COHOMOLOGY ISOMORPHISMS OF GROUPS VIA THE T-FUNCTOR

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Abstract. DRAFT 1st March 2006. We show how Mislin's theorem on group homomorphisms that induce an isomorphism in cohomology can be proved based on ideas of Alperin and the use of Lannes's T-functor. This enables us to extend the class of groups to include groups of finite virtual cohomological dimension and profinite groups.

1. Introduction

We will prove the following theorem for a wide class of groups including finite groups, compact Lie groups, groups of finite virtual cohomological dimension over $\mathbb{F}_p$ and profinite groups (see section 3 for a precise list).

Theorem 1.1. *Let $f : H \to G$ be a homomorphism such that the induced map $f^* : H^*(G; \mathbb{F}_p) \to H^*(H; \mathbb{F}_p)$ is an isomorphism in high degrees. Then f induces an equivalence of categories $\mathcal{S}_p(H) \to \mathcal{S}_p(G)$, where S_p denotes the category of p-subgroups with the morphisms induced by inclusion and conjugation.*

This was proved by Mislin for compact Lie groups [14] and by C.-N. Lee for groups of finite virtual cohomological dimension [10]. Mislin had an *if and only if* statement, but this is no longer true for other classes of groups. These authors also required f^* to be an isomorphism in all degrees, but the difference was dealt with by Mislin in another paper [15].

The proofs used some deep topological methods, so for the generalization to profinite groups we are forced to take a more algebraic approach. This might be of interest in its own right.

Let us recall what an equivalence of categories means in concrete terms. Let $\mathcal{C}$ denote a class of groups closed under subgroups and isomorphisms and write $\mathcal{C}(G)$ for the category with objects the subgroups of G in $\mathcal{C}$ and morphisms the group homomorphisms induced by inclusion and conjugation.

Definition 1.2. The homomorphism $f : H \to G$ is a $\mathcal{C}$-equivalence if and only if the following conditions hold.

(1) If $P \in \mathcal{C}(H)$ then $N_G(f(P)) = f(N_H(P))C_G(f(P))$.
(2) If two $P, Q \in \mathcal{C}(H)$ are such that $f(P), f(Q)$ are conjugate in G then P, Q are conjugate in H.
(3) Every $P \in \mathcal{C}(G)$ is conjugate to a subgroup $f(Q)$ of G for some $Q \in \mathcal{C}(H)$.

Notice that this implies that f induces an isomorphism $P \to f(P)$ for all $P \in \mathcal{C}(H)$.

We will mostly be interested in the classes $\mathcal{S}_p$ of all finite p-groups, $\mathcal{S}_p^{\leq p^n}$ of p-groups of order at most p^n and $\mathcal{A}_p$ of elementary abelian p-groups.

Of course, the first condition for f to be a $\mathcal{C}$-equivalence is equivalent to f inducing an isomorphism $N_H(P)/C_H(P) \to N_G(f(P))/C_G(f(P))$. In the case of a finite group and $\mathcal{S}_p$-equivalence, the third condition can be replaced by the condition that the index $|G : H|$ is not divisible by p. The second condition is then, in fact, a consequence of the other two, by Alperin's Fusion Theorem.

Alperin asked about an algebraic proof of Mislin's Theorem in the case of finite groups. He showed the following proposition, as a consequence of Mislin's Theorem [1].

Proposition 1.3. *Whenever G is finite and $H < G$ induces an isomorphism in mod-p cohomology then:*

(1) *$N_H(P) \leq N_G(P)$ induces an isomorphism in mod-p cohomology for all p-subgroups $P \leq H$.*
(2) *$H/P \leq G/P$ induces an isomorphism in mod-p cohomology for all p-subgroups $P \leq H$ that are normal in G.*

Furthermore, if we could prove this proposition algebraically then for finite groups we could prove Mislin's Theorem algebraically.

We will prove a slight variant on this proposition, where the subgroups P are elementary abelian, and then verify the conditions using Lannes's T-functor. The definition of this functor and the verification of the basic properties that we use are essentially algebraic, concerning modules over the Steenrod algebra (cf. [3]), although not really representation theoretic. Proofs for finite groups that only use representation theory have recently been announced by Hida [6] and Okuyama [16].

2. Descent to Local Subgroups

We will always work in some category of groups and homomorphisms such that if $f : H \to G$ is in the category and $P \leq H$ is in $\mathcal{C}$ then the induced homomorphism $f_P : N_H(P)/P \to N_G(f(P))/f(P)$ is also in the category.

