Warning 7.6. If E is an elementary abelian p-group and M is a kE-module then $\gamma_E(M)$ does depend on the Hopf algebra structure of kE. If we regard kE as the universal enveloping algebra of a restricted Lie algebra with trivial bracket and trivial pth power map, and we use the corresponding comultiplication, then $\gamma_E(M)$ may change. For example, restrict the module of Example 3.3 to a Sylow 2-subgroup, which is elementary abelian of order four. Then by Theorem 7.2, we have $\gamma_E(M) = \sqrt{2}$. But if we use the Lie comultiplication then $M \otimes M \cong M \oplus M$, and so $\gamma_E(M) = 2$.

8. Radius of Convergence

Another way of studying the invariant $\gamma_G(M)$ is to consider power series; we begin with a well known lemma from analysis.

Lemma 8.1 (Cauchy, Hadamard)**.** *Let $\phi\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$. Then the radius of convergence r of the power series*

$$f(t) = \sum_{n=0}^{\infty} \phi(n)t^n$$

is given by

$$1/r = \limsup_{n\to\infty} \sqrt[n]{|\phi(n)|}.$$

For $|t| < r$, the convergence is uniform and absolute.

Proof. See for example Conway [12], Theorem III.1.3. □

Corollary 8.2. *Let M be a kG-module. Consider the power series*

$$f_M(t) = \sum_{n=0}^{\infty} \mathsf{c}_n^G(M)t^n,$$

and let r be the radius of convergence of $f_M(t)$. Then

$$1/r = \gamma_G(M).$$

The following theorem will be used in Section 13.

Theorem 8.3 (Pringsheim)**.** *Suppose that $\phi\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, and that the power series*

$$f(t) = \sum_{n=0}^{\infty} \phi(n)t^n$$

has radius of convergence r. Then $t = r$ is a singular point of $f(t)$.

Proof. See Statement (7.21) in Chapter VII of Titchmarsh [29]. □

9. Banach Algebras

We recall the basics of the theory of norms and spectral radius, referring to Chapters 17–18 of Lax [26] for proofs. We always work over the field of complex numbers.

Definition 9.1. A *normed space* is a vector space B over $\mathbb{C}$, together with a norm $B \to \mathbb{R}$, $x \mapsto \|x\|$, satisfying

$$\|x + y\| \leqslant \|x\| + \|y\|, \qquad \|cx\| = |c|\|x\|,$$

for $x, y \in B$, $c \in \mathbb{C}$, such that $\|x\| \geqslant 0$ and $\|x\| = 0$ if and only if $x = 0$. A *Banach space* is a normed space that is complete with respect to the norm.

A (unital) *normed algebra* is an associative algebra A over $\mathbb{C}$ with identity $\mathbb{1}$ that is also a normed space, with the norm satisfying the additional conditions

$$\|\mathbb{1}\| = 1, \qquad \|xy\| \leqslant \|x\|\|y\|$$

for $x, y \in A$. A *Banach algebra* is a normed algebra that is also a Banach space. Note that we are assuming that all our Banach algebras are unital.

If a is an element of a Banach algebra A, we write $\sigma(a)$ for the *spectrum* of a, namely the set of $\lambda \in \mathbb{C}$ such that $\lambda\mathbb{1} - a$ is not invertible in A. It is a non-empty closed bounded subset of $\mathbb{C}$. The *spectral radius* of $a \in A$, denoted $\rho(a)$, is defined to be $\sup\limits_{\lambda \in \sigma(a)} |\lambda|$.

Notice that if A is finite dimensional and $a \in A$, then the spectral radius $\rho(a)$ is just the largest absolute value of an eigenvalue of the linear map induced by multiplying by a.

Let $A(G) = \mathbb{C} \otimes_{\mathbb{Z}} a(G)$, where $a(G)$ is the *Green ring* or *representation ring* of finitely generated kG-modules. Following [6], we write elements of $A(G)$ in the form $\sum_i a_i[M_i]$ where $a_i \in \mathbb{C}$ and M_i are indecomposable kG-modules. If $M = \bigoplus_i M_i^{n_i}$ is a kG-module, we write $[M]$ for $\sum_i n_i[M_i] \in A(G)$. Multiplication is extended bilinearly from $[M][N] = [M \otimes N]$.

We write $a(G,1)$ for the ideal of $a(G)$ spanned by the elements $[P]$ with P projective, and $a_0(G,1)$ for the linear span of the elements of the form $[M_2]-[M_1]-[M_3]$ where $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence. Then defining $A(G,1) = \mathbb{C} \otimes_{\mathbb{Z}} a(G,1)$ and $A_0(G,1) = \mathbb{C} \otimes_{\mathbb{Z}} a_0(G,1)$, we have

$$A(G) = A(G,1) \oplus A_0(G,1).$$

We put a norm on $A(G)/A(G,1) \cong A_0(G,1)$ by setting

$$\left\| \sum_i a_i[M_i] \right\| = \sum_i |a_i| \dim \mathsf{core}_G(M_i) = \sum_{M_i \text{ non-projective}} |a_i| \dim M_i.$$

The reason for choosing this particular norm is that it has two good properties:

(i) If M is a kG-module then $\|[M]\| = \dim \mathsf{core}_G(M)$; thus $\|\cdot\|$ is additive on direct sums of genuine (as opposed to virtual) modules.

(ii) If H is a subgroup of G then restriction can only reduce the norm: $\|M{\downarrow}_H^G\| \leqslant \|M\|$.

This makes $A(G)/A(G,1)$ into a normed algebra, which we may complete with respect to the norm to obtain a commutative Banach algebra which we shall denote $\hat{A}_1(G)$. Thus $A(G)/A(G,1)$ is a dense subalgebra of $\hat{A}_1(G)$.

Warning 9.2. If p does not divide $|G|$ then $\hat{A}_1(G) = 0$, which is not a Banach algebra because it does not satisfy the condition $\|\mathbb{1}\| = 1$. In this paper we always implicitly assume that the characteristic of the field k divides the order of the group.