We have in mind things like finite groups, compact Lie groups, profinite groups or groups of finite virtual cohomological dimension.

Definition 2.1. An F-equivalence is an element of some given class of morphisms (F) that includes all isomorphisms and is closed under composition. The ones that we will use arise in the following way. Let F be a contravariant functor from our category of groups to some category. With a slight abuse of notation, we say that $f : H \to G$ is an F-equivalence if the induced map $f^* : F(G) \to F(H)$ is an isomorphism. We have in mind some sort of cohomology theory as F.

Definition 2.2. We say that F-equivalence is $\mathcal{C}$-local if whenever $f : H \to G$ is an F-equivalence and $P \in \mathcal{C}(H)$ then the induced homomorphism $f_P : N_H(P)/P \to N_G(f(P))/f(P)$ is an F-equivalence.

Lemma 2.3. *If F-equivalence is $\mathcal{A}_p$-local then it is $\mathcal{S}_p$-local.*

Proof. We prove that F is $\mathcal{S}_p^{\leq p^n}$-local by induction on n. The case $n = 0$ is clear. Suppose that $f : H \to G$ is an F-equivalence and we know that F is $\mathcal{S}_p^{\leq p^n}$-local.

Let $P \leq H$ be a p-subgroup of order p^{n+1} and let $E = Z_p(P)$, the maximal elementary p-subgroup of the centre of P, so $E \neq 1$.

Now $f_E : N_H(E)/E \to N_G(f(E))/f(E)$ is an F-equivalence by hypothesis and $|P/E| \leq p^n$, so, by induction, $f_{P/E} : N_{N_H(E)/E}(P/E) \to N_{N_G(f(E))/f(E)}(f(P)/f(E))$ is an F-equivalence.

But $N_{N_H(E)/E}(P/E) = N_H(P)/E$ and similarly for G, so $f_P : N_H(P)/P \to N_G(f(P))/f(P)$ is an F-equivalence. □

Definition 2.4. We say that F-equivalence detects fusion of elementary abelian p-subgroups if whenever $f : H \to G$ is an F-equivalence then f is an $\mathcal{A}_p$-equivalence.

The next lemma corresponds closely to the argument at the end of [1].

Lemma 2.5. *Suppose that F-equivalence is $\mathcal{A}_p$-local and detects fusion of elementary abelian p-subgroups. Then any F-equivalence must be an $\mathcal{S}_p$-equivalence.*

Proof. We give a proof that an F-equivalence is an $\mathcal{S}_p^{\leq p^n}$-equivalence by induction on n. This is clearly true for $n = 0$. Suppose that $f : H \to G$ is an F-equivalence and we know that an F-equivalence must be an $\mathcal{S}_p^{\leq p^n}$-equivalence for some given n.

By 2.3, F-equivalence is $\mathcal{S}_p$-local, so each $f_P : N_H(P)/P \to N_G(f(P))/f(P)$ is an F-equivalence and, by induction, each f_P is an $\mathcal{S}_p^{\leq p^n}$-equivalence.

For the first condition, given $P \leq H$ of order p^{n+1} let $E = Z_p(P)$, the maximal central elementary abelian p-subgroup of P. We know that $f_E : N_H(E)/E \to N_G(f(E))/f(E)$ is an $\mathcal{S}_p^{\leq p^n}$-equivalence so, in particular,
$N_{N_G(f(E))/f(E)}(f_E(P/E)) = f_E(N_{N_H(E)/E}(P/E))C_{N_G(f(E))/f(E)}(f_E(P/E))$. Thus $N_G(f(P)) = f(N_H(P))\tilde{C}_{N_G(f(P))}(f_E(P/E))$, where $\tilde{C}_{N_G(f(P))}(f_E(P/E))$ is the inverse image of $C_{N_G(f(E))/f(E)}(f_E(P/E))$ in $N_G(f(P))$.

But the kernel of $\tilde{C}_{N_G(f(P))}(f_E(P/E)) \to C_{N_G(f(E))/f(E)}(f_E(P/E))$ is isomorphic to $\mathrm{Hom}(N_G(f(P)), f(E))$, so is an elementary abelian p-subgroup that is normal in $N_G(f(P))$. Since f detects fusion of elementary abelian p-subgroups we can deduce that this kernel is contained in $f(H)$ and so $N_G(f(P)) \cong f(N_H(P))C_G(f(P))$, as required.

For the second condition, suppose that $P, Q \leq H$ are finite p-subgroups of order p^{n+1} such that $f(P)$ and $f(Q)$ are conjugate in G. Then $f(Z_p(P))$ is conjugate to $f(Z_p(Q))$ in G, so $Z_p(P)$ is conjugate to $Z_p(Q)$ in H by hypothesis, say $Z_p(P) = Z_p(Q)^h$ for $h \in H$. Let $R = Q^h$, so $Z_p(R) = Z_p(P) = E$, and $f(P), f(R) \leq N_G(E)$ are conjugate in G and hence in $N_G(E)$, since Z_p is characteristic.

But $f_E : N_H(E)/E \to N_G(f(E))/f(E)$ is an $\mathcal{S}_p^{\leq p^n}$-equivalence and $f_E(P/E)$ is conjugate to $f_E(R/E)$ in $N_G(f(E))/f(E)$, so P/E is conjugate to R/E in $N_H(E)$, thus P is conjugate to R and hence to Q in H.

For the third condition let $P < G$ be a finite p-subgroup of order p^{n+1}. Then $E = Z_p(P)$ is conjugate to a subgroup of $f(H)$, so we may assume that $E \leq f(H)$. There must be an elementary abelian p-subgroup $E' \leq H$ that maps isomorphically to E. Since $f_{E'} : N_H(E')/E' \to N_G(E)/E$ is an $\mathcal{S}_p^{\leq p^n}$-equivalence, there is a $P' \leq H$ containing E' such that $f_{E'}(P'/E') = P/E$ and thus $f(P') = P$. □

3. Cohomology

We will write $H^*(-)$ for $H^*(-;\mathbb{F}_p)$ and an H^*-equivalence will be a map that induces an isomorphism in cohomology. A map that induces an isomorphism in cohomology in high degrees, that is there is some number N depending on the map such that the induced map in cohomology is an isomorphism in degrees greater than N, will be termed an $H^{\gg}$-equivalence.

We will consider the following classes of groups:

(1) finite,
(2) compact Lie,
(3) discrete of finite virtual cohomological dimension over $\mathbb{F}_p$,

(4) discrete and acting on a finite dimensional complex with finite stabilizers and compact quotient,
(5) profinite.

In each case we use the appropriate cohomology theory, so, in particular, we use the Galois or continuous cohomology for profinite groups.

Lemma 3.1. *For all these classes of groups $H^{\gg}$-equivalence detects fusion of elementary abelian subgroups.*

Proof. This is a consequence of Quillen's Stratification Theorem [17], provided that if we are in case (3) we know that the groups concerned have only finitely many conjugacy classes of elementary abelian p-subgroups or that the cohomology is finitely generated. See [1] Theorem 2 for the finite case. It also follows in all cases from considering the degree 0 part of 3.5 or 4.1. □

Corollary 3.2. *If $f : H \to G$ induces a surjection in mod-p cohomology in high degrees then* $\operatorname{Ker} f$ *contains no p-torsion and if H is compact Lie then the induced map $H \to f(H)$ is an H^*-equivalence. For compact Lie groups, if f is an H^*-equivalence (or an $H^{\gg}$-equivalence) then both $H \to f(H)$ and $f(H) \to G$ are H^*-equivalences (or $H^{\gg}$-equivalences).*

Proof. Since f must induce an injection of varieties, $\operatorname{Ker} f$ can contain no elementary abelian p-subgroup and so it can contain no p-torsion.

If H is compact Lie this implies that $\operatorname{Ker} f$ must be a finite p'-group, so has no cohomology and thus the Lyndon-Hochschild-Serre spectral sequence shows that $H \to f(H)$ must be an H^*-equivalence.

If f itself is an H^*- (or $H^{\gg}$-) equivalence then this forces $F(H) \to G$ to be an H^*- (or $H^{\gg}$-) equivalence. □

Lemma 3.3. *If $f : H \to G$ is a surjection and* $\operatorname{Ker} f$ *is a finite p'-group (or even a profinite p'-group) then f is an $\mathcal{S}_p$-equivalence.*

Proof. Condition (3) of the definition is clear. For condition (2) suppose that $f(P) = f(Q)^g$ for some $g \in G$. Then $g = f(h)$ for some $H \in H$, thus both P and Q^h are Sylow p-subgroups of $f^{-1}(f(P))$, so are conjugate. Notice that, if we want, we can take this conjugating element to be in $\operatorname{Ker} f$.