The role of the invariant $\gamma_G(M)$ in this context is that by Theorem 4.4 we have

$$\gamma_G(M) = \lim_{n\to\infty} \sqrt[n]{\|[M]^n\|}. \tag{9.3}$$

Proposition 9.4 (Spectral radius formula, Gelfand [21])**.** *If A is a Banach algebra and $a \in A$ then the spectral radius of a is related to the norm by the formula*

$$\rho(a) = \lim_{n\to\infty} \sqrt[n]{\|a^n\|}. \tag{9.5}$$

Proof. See for example §17.1 of Lax [26]. □

Theorem 9.6. *If M is a kG-module then $\gamma_G(M) = \rho([M])$, where $[M]$ is the corresponding element of $\hat{A}_1(G)$.*

Proof. This follows from (9.3) and Proposition 9.4. □

Lemma 9.7. *Let a be an element in a Banach algebra A, with $\rho(a) = r$. Then r (as a real number) is an element of $\sigma(a) \subseteq \mathbb{C}$ if and only if $\rho(\mathbb{1}+a) = 1+r$.*

Proof. It is clear that $\sigma(\mathbb{1}+a)$ is the set of $1+\lambda$ with $\lambda \in \sigma(a)$. So $\sigma(\mathbb{1}+a)$ is contained in a disc of radius r centred at $1 \in \mathbb{C}$. The only point in this disc at distance $1+r$ from the origin is the real number $1+r$. Now using the fact that $\sigma(a)$ is closed, we see that the spectral radius of $\mathbb{1}+a$ is $1+r$ if and only if $1+r \in \sigma(\mathbb{1}+a)$, namely if and only if $r \in \sigma(a)$. □

Theorem 9.8. *Let M be a kG-module. Then the real number $\gamma_G(M)$ is an element of $\sigma([M]) \subseteq \mathbb{C}$.*

Proof. By Theorem 4.6, we have $\gamma_G(k \oplus M) = 1 + \gamma_G(M)$. By Theorem 9.6, it follows that on $\hat{A}_1(G)$ we have $\rho(\mathbb{1} + [M]) = 1 + \rho([M])$. By Lemma 9.7 this implies that $\gamma_G(M) \in \sigma([M])$. $\square$

The way to connect spectral radius with the species of the Green ring in the sense of Benson and Parker [6] is the following.

Theorem 9.9. *An element a of a commutative Banach algebra A is invertible if and only if $\phi(a) \neq 0$ for all algebra homomorphisms $\phi\colon A \to \mathbb{C}$.*

Proof. See Theorem 3 in Chapter 18 of Lax [26]. $\square$

Remark 9.10. Note that if A is a commutative Banach algebra and $\phi\colon A \to \mathbb{C}$ is an algebra homomorphism then for all $a \in A$ we have $|\phi(a)| \leqslant \|a\|$. It follows that ϕ is automatically continuous with respect to the norm. See Theorem 1 in Chapter 18 of Lax [26].

Corollary 9.11. *If a is an element of a commutative Banach algebra then $\sigma(a)$ is the set of values of $\phi(a)$ as ϕ runs over the algebra homomorphisms $A \to \mathbb{C}$. The spectral radius $\rho(a)$ is equal to* $\sup\limits_{\phi\colon A \to \mathbb{C}} |\phi(a)|$.

Proof. It follows from Theorem 9.9 that $\lambda\mathbb{1} - a$ is not invertible if and only if there exists an algebra homomorphism $\phi\colon A \to \mathbb{C}$ such that $\phi(a) = \lambda$. $\square$

Definition 9.12. Recall from [6] that a *species* of $a(G)$ is a ring homomorphism $s\colon a(G) \to \mathbb{C}$. A species of $a(G)$ extends to give an algebra homomorphism $s\colon A(G) \to \mathbb{C}$, and all algebra homomorphisms have this form.

We say that a species s of $a(G)$ is *core-bounded* if for all kG-modules M we have

$$|s([M])| \leqslant \dim \mathsf{core}_G(M).$$

In particular, the extension of a core-bounded species to $A(G)$ vanishes on $A(G, 1)$, and so defines an algebra homomorphism $A(G)/A(G, 1) \to \mathbb{C}$. So for example the Brauer species, namely the ones that vanish on $A_0(G, 1)$, are not core-bounded because they do not vanish on projective modules.

Lemma 9.13. *If s is any core-bounded species then $|s([\Omega^i(k)])| = 1$ for any $i \in \mathbb{Z}$.*

Proof. We have $|s([\Omega k])|^i = |s([\Omega k]^i)| = |s([\Omega^i k])| \leqslant \dim \mathsf{core}_G(\Omega^i k)$. But $\dim \mathsf{core}_G(\Omega^i k)$ grows polynomially in i (see for example [4] §5.3), so $|s([\Omega k])| \leqslant 1$. The same holds for $|s([\Omega^{-1} k])|$; but $|s([\Omega k])|\,|s([\Omega^{-1} k])| = |s([k])| = 1$, so we must have $|s([\Omega k])| = |s([\Omega^{-1} k])| = 1$. The general case follows from the first formula in this proof. $\square$

Example 9.14. Examining the species for $\mathbb{Z}/2 \times \mathbb{Z}/2$ described in Appendix 1 of [6], we see that not every species that vanishes on $A(G, 1)$ is core-bounded. In this example, the quotient of $A(G)$ by the ideal spanned by the indecomposables of even dimension is isomorphic to the group algebra $\mathbb{C}[t, t^{-1}]$, via an isomorphism sending $\Omega(k)$ to t and $\Omega^{-1}(k)$ to t^{-1}. So there are species s_z parametrised by the non-zero $z \in \mathbb{C}$, which factor through this quotient, and satisfy $s_z(\Omega(k)) = z$, $s_z(\Omega^{-1}(k)) = z^{-1}$. Only the ones with z on the unit circle are core-bounded. This is because the dimension of $\Omega^n(k)$ is $2|n| + 1$, whereas if z is not on the unit circle then either the powers z^n or the powers z^{-n} grow exponentially with n in absolute value.

The following proposition shows that there is a natural correspondence between core-bounded species of $a(G)$ and algebra homomorphisms $\hat{A}_1(G) \to \mathbb{C}$.

Proposition 9.15. *For a species $s\colon A(G) \to \mathbb{C}$, the following are equivalent:*

(i) *s is core-bounded.*
(ii) *For all $x \in A(G)$ we have $|s(x)| \leqslant \|x\|$.*
(iii) *s is continuous with respect to the norm.*
(iv) *s extends to an algebra homomorphism $\hat{A}_1(G) \to \mathbb{C}$.*

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear, and the implication (iv) $\Rightarrow$ (i) follows from Remark 9.10. So it remains to prove that (i) $\Rightarrow$ (ii). Suppose that s is core-bounded, and write $x = \sum_i a_i[M_i]$ where the M_i are indecomposable. Then $s(x) = \sum_i a_i s([M_i])$ and so

$$|s(x)| \leqslant \sum_i |a_i|\,|s([M_i])| \leqslant \sum_i |a_i| \dim \mathsf{core}(M_i) = \|x\|. \qquad \square$$

Theorem 9.16. *If M is a kG-module then*

$$\gamma_G(M) = \sup_{s\colon a(G)\to\mathbb{C}} |s([M])|$$

where the supremum runs over the core-bounded species of $a(G)$. Furthermore, there exists a core-bounded species s of $a(G)$ such that $s([M]) = \gamma_G(M)$.

Proof. This equality follows from Theorem 9.6, Corollary 9.11 and Proposition 9.15. The final statement follows from Theorem 9.8. $\square$

Corollary 9.17. *For any kG-module M, the restriction of γ_G to the sub-semiring of $A(G)$ consisting of elements of the form $f([M]) = \sum_{i=0}^n a_i[M]^i$ with the a_i real and non-negative is additive and multiplicative.*

Proof. By Theorem 9.16, we may choose a species s that maximises $|s([M])|$, and such that $s([M]) = \gamma_G(M)$. Such an s also maximises $|s(f([M]))|$. $\square$

Question 9.18. What can be said about the *quasi-nilpotent* elements of $\hat{A}_1(G)$, namely the elements a satisfying $\lim\limits_{n\to\infty} \sqrt[n]{\|a^n\|} = 0$? These are the elements on which all core-bounded species vanish, and they form the Jacobson radical of $\hat{A}_1(G)$. Is this the closure of the nil radical, or are there more subtle ways of producing quasi-nilpotent elements?

Examples in analysis of quasi-nilpotent operators which are not nilpotent can be found in Examples 2.1.6 and 2.1.7 of Kaniuth [24]. Quasi-nilpotent elements also go by other names in the literature. For example in §I.4 of Gelfand, Raikov and Shilov [22] they are called *generalised nilpotent*, while in Rickart [27] they are called *topologically nilpotent*.

Remark 9.19. Many of the properties of γ_G that we have described correspond to well-known properties of the spectral radius in a Banach algebra, although we have chosen an exposition that is self-contained except for Theorem 9.4. This applies to Theorem 1.2 (i), (ii), (iv), (viii), the second inequality of (x), (xi), (xiii) and (xiv). Others require a Banach lattice: these can record the special role played by the linear combinations of modules with real non-negative coefficients, which roughly approximate the image of genuine modules as opposed to virtual ones; see, for example, the book by Schaefer [28]. The first inequality of Theorem 1.2 (x) and also (xii) correspond to facts about Banach lattices, as does Theorem 9.8 ([28, Prop. V 4.1]).

If we knew that $\hat{A}_1(G)$ was a *symmetric* Banach algebra, then Conjecture 5.3 would follow. Symmetric Banach algebras are extensively discussed in §4.7 of Rickart [27] and in §I.8 of Gelfand, Raikov and Shilov [22].

10. Cyclic groups

The computations in this section are based on Green [23]. For the purpose of this section only, let $G = \mathbb{Z}/p$ be the cyclic group of order p, where $p > 2$ is the characteristic of the field k, and let M_j be the indecomposable kG-module of dimension j for $1 \leqslant j \leqslant p$. Then we have

$$M_2 \otimes M_j \cong \begin{cases} M_2 & j = 1 \\ M_{j+1} \oplus M_{j-1} & 2 \leqslant j \leqslant p-1 \\ M_p \oplus M_p & j = p. \end{cases}$$

Let $U_j(x)$ be the Chebyshev polynomial of the second kind, defined by the recurrence relation $U_0(x) = 1$, $U_1(x) = 2x$, $U_j(x) = 2xU_{j-1}(x) - U_{j-2}(x)$ $(j \geqslant 2)$. These polynomials satisfy

$$U_j(\cos\theta) = \frac{\sin(j+1)\theta}{\sin\theta}$$

for all $j \geqslant 0$. For $j \geqslant 1$, the roots of $U_j(x)$ are real and distinct, symmetric about $x = 0$, and given by

$$x = \cos(m\pi/(j+1)), \quad 1 \leqslant m \leqslant j.$$

We define $f_j(x) = U_{j-1}(x/2)$. So $f_1(x) = 1$, $f_2(x) = x$, and $xf_j(x) = f_{j+1}(x) + f_{j-1}(x)$ $(j \geqslant 2)$. Then we have

$$f_j\left(\frac{\sin 2\theta}{\sin\theta}\right) = \frac{\sin j\theta}{\sin\theta}.$$

Note that $f_p(x)$ is an irreducible polynomial in x^2. For example, we have $f_3(x) = x^2 - 1$ and $f_5(x) = x^4 - 3x^2 + 1$. The roots of $f_p(x)$ are given by $x = 2\cos(m\pi/p)$ $(1 \leqslant m \leqslant p-1)$.

In the ring $a(G)/a(G,1)$, we have $[M_2][M_j] = [M_{j+1}] + [M_{j-1}]$ and $[M_p] = 0$. It follows that $f_j([M_2]) = [M_j]$ and $f_p([M_2]) = 0$. We therefore have

$$a(G)/a(G,1) \cong \mathbb{Z}[X]/(f_p(X))$$

where X corresponds to $[M_2]$. The core-bounded species of $a(G)$ are just the non-Brauer species, and are given by

$$s_m \colon [M_2] \mapsto 2\cos(m\pi/p) \qquad (1 \leqslant m \leqslant p-1).$$

Theorem 10.1. *In the case of the indecomposable kG-module M_j $(1 \leqslant j \leqslant p-1)$ for the cyclic group G of order p, the characteristic of k, we have*

$$\gamma_G(M_j) = \frac{\sin(j\pi/p)}{\sin(\pi/p)} = f_j(\gamma_G(M_2))$$

where

$$\gamma_G(M_2) = \frac{\sin(2\pi/p)}{\sin(\pi/p)} = 2\cos(\pi/p).$$

In fact, γ_G is additive and multiplicative on modules, so this determines γ_G on any module.

Proof. It follows from Theorem 9.16 that for fixed j we need to maximise $|s_m([M_j])|$ over the core-bounded species s_m. By the discussion above,

$$s_m([M_j]) = s_m(f_j([M_2]) = f_j(s_m([M_2])) = f_j(2\cos(m\pi/p)) = U_{j-1}(\cos m\pi/p) = \frac{\sin(jm\pi/p)}{\sin(m\pi/p)}.$$

Express the sines in terms of $e^{im\pi/p}$, expand as a geometric series and pair conjugate terms. If j is odd, say $j = 2r+1$, the result is $1 + \sum_{s=1}^{r} \cos(sm\pi/p)$; the case when j is even is similar and is left to the reader.