For condition (1), let $g \in N_G(f(P))$. Then $g = f(h)$ for some $h \in H$ and, as before P and P^h are conjugate by some $k \in \operatorname{Ker} f$. We can now replace h by hk to obtain an element of $N_H(P)$ that maps to g. □

Lemma 3.4. *If $f : H \to G$ is an H^*-equivalence and $P \leq H$ is a finite p-subgroup that is central in G then $f_P : H/P \to G/f(P)$ is an H^*-equivalence.*

Proof. In the cases where we have classifying spaces (so not profinite groups) we have the following argument. There is a fibration $BG \to B(G/P) \to K(P, 2)$, where $K(P, 2)$ is an Eilenberg-Mac Lane space (cf. [2] II 3.7). There is a similar fibration for H and f induces a map between them. The Comparison Theorem for spectral sequences applied to the Serre spectral sequences for these fibrations yields the result.

For profinite groups, for each open normal subgroup $N \leq G$ that intersects P only in 1, consider the spectral sequence for the fibration $B(G/N) \to B(G/NP) \to K(P, 2)$ and denote the terms by $E_*^{*,*}(G/N)$. Let $E_r^{*,*}(G)$ $(r \geq 2)$ denote the direct limit of the $E_r^{*,*}(G/N)$. It comes with differentials and its homology is $E^{*,*}_{r+1}(G)$. Also $E^{*,*}_*(G)$ converges to $E^{*,*}_\infty(G)$, which is the direct limit of the $E^{*,*}_\infty(G/N)$ and thus consists of the composition factors in some filtration of the direct limit of the $H^*(G/N)$, which is $H^*(G)$. The same discussion applies to H and the map $f_N : H/f^{-1}(N) \to G/N$ induces a map between $E^{*,*}_*(G/N)$ and $E^{*,*}_*(H/f^{-1}(N))$ and hence between $E^{*,*}_*(H)$ and $E^{*,*}_*(G)$. We can now use the same argument as before.

In the cases where we have cohomology with coefficients in a module for the group (so not infinite compact Lie groups) there is another more algebraic argument, but we must suppose that f is injective. In view of 3.2 and 3.3 this is sufficient for finite groups.

We may assume that $H \le G$. Let M be the cokernel of the natural map of $\mathbb{F}_pG$-modules $\mathbb{F}_p \to \mathrm{Coind}_H^G \mathbb{F}_p$. Our conditions imply that $H^*(G; M) = 0$ and we need to show that $H^*(G/P; M) = 0$.

If $H^*(G/P; M) \neq 0$ let m be the smallest degree in which it is not 0. Consider the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G/P; H^q(P; M)) \Rightarrow H^{p+q}(G; M) = 0.$$

But $E_2^{p,q} \cong H^p(G/P; H^q(P) \otimes M) \cong H^p(G/P; M) \otimes H^q(P)$, so $E_2^{p,q} = 0$ for $p < m$. Thus $E_r^{p,q} = 0$ for $p < m$ and $r \ge 2$ and $E_2^{m,0} \neq 0$ survives to $E_\infty^{m,0}$, a contradiction. □

Let $\mathcal{U}$ denote the category of unstable modules over the mod-p Steenrod algebra. For any elementary abelian p-group V, the T-functor $T_V : \mathcal{U} \to \mathcal{U}$ is characterized by the adjunction

$$\mathrm{Hom}_{\mathcal{U}}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^*(V) \otimes N).$$

In fact, if $\mathcal{K}$ denotes the category of unstable algebras over the Steenrod algebra then T_V restricts to a functor $\mathcal{K} \to \mathcal{K}$, and the adjunction remains valid with $\mathcal{U}$ replaced by $\mathcal{K}$.

This functor is exact, commutes with direct limits and takes finitely generated algebras to finitely generated algebras.

We say that a module $M \in \mathcal{U}$ is locally finite if each element is contained in a finite dimensional submodule in $\mathcal{U}$. It turns out that the natural map $M \to T_V M$ is an isomorphism if and only if M is locally finite. More information can be found in [8].

Let $\mathrm{Rep}(V, G)$ denote the G-conjugacy classes of homomorphisms $\rho : V \to G$. The natural map $c_\rho : V \times C_G(\mathrm{Im}\,\rho) \to G, \quad (v, g) \mapsto \rho(v)g$ induces $c_\rho^* : H^*(G) \to H^*(V) \otimes H^*(C_G(\mathrm{Im}\,\rho))$, and this is adjoint to $\mathrm{ad}(c_\rho^*) : T_V H^*(G) \to H^*(C_G(\mathrm{Im}\,\rho))$.