Thus we want to maximise the sum of the elements of a certain class of r-element subsets of $\{\cos(t\pi/p) : 1 \leqslant t \leqslant p-1\}$. Clearly, the maximum over all r-element subsets is obtained by choosing the r largest elements, i.e., $t = 1, \ldots, r$, which in our case is attained with $m = 1$.

Note that $m = 1$ yields the maximum for all the M_j. Since each $s_1([M_j])$ is a positive number, s_1 also yields the maximum on all sums of the M_j, i.e., on all modules. We have $\gamma_G(M) = s_1([M])$ for all modules and the last part of the theorem follows. □

11. Methods of Calculation

We recall some basic facts from Banach theory that we will use.

A linear operator (i.e., linear map) T from a Banach space B to itself is said to be bounded if $\|T\|_{\mathsf{op}} := \sup\{\|Tx\| : \|x\| = 1\}$ is finite. The space of all such bounded operators forms a Banach space $\mathcal{B}(B)$ with norm $\|\cdot\|_{\mathsf{op}}$. If B is finite dimensional then $\sigma(T)$ is just the finite set of eigenvalues, so the set of roots of the characteristic polynomial of T, and $\rho(T)$ is the largest of the absolute values of these.

If we start with a Banach algebra A, then any $a \in A$ yields a bounded linear operator $T_a \in \mathcal{B}(A)$ by $T_a x = ax$ for $x \in A$. It is easy to check that $\|T_a\|_{\mathsf{op}} = \|a\|$; we will usually omit the subscript op and often identify T_a with a. The spectra might differ, but it follows from Proposition 9.4 that the spectral radii agree: $\rho(T_a) = \rho(a)$.

We saw in Lemma 9.13 that for a core-bounded species s we have $s([\Omega k]) = \lambda$ for some complex number λ with $|\lambda| = 1$. It follows that s vanishes on the ideal of $\hat{A}_1(G)$ generated by $[\Omega k] - \lambda[k]$ and so factors through the quotient Banach algebra by the closure of this ideal, which we denote by $\hat{A}_1(G)/(\Omega - \lambda)$. Thus we can find $\gamma_G(M) = \rho(T_M)$ by calculating it on each of these quotients and taking the maximum value.

Because of the form of the definition of $\gamma_G(M)$, we can calculate it on any subalgebra of $\hat{A}_1(G)$ that contains all the tensor powers of M and similarly for $\hat{A}_1(G)/(\Omega-\lambda)$. If we consider the operator T_M, we can even restrict to any Banach subspace that contains some tensor power $M^{\otimes n}$ of M and that is closed under tensor product with $M^{\otimes n}$, by Theorem 1.2 (xiv).

Our strategy will be to use these observations to reduce to the finite dimensional case.

Of course, an operator A on a finite dimensional vector space with given basis can be represented by a square matrix $(A_{i,j})$. There are various possible norms that we can use on matrices. One is the operator norm induced from a norm on the vector space. Another, which we will use later, is $\|A\|_{\max} = \max_{i,j}\{|A_{i,j}|\}$; this is a Banach space norm, but it is not submultiplicative. However, any two vector space norms on a finite dimensional vector space, in this case the vector space of $n \times n$ matrices, are commensurate, i.e., there is a positive real number c such that $c^{-1}\|x\|_1 \leqslant \|x\|_2 \leqslant c\|x\|_1$ for all x. It follows that $\lim_{n\to\infty} \sqrt[n]{\|x\|}$ is independent of the norm, so yields $\rho(x)$, regardless of whether the norm is submultiplicative or not.

Note that if the entries $A_{i,j}$ in the matrix for A are integers then $\rho(A)$ must be an algebraic integer.

The following standard lemma will be useful later, when we look more closely at the quotient $\hat{A}_1(G)/(\Omega - \lambda)$. For a matrix B, we write $|B|$ for the matrix of absolute values of the entries of B. If A_1 and A_2 are matrices of the same size with real entries, we write $A_1 \leqslant A_2$ to indicate that each entry of A_1 is less than or equal to the corresponding entry of A_2.

Lemma 11.1. *Let A be a square matrix with non-negative real entries, and let B be a complex matrix of the same size satisfying $|B| \leqslant A$. Then $\rho(B) \leqslant \rho(A)$.*

Proof. We have in general $|XY| \leqslant |X|\,|Y|$ whenever the product is defined, and hence $|B^n| \leqslant A^n$. Thus $\|B^n\|_{\max} \leqslant \|A^n\|_{\max}$. Taking nth roots and then the limit as n tends to infinity yields the result. □

Finally, we formulate a result that depends heavily on the fact that our norm is additive on modules; it could be generalised to a Banach lattice with a norm that is additive on the positive cone.

Proposition 11.2. *Suppose that we have a bounded operator T on $\hat{A}_1(G)$ that takes kG-modules to kG-modules and for some $m \in \mathbb{N}$ we have kG-modules $S_1, \dots, S_m$ and $Y_1, \dots, Y_m$ with none of the S_i projective. Suppose that there are non-negative integers $A_{i,j}$ such that*

$$T[S_i] = \sum_j A_{i,j}[S_j] + [Y_i], \quad i = 1, \dots, m$$

and consider the matrix $A = (A_{i,j})$. Then $\rho(T) \geqslant \rho(A)$.

Proof. By induction, for any $n \geqslant 1$ there are kG-modules $Z_{i,n}$ such that

$$T^n[S_i] = \sum_j (A^n)_{i,j}[S_j] + [Z_{i,n}], \quad i = 1, \dots, m.$$

Note that this does not require the $[S_i]$ to be linearly independent. Because the norm is additive on sums of modules with non-negative coefficients, we obtain

$$\|T^n\|_{\mathsf{op}}\|[S_i]\| \geqslant \|T^n[S_i]\| = \|\sum_j (A^n)_{i,j}[S_j] + [Z_{i,n}]\| \geqslant \max_j\{(A^n)_{i,j}\}\|[S_j]\| \geqslant \max_j\{(A^n)_{i,j}\},$$

since S_j is not projective, so $\|[S_j]\| \geqslant 1$. It follows that for some i we have

$$\|T^n\|_{\mathsf{op}}\|[S_i]\| \geqslant \max_{i,j}\{(A^n)_{i,j}\} = \|A^n\|_{\max}.$$

Taking nth roots and then the limit as n tends to infinity yields $\rho(T) \geqslant \rho(A)$. □

12. Modules M with $\gamma_G(M)=\sqrt{2}$

In Theorem 5.8, we showed that if M is neither projective nor endotrivial then we have $\gamma_G(M) \geqslant \sqrt{2}$. In this section, we investigate the case of equality.

Lemma 12.1. *For any kG-module we have:*

(i) *If $M = E \oplus N$ with E endotrivial and $\gamma_G(M \otimes M^*) < 4$ then N is projective.*
(ii) *If $M \otimes M^*$ is endotrivial then so is M.*

Proof. For part (i), we have $M \otimes M^* \cong k \oplus (E \otimes N^*) \oplus (E^* \otimes N) \oplus (N \otimes N^*)$. Theorem 3.5 shows that $\gamma_G(M \otimes M^*) \geqslant 4$ unless at least one of the terms in the sum is projective. If this is the case then the tensor product of the modules in the sum is projective. This tensor product is $N^{\otimes 2} \otimes N^{*\otimes 2}$ plus projectives. Lemmas 2.2 and 2.3 show that if this is projective then N is projective.

For the second part, notice that if $M \otimes M^*$ is endotrivial then p cannot divide the dimension of M. Thus k is a summand of $M \otimes M^*$, and since $M \otimes M^*$ is endotrivial, the complementary summand is projective. $\square$

Proposition 12.2. *If $1 < \gamma_G(M \otimes M^*) < 1 + \sqrt{2}$, then p divides the dimension of M.*

Proof. Suppose that the dimension of M is not divisible by p. By Theorem 7.2, without loss of generality we may assume that G is elementary abelian. By Lemma 2.11, we may also assume that k is algebraically closed.

We first show that we may suppose that M is indecomposable. Otherwise, choose an indecomposable summand M_1 of M with dimension not divisible by p. Then $\gamma_G(M_1 \otimes M_1^*) < 1 + \sqrt{2}$, and we claim that $1 < \gamma_G(M_1 \otimes M_1^*)$ so that we may replace M by M_1. If not, then $M_1 \otimes M_1^*$ is endotrivial by Theorem 7.5, so M_1 is endotrivial by Lemma 12.1 (ii). Now apply Lemma 12.1 (i) to M to see that $M = M_1 \oplus (\mathsf{proj})$, so $\gamma_G(M_1 \otimes M_1^*) = \gamma_G(M \otimes M^*) > 1$, a contradiction. Thus we may assume that M is indecomposable.

We have $M \otimes M^* \cong k \oplus X$ with X non-projective. So using Theorem 4.6 we have

$$1 + \sqrt{2} > \gamma_G(M \otimes M^*) = \gamma_G(k \oplus X) = 1 + \gamma_G(X).$$

By Theorem 5.8, the only possibility is $\gamma_G(X) = 1$, and by Theorem 7.5, X is endotrivial. By Proposition 7.4 we have $X \cong \Omega^r k \oplus (\mathsf{proj})$. We know that X is self dual, hence if G is not cyclic we have $r = 0$ and so $M \otimes M^* \cong k \oplus k \oplus (\mathsf{proj})$. This contradicts Theorem 2.1 of Benson and Carlson [5] (this is where we need to use the statements that M is indecomposable of dimension not divisible by p, and k is algebraically closed). If G is cyclic then Ω has period two and the only other possibility is $r = 1$. Thus $M \otimes M^* \cong k \oplus \Omega k \oplus (\mathsf{proj})$. This contradicts our assumption that p does not divide the dimension of M, so the lemma is proved. $\square$

The next proposition involves a number $\alpha \approx 2.839286755\ldots$, which is the unique real root of the polynomial $x^3 - 4x^2 + 4x - 2$.

Proposition 12.3. *If M has dimension divisible by p and $M \otimes M^* \otimes M \cong M \oplus M \oplus X$ with X not projective, then $\gamma_G(M \otimes M^*) \geqslant \alpha$.*

Proof. The dimension of X is divisible by p, so by Proposition 2.1 (ii), $X \otimes X^* \otimes X \cong X \oplus X \oplus Y$, but we have no information on whether Y is projective. We have

$$\begin{aligned}(M \otimes M^*) \otimes (X \otimes M^*) &\cong 2(X \otimes M^*) \oplus (X \otimes X^*)\\ (M \otimes M^*) \otimes (X \otimes X^*) &\cong (M \otimes M^* \otimes X \otimes X^*)\\ (M \otimes M^*) \otimes (M \otimes M^* \otimes X \otimes X^*) &\cong 2(X \otimes M^*) \oplus 2(M \otimes M^* \otimes X \otimes X^*) \oplus (Y \otimes M^*).\end{aligned}$$

Set $T = M \otimes M^*$, $S_1 = X \otimes M^*$, $S_2 = X \otimes X^*$ and $S_3 = M \otimes M^* \otimes X \otimes X^*$. None of the S_i are projective. Indeed, for S_2 this is proved in Lemma 2.2. Since S_2 is isomorphic to a summand of $S_1 \otimes M \otimes M^*$, it follows that S_1 is not projective. Finally, $S_3 \cong S_1 \otimes S_1^*$, so S_3 is not projective by Lemma 2.2 again.

We regard tensoring with T as an operator and, ignoring $Y \otimes M^*$ for the moment, the above isomorphism says that the action of T on the ordered set $\{S_1, S_2, S_3\}$ is recorded in the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial of the matrix A is $x^3 - 4x^2 + 4x - 2$, so $\rho(A) = \alpha$. Applying Proposition 11.2, we obtain

$$\gamma_G(M \otimes M^*) = \rho(T) \geqslant \rho(A) = \alpha. \qquad \square$$

Remark 12.4. The appeal to Proposition 11.2 at the end of the proof of Proposition 12.3 can be expressed more naïvely as follows. We have $T \otimes T \cong 2T \oplus S_1$, $T \otimes S_1 \cong 2S_1 \oplus S_2$, $T \otimes S_2 \cong S_3$, $T \otimes S_3 \cong 2S_1 \oplus 2S_2 \oplus (Y \otimes M^*)$. So ignoring some summands, we see that $T^{\otimes(n+2)}$ has

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A^n \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

as a direct summand. It follows that the number of non-projective direct summands of $T^{\otimes(n+2)}$ is at least the sum of the entries in $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A^n$. By Frobenius–Perron theory, this number is bounded below by a constant multiple of α^n, since α is the largest real root of this matrix. So we have $\gamma_G(T) \geqslant \alpha$.