The basic result on the T-functor applied to the cohomology of groups is the following theorem of Lannes.

Theorem 3.5. *For all the classes of groups mentioned above, except that if the group is profinite we must assume that its cohomology ring is finitely generated over $\mathbb{F}_p$, the map*

$$T_V H^*(G) \to \prod_{\rho \in \mathrm{Rep}(V,G)} H^*(C_G(\mathrm{Im}\,\rho)),$$

with components $\mathrm{ad}(c_\rho^*)$ is an isomorphism of algebras over the Steenrod algebra.

The proof is given in [7] for cases (1),(2) and (3); this is unpublished, but see [3]. Case (4) appears in [4] and profinite groups with finitely generated cohomology in [5]. There is also a version for general profinite groups given in the next section.

Corollary 3.6. *$H^{\gg}$-equivalence is p-local and we have proved Theorem 1.1.*

Proof. By 2.5 and 3.1, we only need to show that $H^{\gg}$-equivalence is elementary p-local. Let $f : H \to G$ be an $H^{\gg}$-equivalence and let $E \le H$ be p-elementary. We claim, following [15], that $f_E : C_G(E) \to C_H(f(E))$ is an H^* equivalence, not just an $H^{\gg}$-equivalence. Choose $V \cong E$.

The map $f^* : H^*(G) \to H^*(H)$ has kernel and cokernel bounded in degree, hence locally finite. We write this as an exact sequence $0 \to K \to H^*(G) \to H^*(H) \to C \to 0$.

For the moment we assume that we are not working with profinite groups with infinitely many conjugacy classes of elementary abelian p-subgroups, postponing this case to the next section.

The natural map $H^*(G) \to T_V H^*(G)$ followed by projection onto the factor corresponding to $\rho = 1$ in 3.5 is the identity, so there is a natural isomorphism from $\prod_{\rho\in \mathrm{Rep}(V,G),\rho\neq 1} H^*(C_G(\mathrm{Im}\,\rho))$ to the cokernel. We have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & H^*(G) & \longrightarrow & H^*(H) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & T_V H^*(G) & \longrightarrow & T_V H^*(H) & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \prod_{1\neq\rho\in \mathrm{Rep}(V,G)} H^*(C_G(\mathrm{Im}\,\rho)) & \longrightarrow & \prod_{1\neq\rho\in \mathrm{Rep}(V,H)} H^*(C_H(\mathrm{Im}\,\rho)). & & & &
\end{array}$$

Since the first two rows are exact and so are the columns it follows that the bottom arrow is an isomorphism.

By 3.1, $\mathrm{Rep}(V,H) \to \mathrm{Rep}(V,G)$ is a bijection. Picking a ρ with image E we see from 3.5 or 4.1 that the induced map $C_H(E) \to C_G(f(E))$ an H^*-equivalence.

By 3.4, $C_H(E)/E \to C_G(f(E))/f(E)$ is also an H^*-equivalence. We also know from 3.1 that the induced map $N_H(E)/C_H(E) \to N_G(f(E))/C_G(f(E))$ is an H^*-equivalence. Combining these using the Lyndon-Hochschild-Serre spectral sequence and the Comparison Theorem for spectral sequences we finally obtain that $f_E : N_H(E)/E \to N_G(f(E))/f(E)$ is an H^*-equivalence. □

4. Profinite Groups

When a profinite group has infinitely many conjugacy classes of elementary abelian subgroups we need to take more care with the T-functor.

We give each $H^*(C_G(\mathrm{Im}\,\rho))$ the discrete topology and then give their product $\prod_{\rho\in \mathrm{Rep}(V,G)} H^*(C_G(\mathrm{Im}\,\rho))$ the product topology.

Proposition 4.1. *For any profinite group G the natural map $H^*(G) \to \prod_{\rho\in \mathrm{Rep}(V,G)} H^*(C_G(\mathrm{Im}\,\rho))$ is injective with dense image.*

Proof. Let $\mathrm{Hom}(V,G)$ denote the set of all homomorphisms from V to G; G acts on it by conjugation. Thus $\mathrm{Hom}(V,G)$ is naturally a profinite G-set and $\mathrm{Rep}(V,G) = \mathrm{Hom}(V,G)/G$. Notice that $C_G(\mathrm{Im}\,\rho)$ is the stabilizer of $\rho \in \mathrm{Hom}(V,G)$.

For any G-set X let $O_x(G,X) = Gx$ denote the orbit of $x \in X$. By abuse of notation we will often we will allow $x \in X/G$ and often write just O_x. Let $F(X)$ denote the group of continuous functions $X \to \mathbb{F}_p$, with G acting by $(gf)(x) = f(g^{-1}x)$, $g \in G, f \in F(X), x \in X$. Then $F(O_x) \cong \mathrm{Coind}^G_{\mathrm{Stab}_G(x)} \mathbb{F}_p$, so $H^*(G; F(O_x)) \cong H^*(\mathrm{Stab}_G(x))$ (see [19]).

Thus the result of 3.5 for finite groups G can be written in a coordinate-free way as

$$T_V H^*(G) \cong \prod_{\rho \in \mathrm{Rep}(V,G)} H^*(G, F(O_x)) \cong H^*(G, F(\mathrm{Hom}(V, G))).$$

Now consider a profinite group G and let N denote an open normal subgroup. Thus $T_V H^*(G/N) \cong H^*(G/N; F(\mathrm{Hom}(V, G/N)))$.

Now $\mathrm{Hom}(V, G) \cong \varprojlim \mathrm{Hom}(V, G/N)$ so $F(\mathrm{Hom}(V, G)) \cong \varinjlim F(\mathrm{Hom}(V, G/N))$ and thus $H^*(G, F(\mathrm{Hom}(V, G))) \cong \varinjlim H^*(G/N, F(\mathrm{Hom}(V, G/N)))$ (see [18] 6.5.5). Since $H^*(G) \cong \varinjlim H^*(G/N)$ and T_V commutes with direct limits, we find that $T_V H^*(G) \cong H^*(G; F(\mathrm{Hom}(V, G)))$.

The dual of a theorem of Mel'nikov [12], given explicitly in the form that we will use in [20], states that for any profinite G-set X the natural map induced by the restrictions $H^*(G; F(X)) \to \prod_{x \in X/G} H^*(G; F(O_x))$ is injective with dense image, so we are done. □

Corollary 4.2. *The result of 3.6 is also true for profinite groups.*

Proof. The only problem in the proof of 3.6 is that in the diagram we do not know that the two bottom vertical arrows are surjections and we still need to deduce that the bottom horizontal map is an isomorphism, that is that each $f_E^* : H^*(C_G(E)) \to H^*(C_H(E))$ is an isomorphism.

The density statement in 4.1 is equivalent to the following: if Y is a finite subset of $\mathrm{Rep}(V, G)$ then the map $T_V H^*(G) \to \prod_{\rho \in Y} H^*(C_G(\mathrm{Im}\,\rho))$ is surjective. The surjectivity of f_E^* follows.

In order to show injectivity, suppose that $x \in H^*(C_G(E))$ is such that $f_E^*(x) = 0$, where $E = \mathrm{Im}\,\rho$. This x must be the inflation of some $x' \in H^*(C_{G/N}(E))$ for some open normal subgroup N of G. Let y be the image of x' in $H^*(C_{H/f^{-1}(N)}(f(E)))$. Since y must inflate to 0 in $H^*(C_H(f(E)))$ there is some open normal subgroup M of G, contained in N such that the inflation of y to $H^*(C_{H/f^{-1}(M)}(f(E)))$ is already 0. Let x'' be the inflation of x to $H^*(C_{G/M}(E))$.

According to 3.5, there is a $z \in T_V H^*(G/M)$ such that the ρ-coordinate of z is x'' and the other coordinates are 0. The inflation of z to $T_V H^*(G)$ is non-zero, because its ρ-coordinate is non-zero, but its image in $T_V H^*(H)$ is zero, by construction. This contradicts the fact that the map in 4.1 is injective. □

5. Remarks

The fact that the T-functor preserves noetherian rings can be used to show that all our constructions preserve groups with noetherian cohomology rings, provided that the construction of 3.4 preserves noetherian cohomology. This is used in [13] to show that a pro-p group with noetherian cohomology has only a finite number of conjugacy classes of finite p-subgroups.

In fact we do not need a group homomorphism $f : H \to G$, we only need a map between the p-completed classifying spaces, as was shown by Chun-Nip Lee [9], [11]. The methods of the present paper readily extend to prove this generalization.

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