Theorem 12.5. *For any non-projective kG-module M, the following conditions are equivalent.*

(i) $1 < \gamma_G(M \otimes M^*) < 1 + \sqrt{2}$.
(ii) $\gamma_G(M \otimes M^*) = 2$.
(iii) $M \otimes M^* \otimes M \cong M \oplus M \oplus (\mathsf{proj})$.
(iv) $[M \otimes M^*]^2 = 2[M \otimes M^*]$ *in* $a(G)/a(G, 1)$.

Proof. We begin by showing that (i) implies (iii).

If the dimension of M is divisible by p then by Proposition 2.1 we have

$$M \otimes M^* \otimes M \cong M \oplus M \oplus X.$$

If X is not projective then by Proposition 12.3 we have $\gamma_G(M \otimes M^*) \geqslant \alpha > 1 + \sqrt{2}$, contradicting (i).

On the other hand, if the dimension of M is not divisible by p then Proposition 12.2 shows that $\gamma_G(M \otimes M^*)$ cannot lie between 1 and $1+\sqrt{2}$, also contradicting (i).

To see that (iii) implies (iv), tensor with M^*. Then it is straightforward to see that (iv) implies (ii) and (ii) implies (i). $\square$

Corollary 12.6. *If a kG-module M satisfies $1 < \gamma_G(M) < \sqrt{1+\sqrt{2}} \approx 1.553773974\ldots$ then it satisfies the conditions of Theorem 12.5.*

Proof. The inequalities in the corollary imply those in Theorem 12.5 (i), by Theorem 5.1 and Lemma 12.1 (ii). $\square$

Remarks 12.7. (i) If Conjecture 5.3 holds, then the converse of Corollary 12.6 also holds, provided M is not projective.

(ii) If M satisfies $M \otimes M^* \otimes M \cong M \oplus M \oplus (\mathsf{proj})$ then the restriction of M to any cyclic subgroup of G of order p is projective. This can be seen by noting that the module M is self-dual on restriction to any such cyclic subgroup C of G, and the relation $[M]^3 = 2[M]$ in $a(G)/a(G,1)$ yields that $\gamma_C(M)$ is equal to 0 or $\sqrt{2}$; the latter value can be seen to be impossible from the calculations in Section 10. It is not clear whether there are any such modules except in the case that $p = 2$ and a Sylow 2-subgroup of G is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

We end this section with a related conjecture, based on the known examples of modules M with $\gamma_G(M) < 2$. Such modules are quite hard to construct.

Conjecture 12.8. *If M is a kG-module with $\gamma_G(M) < 2$ then $\gamma_G(M) = 2\cos(\pi/m)$ for some integer $m \geqslant 2$.*

Remark 12.9. The case $m = 2$ gives $\gamma_G(M) = 0$, $m = 3$ gives $\gamma_G(M) = 1$, and the case $m \geqslant 4$ gives $\gamma_G(M) \geqslant \sqrt{2}$. This fits well with Theorems 5.8, 7.5 and 12.6. The two dimensional representation of $\mathbb{Z}/p$ in characteristic p is an example with $\gamma_G(M) = 2\cos(\pi/p)$, see Theorem 10.1. It follows from the computations in Alperin [1] and Section 3 of Craven [15] that if M is the two dimensional natural module for $SL(2,q)$ with q a prime power, then M is algebraic and $\gamma_G(M) = 2\cos(\pi/q)$. This can also be seen by using the weight theory of tilting modules.

Question 12.10. What are the general properties of modules M with $\gamma_G(M) < 2$?

13. Eventually Recursive Functions

Definition 13.1. We say that a function $\phi\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ is *eventually recursive* of degree d if there exists a homogeneous linear recurrence relation with constant coefficients, in other words a recurrence relation of the form

$$\phi(n) + a_1\phi(n-1) + \cdots + a_d\phi(n-d) = 0$$

with $a_i \in \mathbb{Z}$, which is satisfied for all large enough integers n. The recurrence relation of minimal degree eventually satisfied by $\phi(n)$ is uniquely determined, and the corresponding polynomial

$$z^d + a_1 z^{d-1} + \cdots + a_d$$

has $a_d \neq 0$ and is called the *characteristic polynomial* of the recurrence relation.

The following is a standard theorem from the theory of recurrence relations.

Theorem 13.2. *If ϕ is eventually recursive and r is the radius of convergence of the generating function $f(t) = \sum_{n=0}^{\infty} \phi(n)t^n$ then $1/r$ is an algebraic integer. It is the largest positive real root of the characteristic polynomial.*

Proof. It is easy to check that $(1 + a_1t + \cdots + a_dt^d)f(t)$ is a polynomial in t, and therefore $f(t)$ is a rational function with denominator equal to $1 + a_1t + \cdots + a_dt^d$. The poles of this rational function are the $1/\lambda_i$ where the roots of the characteristic polynomial are λ_i. The radius of convergence r is therefore the smallest of the $1/|\lambda_i|$. Now apply Pringsheim's Theorem 8.3 to $f(t)$ to see that r is a root of the characteristic polynomial. □

Based on a large number of computations using Magma, some of which are presented in Section 15, we make the following conjecture.

Conjecture 13.3. *If M is a kG-module then the function $\phi(n) = \mathsf{c}_n^G(M)$ is eventually recursive.*

Remarks 13.4.
(i) If M is an algebraic module then Conjecture 13.3 holds for M.
(ii) If Conjecture 13.3 holds for a module M, then by Theorem 13.2, $\gamma_G(M)$ is an algebraic integer.
(iii) In Example 2.12, the minimal equation for M modulo projectives is $x^2(x-2)^3 = 0$. The largest solution to this has $|x| = 2$, and so $1/r = \gamma_G(M) = 2$. The associated recurrence relation is $\mathsf{c}_n^G(M) - 8c_{n-3}^G(M) = 0$ for $n \geqslant 5$.
(iv) In Example 5.5, the module M is not algebraic, but nonetheless, the relation
$$M \otimes M \cong M^* \oplus \Omega(M^*)$$
can be used to produce a recurrence relation for $\mathsf{c}_n^G(M)$ for large enough n.

The consequence of Conjecture 13.3 that $\gamma_G(M)$ is an algebraic integer is at least consistent with the last result of this section.

Proposition 13.5. *The invariant $\gamma_G(M)$ can take only countably many values (for all primes, fields, finite groups and finitely generated modules).*

Proof. We know that γ_G does not vary with field extension, by Lemma 2.11, and any representation over any field can be realised over a finitely generated field—the subfield generated over $\mathbb{F}_p$ by the entries of the matrices in some matrix form of the representation—so we only need to consider finitely generated fields. A finitely generated field is countable and there are only countably many such fields. To see this, add the generators one by one. If a generator is algebraic over the field produced at the previous stage then there are only countably many possibilities for its minimal polynomial and hence for the new field. If it is transcendental, the new field is already determined. Thus, for a given group, field and dimension, there are only countably many representations (consider matrices again) and so only countably many possible values of γ_G. There are also only countably many possibilities for group, field and dimension. □

This proof also works for any other invariant of kG-modules that is preserved under field extension.

14. Omega-Algebraic Modules

Definition 14.1. A kG-module M is called *Omega-algebraic* if the non-projective indecomposable direct summands of the modules $M^{\otimes n}$ fall into finitely many orbits of the syzygy functor Ω.

An example of an Omega-algebraic module may be found in Example 5.5. We'll see more examples in the next section, as well as evidence that not all modules are Omega-algebraic. A weaker form of Conjecture 13.3 is the following.

Conjecture 14.2. *If M is an Omega-algebraic kG-module then the function $\phi(n) = \mathsf{c}_n^G(M)$ is eventually recursive.*

We can calculate $\gamma_G(M)$ for an Omega-algebraic module M as follows, using the methods of Section 11. First we restrict to the subspace V of $\hat{A}_1(G)$ generated by all the indecomposable summands of all the tensor powers of M, together with all their syzygies. Choose representatives of the Ω-orbits of these indecomposable summands, say $M_1, \dots, M_d$; by hypothesis there are only finitely many. Each $M \otimes M_i$ decomposes as a direct sum of modules of the form $\Omega^m(M_j)$. The operation of tensoring with M gives us a $d \times d$ matrix $X(\Omega)$, whose entries are Laurent polynomials in the operator Ω which have non-negative coefficients.

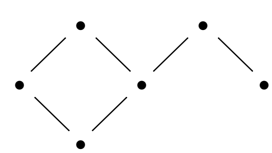
Now, for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, form the quotient $V/(\Omega - \lambda)$ of V by the linear span of the elements $[\Omega^{m+1}(M_j)] - \lambda[\Omega^m(M_j)]$ with $m \in \mathbb{Z}$, $1 \leqslant j \leqslant d$. This quotient is a finite dimensional vector space with basis the images of the $[M_j]$; the matrix corresponding to tensoring with M is $X(\lambda)$, meaning that we substitute λ for Ω. Since the Laurent polynomials in Ω have non-negative coefficients we have $|X(\lambda)| \leqslant X(1)$, so we can apply Lemma 11.1 to see that $\rho(X(\lambda)) \leqslant \rho(X(1))$. It follows, using the discussion at the beginning of Section 11, that $\gamma_G(M)$ is the largest eigenvector of the matrix $X(1)$. In particular, it is an algebraic integer.

15. Some examples

Example 5.5 is an example of an Omega-algebraic module which is not algebraic. Here are some more complicated examples. The computations use the methods outlined in the previous section and in Section 11.

Example 15.1. Let $G = \langle g, h \rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ and $k = \mathbb{F}_3$. Let M_6 be the six-dimensional kG-module given by the following matrices, which has the diagram shown:

$$g \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



For an explanation of the diagrams in this section, see Example 5.5. Then M_6 is Omega-algebraic. Letting $M_3 = k_{\langle g \rangle}{\uparrow}^G$ and P be the projective indecomposable module, we have

$$M_6 \otimes M_6 \cong \Omega(M_6) \oplus \Omega^{-1}({M_6}^*) \oplus M_3 \oplus P$$

$$M_6 \otimes {M_6}^* \cong \Omega^{-1}(M_6) \oplus \Omega({M_6}^*) \oplus M_3 \oplus P$$
$$M_6 \otimes M_3 \cong \Omega(M_3) \oplus \Omega(M_3) \oplus \Omega(M_3).$$

The rows of the table in Figure 15.1 give the result of tensoring a module in the first column with M_6, and writing it as a direct sum of the syzygies of the modules listed in the first row of the table.

	M_6	${M_6}^*$	M_3	P
M_6	Ω	Ω^{-1}	1	1
${M_6}^*$	Ω^{-1}	Ω	1	1
M_3	0	0	3Ω	0
P	0	0	0	6

FIGURE 15.1. Table for Example 15.1

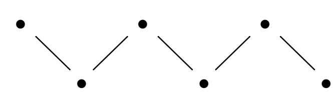
By the argument described at the end of Section 14, we delete the projectives in this table and replace Ω by 1, to obtain the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Then we find the eigenvalues of this matrix, which are 3, 2 and 0. The largest of these is 3, so $\gamma_G(M_6) = 3$.

Example 15.2. Let $G = \langle g, h \rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ and $k = \mathbb{F}_3$. This time, let M_6 be the self-dual six dimensional module given by the following matrices, which has the diagram shown:

$$g \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Then M_6 is Omega-algebraic. There are modules M_{15}, M_{21}, M_{27}, ${M_{27}}^*$, M_{39}, ${M_{39}}^*$ and the projective indecomposable module P, such that the rows of the table in Figure 15.2 give the result of tensoring with M_6.

If we throw out the projectives, and replace Ω by 1, the largest real eigenvalue of the remaining matrix will give the value of $\gamma_G(M)$. In this example, the eigenvalues are 4, 3, 1, 0, 0, 0, -2, -2, so $\gamma_G(M) = 4$.

Example 15.3. Let $G = \mathbb{Z}/5 \times \mathbb{Z}/5$, and $k = \mathbb{F}_5$. Let M_3 be the three dimensional module $kG/\mathsf{Rad}^2(kG)$. Then M_3 is Omega-algebraic. More precisely, there are modules M_6, M_8, M_{10}, M_{15}, M_{30}, M_{35}, M_{45} and M_{65} such that M_8, M_{35} and M_{65} are self-dual, and the rows of the table in Figure 15.3 give the effect of tensoring with M_3.

Replacing Ω by 1, the characteristic polynomial is

$$(x^2 - 3x + 1)(x^6 - 4x^3 - 1)(x^8 + 3x^7 + 8x^6 + 6x^5 + 5x^4 - x^3 + 12x^2 + 7x + 9).$$

	M_6	M_{15}	M_{21}	M_{27}	${M_{27}}^*$	M_{39}	${M_{39}}^*$	M_{66}	P
M_6		1	1						
M_{15}	2					1	1		
M_{21}	1			1	1			1	
M_{27}				2	2				6
${M_{27}}^*$				2	2				6
M_{39}	Ω		1	1	1		2Ω		6
${M_{39}}^*$	Ω^{-1}		1	1	1	$2\Omega^{-1}$			6
M_{66}			2	2	2	Ω^{-1}	Ω		18
P									6

FIGURE 15.2. Table for Example 15.2

	k	M_3	${M_3}^*$	M_6	${M_6}^*$	M_8	M_{10}	${M_{10}}^*$	M_{15}	${M_{15}}^*$	M_{30}	${M_{30}}^*$	M_{35}	M_{45}	${M_{45}}^*$	M_{65}
k		1														
M_3			1	1												
${M_3}^*$	1					1										
M_6						1	1									
${M_6}^*$			1							1						
M_8		1			1				1							
M_{10}								Ω^{-1}	1							
${M_{10}}^*$												1				
M_{15}										1	1					
${M_{15}}^*$								1					1			
M_{30}							2			Ω^{-1}			1			
${M_{30}}^*$							Ω			2					1	
M_{35}									2			1		1		
M_{45}									Ω^{-1}		1				1	
${M_{45}}^*$									Ω^{-1}				1			1
M_{65}							Ω^{-1}		Ω					1		

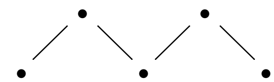
FIGURE 15.3. Table for Example 15.3

So writing τ for $(1+\sqrt{5})/2$, the eigenvalues are $1+\tau$ and $2-\tau$; τ and $1-\tau$ times cube roots of unity; and the roots of a degree eight irreducible with no real roots. The largest real eigenvalue is $1+\tau \approx 2.618$, and so $\gamma_G(M) = 1+\tau$. This happens to be the same as $\gamma_H(M)$ for each cyclic subgroup H of G in this case. This is because the restriction is $k \oplus M_2$, so $\gamma_H(M)$ can be computed using Theorems 4.6 and 10.1.

The next example shows how we can make use of examples already computed.

Example 15.4. Let $G = \mathbb{Z}/3 \times \mathbb{Z}/3 = \langle g, h \rangle$ and $k = \mathbb{F}_3$. Let M_5 be the five dimensional kG-module given by the following matrices, which has the diagram shown:

$$g \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Consider the restriction of M_5 to the cyclic subgroup generated by g. This is a sum of the trivial module and two copies of the indecomposable module of dimension 2. By Theorem 10.1 we have $\gamma_{\langle g \rangle}(M_5) = 1 + 4\cos(\pi/3) = 3$ and by Theorem 2.10 $\gamma_G(M_5) \geqslant 3$.

There is a short exact sequence

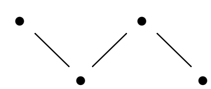
$$0 \to M_5 \to \Omega^{-1}(M_3) \to k \to 0,$$

where M_3 is the three dimensional module of Example 5.5. Since $\gamma_G(M_3) = 2$, by Theorem 5.4 we have $\gamma_G(\Omega^{-1}(M_3)) = 2$. So using Corollary 5.6 we have $\gamma_G(M_5) \leqslant 3$. Combining these bounds, we have $\gamma_G(M_5) = 3$. This example is quite difficult to compute directly.

Not all modules are Omega-algebraic. For example, the module of Example 5.5 inflated to $(\mathbb{Z}/3)^3$, with one of the factors acting trivially, is not Omega-algebraic. It is less clear whether faithful modules can fail to be Omega-algebraic, but some evidence is provided by the following example.

Example 15.5. Calculations using Magma give the following. Let M be the self-dual four dimensional module for $G = \langle g, h \rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ over $\mathbb{F}_3$ given by the following matrices, which has the diagram shown:

$$g \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



The beginning of the pattern is as follows:

$$\begin{aligned} M \otimes M &\cong k \oplus V_2 \oplus W & \qquad M \otimes X &\cong 2X \oplus X^* \oplus (\mathsf{proj}) \\ M \otimes V_2 &\cong M \oplus V_3 & M \otimes X^* &\cong 2X^* \oplus X \oplus (\mathsf{proj}) \\ M \otimes W &\cong M \oplus X \oplus X^* \end{aligned}$$

Here, V_2, W and V_3 are self-dual of dimensions 5, 10 and 16, X has dimension 18, and its dual satisfies $X^* \cong \Omega(X) \cong \Omega^{-1}X$.

Thereafter, at least as far as we have calculated, there are self-dual indecomposable modules denoted V_i for all $i \geqslant 1$, where $V_1 = M$ and V_2 and V_3 are the same modules as above. The dimension of V_i is $3i + a(i)$, where $a(i)$ depends only on the residue class of i modulo 6.

$i \pmod 6$	0	1	2	3	4	5
$a(i)$	11	1	-1	7	2	7

The tensor product of M with V_i for $i \geqslant 2$ appears to be given by

$$M \otimes V_i \cong \begin{cases} V_{i-1} \oplus V_{i+1} \oplus (\mathsf{proj}) & \text{if } 3 \nmid i \\ V_{i-1} \oplus V_{i+1} \oplus X \oplus X^* \oplus (\mathsf{proj}) & \text{if } 3 \mid i. \end{cases}$$

We have verified this pattern for $2 \leqslant i \leqslant 35$. Notice that $[X \oplus X^*]$ is an eigenvector of multiplication by $[M]$ with eigenvalue 3; this does not depend on having correctly spotted the pattern. Thus $\gamma_G(M) \geqslant 3$.

Now consider the diagram that describes M. If we remove the right hand vertex we obtain the diagram for the three dimensional module of Example 5.5, which we will denote by M_3. We know that $\gamma_G(M_3) = 2$ and there is a short exact sequence

$$0 \to k \to M \to M_3 \to 0.$$

By Corollary 5.6 we have $\gamma_G(M) \leqslant \gamma_G(M_3) + \gamma_G(k) = 3$. Thus $\gamma_G(M) \leqslant 3$. Combining this with the previous bound gives a proof that $\gamma_G(M) = 3$.

It appears that this example is not Omega-algebraic, but it still seems to satisfy Conjecture 13.3, although we have not been able to write down a proof of this.

We end with a conjecture which is closely related to Conjecture E of Craven [14], and which is motivated by extensive computations using Magma.

Conjecture 15.6. *If M is an absolutely indecomposable module for $\mathbb{Z}/p \times \mathbb{Z}/p$ and the dimension of M is divisible by p then M is Omega-algebraic.*

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Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, Scotland, United Kingdom

School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom