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Cohomology of p-Adic Analytic Groups

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1 Introduction

The purpose of this article is to give an exposition on the cohomology of compact p-adic analytic groups. The cohomology theory of profinite groups was initiated by J. Tate and developed by J-P. Serre [23] in the sixties, with applications to number theory. In his extraordinary work on p-adic analytic groups [17], M. Lazard also considered their cohomology and proved two striking theorems: Lazard's first theorem states that a compact p-adic analytic group G is a virtual Poincaré duality group; his second theorem states that the rational cohomology of G coincides with the G-stable cohomology of its associated $\mathbb{Q}_p$-Lie algebra $L(G)$. Our main goal is to discuss these results of Lazard in the spirit of the treatment of the structure of p-adic analytic groups in [10]. We also wish to emphasize the close parallels with the theory of discrete duality groups. In order to achieve this goal we need to set up the appropriate homological algebra.

For a profinite group G there are various choices of modules one can consider. They are all required to have a topology and G must act continuously. We denote by $\mathfrak{D}_p(G)$ the category of discrete G-modules whose objects are p-torsion modules. The dual category of compact G-modules whose objects are inverse limits of finite discrete p-torsion G-modules will be denoted by $\mathfrak{C}_p(G)$. Moreover, these categories are (anti-)equivalent by the Pontrjagin duality operation ([6], Prop. 2.3). The role played by the single category of left $\mathbb{Z}[G]$-modules for a discrete group G, is now played by the two

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categories $\mathfrak{D}_p(G)$ and $\mathfrak{C}_p(G)$.

It turns out that $\mathfrak{D}_p(G)$ has enough injectives and $\mathfrak{C}_p(G)$ has enough projectives. So one has a natural notion of cohomology $H^\bullet(G, _)$ with coefficients in $ob(\mathfrak{D}_p(G))$ (as originally introduced by Tate) and also of homology $H_\bullet(G, _)$ with coefficients in $ob(\mathfrak{C}_p(G))$ (as described in [6]). Moreover, these functors are linked by Pontrjagin duality, i.e., one has a natural isomorphism

$$H^\bullet(G, _^*) \simeq H_\bullet(G, _)^*, \tag{1.1}$$

where * denotes the Pontrjagin duality operation (3.4.6).

For certain profinite groups it is possible to enlarge the coefficient category for the cohomology functor. By analogy with the discrete case we call these groups of type p-$\mathbf{FP}_\infty$. These are the profinite groups G for which $\mathbb{Z}_p$ has a projective resolution $P_\bullet \to \mathbb{Z}_p$ in $\mathfrak{C}_p(G)$ with each P_k finitely generated as $\mathbb{Z}_p[\![G]\!]$-module. Here, $\mathbb{Z}_p[\![G]\!] = \varprojlim \mathbb{Z}_p[G/U]$ denotes the compact[1] $\mathbb{Z}_p$-group algebra of G ([10], §7). Although Lazard did not introduce this notion, he showed that compact p-adic analytic groups are of type p-$\mathbf{FP}_\infty$.

In order to have a cohomology and homology functor defined on the same category, we introduce the Pontrjagin category $\mathfrak{P}_p(G)$ which contains $\mathfrak{D}_p(G)$ and $\mathfrak{C}_p(G)$ as full subcategories. It is an exact category with the property that $\mathbf{Hom}_{\mathfrak{P}_p(G)}(A, B)$ is a topological $\mathbb{Z}_p$-module (with respect to the compact-open topology). Moreover, Pontrjagin duality provides a duality operation * on $\mathfrak{P}_p(G)$. For profinite groups of type p-$\mathbf{FP}_\infty$ it is possible to define homology and cohomology with coefficients in $\mathfrak{P}_p(G)$, and moreover the cohomology and homology groups naturally carry a topology.

One of the most striking results of Lazard's work on p-adic analytic groups is that torsion-free p-adic analytic pro-p groups satisfy *Poincaré duality*[2] (at the prime p), i.e., the cohomology $H^\bullet(G, \mathbb{F}_p)$ of G with coefficients in the trivial G-module $\mathbb{F}_p$ is a finite-dimensional $\mathbb{F}_p$-algebra under cup product, such that

(1) $dim_{\mathbb{F}_p}(H^n(G, \mathbb{F}_p)) = 1$, where n equals the dimension of G,

(2) $H^j(G, \mathbb{F}_p) = 0$ for $j > n$,

(3) the bilinear form

$$H^k(G, \mathbb{F}_p) \otimes H^{n-k}(G, \mathbb{F}_p) \longrightarrow H^n(G, \mathbb{F}_p) \tag{1.2}$$

induced by cup product is non-degenerate.

Discrete Poincaré duality groups were defined and studied by R. Bieri and B. Eckmann ([3]) some years later. A discrete group G of type FP_∞ of cohomological dimension n is called a *Poincaré duality group*, if

$$H^k(G, \mathbb{Z}[G]) = \begin{cases} \mathbb{Z} & \text{for } k = n, \\ 0 & \text{for } k \neq 0. \end{cases} \tag{1.3}$$

[1] In [10] this algebra is called the *completed group algebra*.

[2] This is the definition given by J-P. Serre ([23], Chap. I, §4.5).

For these groups one has a natural isomorphism between cohomology and homology with a twist by the dualizing module ([5], Chap.VIII, Thm. 10.1). One has a natural isomorphism

$$H^{n-k}(G, _) \simeq H_k(G, D(G) \otimes _), \tag{1.4}$$

where $D(G)$ denotes the dualizing module.

In order to have a notion of Poincaré duality for profinite groups (not just pro-p groups), we translate this definition to the profinite setting, i.e., a profinite group G of type p-$\mathbf{FP}_\infty$ of finite cohomological dimension n is called a *Poincaré duality group* in dimension n at the prime p, if

$$\mathbf{H}^k(G, \mathbb{Z}_p[\![G]\!]) = \begin{cases} \mathbb{Z}_p & \text{for } k = n, \\ 0 & \text{for } k \neq 0. \end{cases} \tag{1.5}$$

If G is a pro-p group this coincides with the definition above (1.2). Since it is possible to define the cohomology and homology functor on the same category, the theory is completely analogous to the theory for discrete Poincaré duality groups, i.e., (1.4) holds with $D(G)$ replaced by the compact dualizing module $D_p(G)$. This setup makes it easy to show Poincaré duality for p-torsion-free compact p-adic analytic groups using only results of Serre on the cohomological dimension [24] and the calculation of the cohomology of a uniformly powerful[3] pro-p group with coefficients in $\mathbb{F}_p$. This avoids referring to duality theorems due to Tate and J.L. Verdier as in the original proof given by Lazard [17], Prop. V.2.5.6.7. There are several possibilities for calculating the cohomology ring $H^\bullet(G, \mathbb{F}_p)$ for a uniformly powerful pro-p group ([17], Chap.V, (2.2), [26]], [27], Chap. II). However, the conceptual approach which provides a method to calculate (or at least to approximate) $H^\bullet(G, M)$ for a finitely generated pro-p group G and $M \in ob(\mathfrak{D}_p(G))$, $p.M = 0$, is given by the May spectral sequence (Thm. 5.1.12). For a uniformly powerful pro-p group P and $M = \mathbb{F}_p$ this spectral sequence collapses at the E_1-term and yields the desired result. Since we could not find this application anywhere in the literature,[4] we will discuss this technique in Section 5. The approach used by Lazard is certainly related to this technique ([9], Chap. 1, Thm. 1).

Finally, we discuss a result of Lazard which relates the rational cohomology $\mathbb{H}^\bullet(G, _)$ of a compact p-adic analytic group G to the Lie algebra cohomology $H^\bullet(L(G), _)$ of the $\mathbb{Q}_p$-Lie algebra $L(G)$ associated to G ([17], (V.2.4)). For a compact p-adic analytic group G it is possible to define cohomology with coefficients in $\mathfrak{fd}_p(G)$, the abelian category of finite-dimensional $\mathbb{Q}_p$-vector spaces with continuous left G-action. Every $\mathbb{Z}_p[\![G]\!]$-module $W \in ob(\mathfrak{fd}_p(G))$ can be considered as an $L(G)$-module for the $\mathbb{Q}_p$-Lie algebra $L(G)$ (Prop. 5.2.3). To distinguish between the different categories we use the symbol $L(W)$ if we consider W as an $L(G)$-module. The Lie algebra cohomology $H^\bullet(L(G), L(W))$ has a natural (right) G-action. Although not explicitly stated, Lazard establishes[5] a natural isomorphism

$$\mathbb{H}^\bullet(G, W) \simeq H^\bullet(L(G), L(W))^G \tag{1.6}$$

[3] This is the notion introduced in ([10]). Lazard called these groups *équi-p-valué*.

[4] For a finite powerful p-group this spectral sequence was considered in [9], (6.4).

(Thm. 5.2.4). This isomorphism can be used to study the action of a p-torsionfree compact p-adic analytic group G on its compact dualizing module $D_p(G)$. One obtains:

(1) The representation $\phi_{D_p(G)}\colon G \to \mathrm{End}_{\mathbb{Z}_p}(D_p(G)) = \mathbb{Z}_p^*$ is given by

$$\phi_{D_p(G)}(g) = det(Ad(g)), \tag{1.7}$$

where $Ad\colon G \to \mathrm{End}_{\mathbb{Z}_p}(L(G))$ denotes the (right) adjoint representation.

(2) The p-torsion-free compact p-adic analytic group G is virtually orientable if and only if $L(G)$ is a unimodular $\mathbb{Q}_p$-Lie algebra.

(Cor. 5.2.5, Cor. 5.2.6). Furthermore, a well known vanishing result for Lie algebra cohomology implies a vanishing result for rational cohomology of semi-simple compact p-adic analytic groups (Cor. 5.2.7).

2 Some homological algebra

Since our main subject of interest is the cohomology of profinite groups, we will have to use concepts and methods from homological algebra rather frequently. We assume that the reader is familiar with the standard notions such as categories, functors, natural transformations, additive and abelian categories, ([[8], [[18], [[14]). In Section §3 we describe the natural categories of representations for profinite groups. They are abelian categories, or exact categories, with the property that every kernel and cokernel of a morphism has a canonical representative. We call such an abelian (resp. exact) category an abelian (resp. exact) *module category*. We therefore start in §2.1 with a discussion of certain categories of topological $\mathbb{Z}_p$-modules and their natural duality operation, known as Pontrjagin duality. In §2.2 we give a short description of the construction of homological and cohomological functors and the connections between them. In §2.3 we discuss the cohomology of augmented algebras with a group action. Spectral sequences are considered in §2.4. Finally, we introduce a lemma of Serre, which is one of the main techniques for the construction of projective resolutions. For further information the reader is referred to the standard literature ([8], [18], [14]).

2.1 The categories $\mathfrak{F}_p$, $\mathfrak{C}_p$, $\mathfrak{D}_p$ and $\mathfrak{P}_p$

The category of left R-modules, ${}_R\mathrm{Mod}$, for a given ring R is an abelian module category. However, the category of topological R-modules (with morphisms defined to be continuous) for a topological ring R is in general not abelian. For $R = \mathbb{Z}_p$ we consider the categories $\mathfrak{F}_p$, $\mathfrak{C}_p$, $\mathfrak{D}_p$ and $\mathfrak{P}_p$.

By $\mathfrak{F}_p$ we denote the category whose objects are finite $\mathbb{Z}_p$-modules. For $A, B \in ob(\mathfrak{F}_p)$ we define

$$\mathrm{hom}_{\mathfrak{F}_p}(A, B) = \mathrm{Hom}_{\mathbb{Z}_p}(A, B). \tag{2.1.1}$$

[5] The proof of Lazard actually shows (1.6). However, he did not state this theorem in that form. In order to identify the Lie algebra cohomology he introduced the *stable rational cohomology*.

Thus $\mathfrak{F}_p$ is a full subcategory of ${}_{\mathbb{Z}_p}\mathrm{Mod}$, and it is easily seen that $\mathfrak{F}_p$ is an abelian module category. However, it has neither enough projectives nor enough injectives.

Let $\mathfrak{D}_p$ denote the category whose objects are p-torsion modules, i.e., $M \in {}_{\mathbb{Z}_p}\mathrm{Mod}$ is contained in $ob(\mathfrak{D}_p)$ if and only if for all $m \in M$ there exists some $k \in \mathbb{N}_0$ such that $p^k.m = 0$. As morphisms we define

$$\hom_{\mathfrak{D}_p}(A, B) = \mathrm{Hom}_{\mathbb{Z}_p}(A, B), \tag{2.1.2}$$

$A, B \in ob(\mathfrak{D}_p)$. So $\mathfrak{D}_p$ is again a full subcategory of ${}_{\mathbb{Z}_p}\mathrm{Mod}$, and it is an abelian module category. This category has enough injectives ([11]), but not enough projectives. Note that it is closed under direct limits and that $\mathfrak{F}_p$ is a full subcategory of $\mathfrak{D}_p$.

By $\mathfrak{C}_p$ we denote the category whose objects are the profinite $\mathbb{Z}_p$-modules. As morphisms we take

$$\hom_{\mathfrak{C}_p}(A, B) = \mathbf{Hom}_{\mathbb{Z}_p}(A, B) = \{\alpha \in \mathrm{Hom}_{\mathbb{Z}_p}(A, B) \mid \alpha \text{ continuous}\}, \tag{2.1.3}$$

$A, B \in \mathfrak{C}_p$. The category $\mathfrak{C}_p$ is a full subcategory of the category ${}_{\mathbb{Z}_p}\mathrm{TopMod}$ of topological $\mathbb{Z}_p$-modules. It is easy to check that it is an abelian module category. Note that a profinite $\mathbb{Z}_p$-module is just an abelian pro-p group. Since there exist free abelian pro-p groups, $\mathfrak{C}_p$ has enough projectives, but not enough injectives ([6], Lemma 1.6). The set $\mathbf{Hom}_{\mathbb{Z}_p}(A, B)$ carries naturally a topology—the compact-open topology—making $\mathbf{Hom}_{\mathbb{Z}_p}(A, B)$ a complete topological $\mathbb{Z}_p$-module. We therefore call $\mathfrak{C}_p$ a *topological abelian module category*. Note that $\mathfrak{C}_p$ is closed under inverse limits.

By $\mathfrak{P}_p$ we denote the category whose objects are either contained in $ob(\mathfrak{D}_p)$ or in $ob(\mathfrak{C}_p)$. As morphisms we take

$$\begin{aligned} \hom_{\mathfrak{P}_p}(A, B) &= \mathbf{Hom}_{\mathbb{Z}_p}(A, B) \\ &= \{\alpha \in \mathrm{Hom}_{\mathbb{Z}_p}(A, B) \mid \alpha \text{ continuous and strict}\}. \end{aligned} \tag{2.1.4}$$

Recall that a morphism of topological groups $\alpha\colon A \to B$ is called *strict*, if the canonical topologies (i.e., the quotient topology and induced topology) on $im(\alpha)$ coincide. Thus, for example, for $A \in ob(\mathfrak{D}_p)$, $B \in ob(\mathfrak{C}_p)$, $\phi \in \mathrm{Hom}_{\mathbb{Z}_p}(A, B)$ belongs to $\mathbf{Hom}_{\mathbb{Z}_p}(A, B)$ if and only if ϕ is *continuous* and has *finite image*. Again we consider $\mathbf{Hom}_{\mathbb{Z}_p}(A, B)$ as a topological space with the compact-open topology. This category satisfies all of the axioms for an abelian category except the existence of a direct product. Following D.A. Buchsbaum ([7]), we call such a category an *exact category*.

The category $\mathfrak{P}_p$ contains $\mathfrak{D}_p$ and $\mathfrak{C}_p$ as full subcategories. A short exact sequence in $\mathfrak{P}_p$ is either a short exact sequence in $\mathfrak{D}_p$ or a short exact sequence in $\mathfrak{C}_p$. The reason why we have introduced it here is because it has a duality operation analogous to taking the dual space of a vector space in the category of vector spaces. This operation is usually called the *Pontrjagin duality operation*.

Proposition 2.1.1. *Let $I_p = \mathbb{Q}_p/\mathbb{Z}_p \in ob(\mathfrak{D}_p)$. Then $^*\colon \mathfrak{P}_p \to \mathfrak{P}_p$ given by*

$$^* = \mathbf{Hom}_{\mathbb{Z}_p}(_, I_p) \tag{2.1.5}$$

is a contravariant exact additive functor. The composition $^* \circ {}^*$ is naturally isomorphic to the identity functor on $\mathfrak{P}_p$. Moreover, for $A, B \in ob(\mathfrak{P}_p)$ one has a natural

isomorphism of topological $\mathbb{Z}_p$-modules

$$\mathbf{Hom}_{\mathbb{Z}_p}(A, B) \simeq \mathbf{Hom}_{\mathbb{Z}_p}(B^*, A^*). \tag{2.1.6}$$

Proof. Note that for $A \in ob(\mathfrak{D}_p)$, $\mathbf{Hom}_{\mathbb{Z}_p}(A, I_p)$ carries the topology of pointwise convergence, while for $B \in ob(\mathfrak{C}_p)$, $\mathbf{Hom}_{\mathbb{Z}_p}(B, I_p)$ is a discrete topological $\mathbb{Z}_p$-module. So the first part of the proposition is well known ([6], Prop. 2.3, [13], p. 400). For the second part consider the right and left hand side as contravariant/covariant bifunctors to the category of abelian groups. The morphism function of the functor * yields the desired isomorphism as abelian groups. Then one can easily check, using the canonical neighbourhood basis of $0 \in \mathbf{Hom}_{\mathbb{Z}_p}(A, B)$, that the isomorphism is also a homeomorphism.

Pontrjagin duality can be used quite successfully to identify certain cohomology and homology groups of profinite groups. An immediate application of Pontrjagin duality is also the following:

Corollary 2.1.2. *If $M \in ob(\mathfrak{C}_p)$ is p-torsion-free, then it is a free abelian pro-p group and therefore free (and projective) in $\mathfrak{C}_p$.*

Proof. We have to find a free generating set in M. Let $W = M/p.M$ and $V = W^* \in ob(\mathfrak{D}_p)$. Then V is an $\mathbb{F}_p$-vector space. By Zorn's lemma, V has a basis $X \subset V$. For $x \in X$, let $x^* \in W$ be an element vanishing on $X \setminus \{x\}$ and satisfying $x^*(x) \neq 0$. Then $X^* = \{ x^* \in W \mid x \in X \}$ is a free generating set of W as free elementary abelian pro-p group. Using the p-adic filtration on M then shows the desired result.

2.2 Homological and cohomological functors

Let $\mathfrak{A}$, $\mathfrak{B}$ be abelian or just exact categories and let $\mathbf{\Sigma}(\mathfrak{A})$, $\mathbf{\Sigma}(\mathfrak{B})$ denote the class of all short exact sequences in $\mathfrak{A}$, $\mathfrak{B}$. Let $\mathbf{\Sigma}^* \subseteq \mathbf{\Sigma}(\mathfrak{A})$ be a proper class of short exact sequences.[6] A $\mathbf{\Sigma}^*$*-connected pair* (S, e, T) of covariant functors is a pair of covariant additive functors $S, T : \mathfrak{A} \to \mathfrak{B}$ together with a function e which assigns to each proper exact sequence $\mathbf{s}: 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0 \in \mathbf{\Sigma}^*$ a morphism $e(\mathbf{s}) \in \hom_{\mathfrak{B}}(S(C), T(A))$. This $e(\mathbf{s})$ must have the property that for any short exact sequence $\mathbf{s}': 0 \to A' \xrightarrow{\sigma'} B' \xrightarrow{\tau'} C' \to 0$ and any morphism $(\alpha, \beta, \gamma) \in \hom_{\mathbf{\Sigma}^*}(\mathbf{s}, \mathbf{s}')$ of short exact sequences, the diagram

$$\begin{array}{ccc} S(C) & \xrightarrow{e(\mathbf{s})} & T(A) \\ {\scriptstyle S(\gamma)}\downarrow & & {\scriptstyle T(\alpha)}\downarrow \\ S(C') & \xrightarrow{e(\mathbf{s}')} & T(A') \end{array} \tag{2.2.1}$$

commutes. The function e is called the *connecting homomorphism* of the pair. The diagram (2.2.1) implies that e is a natural transformation if one considers S and T as functors from $\mathbf{\Sigma}^*$ to $\mathfrak{B}$. Moreover, if $\mathfrak{A}$ and $\mathfrak{B}$ are abelian then,

[6] For a definition see [18], XII. If $\mathfrak{A}$ is exact but not abelian, axiom (ii) must be omitted.

– if $\mathbf{s}$ is congruent to $\mathbf{s}'$, then $e(\mathbf{s}) = e(\mathbf{s}')$,

– for $\gamma \in \hom_{\mathfrak{A}}(C', C)$ one has $e(\mathbf{s} \circ \gamma) = e(\mathbf{s}) \circ S(\gamma)$,

– for $\alpha \in \hom_{\mathfrak{A}}(A, A')$ one has $e(\alpha \circ \mathbf{s}) = T(\alpha) \circ e(\mathbf{s})$,

([18], XII.7.2(a)–(c)). Recall that the short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ and $\mathbf{s}'\colon 0 \to A \xrightarrow{\sigma'} B' \xrightarrow{\tau'} C \to 0$ are called *congruent*, if there exists a morphism $\boldsymbol{\beta} = (id_A, \beta, id_C) \in \hom_{\Sigma}(\mathbf{s}, \mathbf{s}')$. It is obvious that β is necessarily an isomorphism.

From this property we conclude that e induces a morphism of abelian groups

$$e\colon Ext^1_{\mathbf{\Sigma}^*}(C, A) \longrightarrow \hom_{\mathfrak{B}}(S(C), T(A)) \tag{2.2.2}$$

and that

$$S(A) \xrightarrow{S(\sigma)} S(B) \xrightarrow{S(\tau)} S(C) \xrightarrow{e(\mathbf{s})} T(A) \xrightarrow{T(\sigma)} T(B) \xrightarrow{T(\tau)} T(C) \tag{2.2.3}$$

is a complex, i.e., the composition of any two successive maps is zero ([18], §XII.7).

A *morphism of* $\mathbf{\Sigma}^*$*-connected pairs* $(f, g)\colon (S, e, T) \to (S', e', T')$ of covariant additive functors is a pair of natural transformations $f\colon S \to S'$, $g\colon T \to T'$, such that the diagram

$$\begin{array}{ccc} S(C) & \xrightarrow{e(\mathbf{s})} & T(A) \\ {\scriptstyle f(C)}\downarrow & & {\scriptstyle g(A)}\downarrow \\ S'(C) & \xrightarrow{e'(\mathbf{s})} & T'(A) \end{array} \tag{2.2.4}$$

commutes for all $\mathbf{s} \in \mathbf{\Sigma}^*$.

A $\mathbf{\Sigma}^*$*-connected sequence* $(T^\bullet, e^\bullet)$ of covariant additive functors is a sequence of covariant additve functors $T^k\colon \mathfrak{A} \to \mathfrak{B}$, $k \in \mathbb{Z}$, such that (T^k, e^k, T^{k+1}) is a $\mathbf{\Sigma}^*$-connected pair for all $k \in \mathbb{Z}$. A *morphism of* $\mathbf{\Sigma}^*$*-connected sequences* $f^\bullet\colon (S^\bullet, e_S^\bullet) \to (T^\bullet, e_T^\bullet)$ is a sequence of natural transformations $f^\bullet = (f^k)_{k\in\mathbb{Z}}$ with the property that $(f^k, f^{k+1})\colon (S^k, e_S^k, S^{k+1}) \to (T^k, e_T^k, T^{k+1})$ is a morphism of $\mathbf{\Sigma}^*$-connected pairs for all $k \in \mathbb{Z}$.

A $\mathbf{\Sigma}^*$-connected sequence $(T^\bullet, e^\bullet)$ of covariant additve functors is called a $\mathbf{\Sigma}^*$*-cohomological functor* if $T^k = 0$ for $k < 0$, and if for all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathbf{\Sigma}^*$, the sequence

$$0 \longrightarrow T^0(A) \xrightarrow{T^0(\sigma)} T^0(B) \xrightarrow{T^0(\tau)} T^0(C) \xrightarrow{e^0(\mathbf{s})} T^1(A) \xrightarrow{T^1(\sigma)} T^1(B) \longrightarrow \cdots \tag{2.2.5}$$

is exact in $\mathfrak{B}$. If $\mathfrak{A}$ has enough $\mathbf{\Sigma}^*$-injectives, we call the $\mathbf{\Sigma}$-cohomological functor $(T^\bullet, e^\bullet)$ *co-effaceable* if $T^i(J) = 0$ for all $\mathbf{\Sigma}^*$-injective objects $J \in ob(\mathfrak{A})$ and $i \geq 1$.

A $\mathbf{\Sigma}^*$-connected sequence $(S^\bullet, e^\bullet)$ of covariant additive functors $S^k\colon \mathfrak{A} \to \mathfrak{B}$ is called a $\mathbf{\Sigma}^*$*-homological functor* if $S^k = 0$ for $k > 0$, and if for all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathbf{\Sigma}^*$ the sequence

$$\cdots \longrightarrow S^{-1}(B) \xrightarrow{S^{-1}(\tau)} S^{-1}(C) \xrightarrow{e_{-1}(\mathbf{s})} S^0(A) \xrightarrow{S^0(\sigma)} S^0(B) \xrightarrow{S^0(\tau)} S^0(C) \longrightarrow 0 \tag{2.2.6}$$

is exact in $\mathfrak{B}$. If $\mathfrak{A}$ has enough $\mathbf{\Sigma}^*$-projectives then the $\mathbf{\Sigma}^*$-homological functor $(S^\bullet, e^\bullet)$ is called *effaceable* if $S^i(P) = 0$ for all $\mathbf{\Sigma}^*$-projective objects $P \in ob(\mathfrak{A})$ and $i \leq -1$.

The following theorem summarizes one of the "general principles of homological algebra".

Theorem 2.2.1. *Let $\mathfrak{A}$, $\mathfrak{B}$ be abelian module categories and let $\mathbf{\Sigma}^* \subseteq \mathbf{\Sigma}(\mathfrak{A})$ be a proper class of short exact sequences.*
(a) Assume that $\mathfrak{A}$ has enough $\mathbf{\Sigma}^*$-injectives and let $(T^\bullet, e_T^\bullet)$ be a co-effaceable $\mathbf{\Sigma}^*$-cohomological functor, $T^k : \mathfrak{A} \to \mathfrak{B}$. Let $(U^\bullet, e_U^\bullet)$ be a non-negative $\mathbf{\Sigma}^*$-connected sequence of covariant additive functors, i.e., $U^k : \mathfrak{A} \to \mathfrak{B}$ and $U^k = 0$ for $k < 0$. Then for every natural transformation $f : T^0 \to U^0$, there exists a unique morphism of connected sequences $f^\bullet : (T^\bullet, e_T^\bullet) \to (U^\bullet, e_U^\bullet)$ with $f^0 = f$. If f^0 is a natural isomorphism and $(U^\bullet, e_U^\bullet)$ is also a co-effaceable cohomological functor, then all f^i are natural isomorphisms.
(b) Assume that $\mathfrak{A}$ has enough $\mathbf{\Sigma}^*$-projectives and let $(S^\bullet, e_S^\bullet)$ be an effaceable $\mathbf{\Sigma}^*$-homological functor, $S^k : \mathfrak{A} \to \mathfrak{B}$. Let $(V^\bullet, e_V^\bullet)$ be a non-positive $\mathbf{\Sigma}^*$-connected sequence of covariant additive functors, i.e., $V^k : \mathfrak{A} \to \mathfrak{B}$ and $V^k = 0$ for $k > 0$. Then for every natural transformation $g : V^0 \to S^0$, there exists a unique morphism of connected sequences $g^\bullet : (V^\bullet, e_V^\bullet) \to (S^\bullet, e_S^\bullet)$ with $g^0 = g$. If g^0 is a natural isomorphism and $(V^\bullet, e_V^\bullet)$ is also an effaceable homological functor, then all g^i are also natural isomorphisms.

Proof. In the notation of [18], $(T^\bullet, e_T^\bullet)$ is a right $\mathbf{\Sigma}^*$-universal and $(S^\bullet, e_S^\bullet)$ a left $\mathbf{\Sigma}^*$-couniversal connected sequence. Hence the claim follows from [18], Thm. XII.7.6, Thm. XII.7.2.

Our construction of homological and cohomological functors relies mainly on the exact homology sequence and the comparison theorem.

Let $\mathfrak{A}$ be an exact module category. A *chain complex* $(A_\bullet, \partial_\bullet)$ is a sequence of modules $A_k \in ob(\mathfrak{A})$, $k \in \mathbb{Z}$, and morphisms $\partial_k : A_k \to A_{k-1}$ such that $\partial_{k-1} \circ \partial_k = 0$. The sequence of mappings $\partial_\bullet = (\partial_k)_{k\in\mathbb{Z}}$ is called the *differential* of the chain complex. The *homology* of a chain complex is given by

$$H_k(A_\bullet, \partial_\bullet) = ker(\partial_k)/im(\partial_{k+1}). \tag{2.2.7}$$

A *mapping of chain complexes* $\sigma_\bullet : (A_\bullet, \partial_\bullet) \to (B_\bullet, \partial'_\bullet)$ is a sequence of mappings $\sigma_k : A_k \to B_k$ such that the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_{k+1} & \xrightarrow{\partial_{k+1}} & A_k & \xrightarrow{\partial_k} & A_{k-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \sigma_{k+1}} & & \downarrow{\scriptstyle \sigma_k} & & \downarrow{\scriptstyle \sigma_{k-1}} & & \\ \cdots & \longrightarrow & B_{k+1} & \xrightarrow{\partial'_{k+1}} & B_k & \xrightarrow{\partial'_k} & B_{k-1} & \longrightarrow & \cdots \end{array} \tag{2.2.8}$$

commutes. The sequence of mappings of chain complexes

$$0 \longrightarrow (A_\bullet, \partial_\bullet) \xrightarrow{\sigma_\bullet} (B_\bullet, \partial'_\bullet) \xrightarrow{\tau_\bullet} (C_\bullet, \partial''_\bullet) \longrightarrow 0 \tag{2.2.9}$$

is called *short exact*, if $0 \to A_k \overset{\sigma_k}{\to} B_k \overset{\tau_k}{\to} C_k \to 0$ is short exact for all $k \in \mathbb{Z}$. The category of chain complexes $\mathbf{C}_\bullet(\mathfrak{A})$ with mappings of chain complexes as morphisms is also exact, and is abelian if $\mathfrak{A}$ is abelian. Moreover, the homology functors $H_n(_)\colon \mathbf{C}_\bullet(\mathfrak{A}) \to \mathfrak{A}$ are covariant additive functors.

Dually, $(B^\bullet, \delta^\bullet)$ is called a *cochain complex* if B^k is a sequence of modules with $\delta^k\colon B^k \to B^{k+1}$, $\delta^{k+1} \circ \delta^k = 0$. The *cohomology* of a cochain complex is given by

$$H^k(B^\bullet, \delta^\bullet) = ker(\delta^k)/im(\delta^{k-1}). \tag{2.2.10}$$

Mappings of cochain complexes and short exact sequences are defined in the same way as for chain complexes. The category of cochain complexes will be denoted by $\mathbf{C}^\bullet(\mathfrak{A})$. It is also exact (abelian if $\mathfrak{A}$ is abelian). By reindexing, chain complexes can be made cochain complexes and vice versa.

Theorem 2.2.2. (Exact homology sequence) Let $\mathfrak{A}$ be an exact module category and $\mathbf{s}_\bullet\colon 0 \to (A_\bullet, \partial_\bullet) \overset{\sigma_\bullet}{\to} (B_\bullet, \partial'_\bullet) \overset{\tau_\bullet}{\to} (C_\bullet, \partial''_\bullet) \to 0$ a short exact sequence of chain complexes. Then there exists an exact sequence

$$\cdots \to H_n(A_\bullet) \overset{\sigma_n}{\to} H_n(B_\bullet) \overset{\tau_n}{\to} H_n(C_\bullet) \overset{e_k(\mathbf{s}_\bullet)}{\to} H_{n-1}(A_\bullet) \overset{\sigma_{n-1}}{\to} H_{n-1}(B_\bullet) \to \cdots, \tag{2.2.11}$$

where σ_n, τ_n denote the induced maps and $e_n(\mathbf{s}_\bullet)$ is given by the switchback rule. Moreover, $e_n(_)$ is a natural transformation of the functors

$$H_n(__{fin}), H_{n-1}(__{in})\colon \mathbf{\Sigma}(\mathbf{C}_\bullet(\mathfrak{A})) \longrightarrow \mathfrak{A} \tag{2.2.12}$$

Proof. ([18], Thm. II.4.1, Prop. II.4.2). Here we have used the abbreviations $__{fin}$ and $__{in}$ for the final and initial term of a short exact sequence, and have used the same symbol for the induced mappings on the homology. For $\mathfrak{A} = {}_{\mathbb{Z}}\mathrm{Mod}$ the switchback rule is given by the following assignment. If $x \in C_n$ with $\partial''_n(x) = 0$, let $[x]$ denote the corresponding homology class in $H_n(C_\bullet, \partial''_\bullet)$. Choose $y \in B_n$ with $\tau_n(y) = x$. Then $\tau_{n-1}(\partial'_n(y)) = \partial''_n(\tau_n(y)) = 0$. Hence there exists $z \in A_{n-1}$ with $\sigma_{n-1}(z) = \partial'_n(y)$. Since $\sigma_{n-2}(\partial_{n-1}(z)) = \partial'_{n-1}(\sigma_{n-1}(z)) = 0$ and as σ_{n-2} is injective, one has $\partial_{n-1}(z) = 0$. Then $e_n(\mathbf{s}_\bullet)([x]) = [z]$ and this assignment is well defined and satisfies (2.2.11) and (2.2.12). Using this, it is now easy to establish a proof for general $\mathfrak{A}$ using only kernels and cokernels of mappings.

In order to simplify notations we omit the differential in the chain complex. Let $A_\bullet$, $B_\bullet$ be chain complexes and $f_\bullet, f'_\bullet\colon A_\bullet \to B_\bullet$ be chain mappings. One says that f and f' are *chain homotopic* if there exists a sequence of mappings $h_\bullet = (h_n)_{n\in\mathbb{Z}}$, $h_n\colon A_n \to B_{n+1}$, such that

$$f_n - f'_n = \partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n. \tag{2.2.13}$$

For short we write $f_\bullet \simeq f'_\bullet$. It is an immediate consequence of the definition that chain homotopic maps induce the same mapping on the homology groups

$$f_\bullet = f'_\bullet\colon H_\bullet(A_\bullet) \longrightarrow H_\bullet(B_\bullet). \tag{2.2.14}$$

The complexes $A_\bullet$ and $B_\bullet$ are called *chain homotopy equivalent*, if there exist chain maps $g_\bullet\colon A_\bullet \to B_\bullet$ and $f_\bullet\colon B_\bullet \to A_\bullet$, such that $f_\bullet \circ g_\bullet \simeq 1_{A_\bullet}$, $g_\bullet \circ f_\bullet \simeq 1_{B_\bullet}$.

Let $M \in ob(\mathfrak{A})$. $P_\bullet \xrightarrow{\varepsilon} M$ is called a *projective resolution* of M, if $P_\bullet$ is a non-negative chain complex of projective modules $P_k \in ob(\mathfrak{A})$ and

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \tag{2.2.15}$$

is exact. Dually, $M \xrightarrow{\eta} J_\bullet$ is called an *injective resolution*, if $J_\bullet$ is a non-positive chain complex of injective modules $J_k \in ob(\mathfrak{A})$ and

$$M \xrightarrow{\eta} J_0 \xrightarrow{\partial_0} J_{-1} \xrightarrow{\partial_{-1}} J_{-2} \longrightarrow \cdots \tag{2.2.16}$$

is exact. The comparison theorem states that projective and injective resolutions are unique up to chain homotopy equivalence. More precisely, one has

Theorem 2.2.3. (Comparison theorem) *Let A, $B \in ob(\mathfrak{A})$ and $\alpha \in \hom_{\mathfrak{A}}(A, B)$.*

(1) Let $P_\bullet \xrightarrow{\varepsilon} A$ be a projective resolution of A and $Q_\bullet$ a chain complex, $\varepsilon\colon Q_0 \to B$ such that $\cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\varepsilon} B \to 0$ is exact. Then there exists a chain mapping $f_\bullet\colon P_\bullet \to Q_\bullet$ making the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & A \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \alpha} \\ \cdots & \longrightarrow & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\varepsilon} & B \end{array} \tag{2.2.17}$$

commute. Moreover, any pair of chain mappings $f_\bullet$, $f'_\bullet$ satisfying (2.2.17) *are chain homotopic.*

(2) Let $A \xrightarrow{\eta} J_\bullet$, $B \xrightarrow{\eta} I_\bullet$ be injective resolutions. Then there exists a chain mapping $g_\bullet\colon J_\bullet \to I_\bullet$ making the diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{\eta} & J_0 & \xrightarrow{\partial_0} & J_{-1} & \xrightarrow{\partial_{-1}} & J_{-2} & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_{-1}} & & \downarrow{\scriptstyle g_{-2}} & & \\ B & \xrightarrow{\eta} & I_0 & \xrightarrow{\partial_0} & I_{-1} & \xrightarrow{\partial_{-1}} & I_{-2} & \longrightarrow & \cdots \end{array} \tag{2.2.18}$$

commute. Moreover, any pair of chain mappings $g_\bullet$, $g'_\bullet$ satisfying (2.2.18) *are chain homotopic.*

Proof. ([1], Thm. 2.4.2, [18], Thm. III.6.1, Thm. III.8.1) This yields:

Theorem 2.2.4. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{Q}$ be exact module categories, $\mathfrak{A}_0 \subseteq \mathfrak{A}$ a full subcategory of $\mathfrak{A}$ and $M \in ob(\mathfrak{A}_0)$. Assume also that M has a projective resolution $P_\bullet \to M$ with $P_k \in ob(\mathfrak{A}_0)$.*

(a) *Let* $T(_,_)\colon \mathfrak{A}_0 \times \mathfrak{B} \to \mathfrak{Q}$ *be a biadditive left/left exact contravariant/covariant bifunctor such that* $T(P,_)\colon \mathfrak{B} \to \mathfrak{Q}$ *is exact for all* $P \in ob(\mathfrak{A}_0)$ *projective in* $\mathfrak{A}$. *Then* $(T_M^\bullet, e^\bullet)$ *with*

$$T_M^\bullet(B) = H^\bullet(T(P_\bullet, B), \delta^\bullet), \tag{2.2.19}$$

and $e^\bullet$ *given by the exact homology sequence, is a cohomological functor. Moreover, it does not depend on the particular choice of projective resolution of* M *in* $\mathfrak{A}_0$.

If $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{B}$ *has enough injectives and* $T(_, J)$ *is exact for all* $J \in ob(\mathfrak{B})$ *injective, then* $(T_M^\bullet, e^\bullet)$ *is co-effaceable. In this case there is a (unique) natural isomorphism*

$$(T_M^\bullet, e^\bullet) \simeq (\mathcal{R}^\bullet T_M, e^\bullet) \tag{2.2.20}$$

of cohomological functors, where $\mathcal{R}^\bullet T_M$ *denote the right derived functors of* $T_M(_) = T(M,_)$.

(b) *Let* $S(_,_)\colon \mathfrak{A}_0 \times \mathfrak{B} \to \mathfrak{Q}$ *be a biadditive right/right exact covariant/covariant bifunctor such that* $S(P,_)\colon \mathfrak{B} \to \mathfrak{Q}$ *is exact for all* $P \in ob(\mathfrak{A}_0)$ *projective in* $\mathfrak{A}$. *Then* $(S_\bullet^M, e_\bullet)$ *with*

$$S_\bullet^M(B) = H_\bullet(S(P_\bullet, B), \partial_\bullet), \tag{2.2.21}$$

and $e_\bullet$ *given by the exact homology sequence is a homological functor. Moreover, it does not depend on the particular choice of projective resolution of* M *in* $\mathfrak{A}_0$.

If $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{B}$ *has enough projectives and* $S(_, Q)$ *is exact for all* $Q \in ob(\mathfrak{B})$ *projective, then* $(S_\bullet^M, e_\bullet)$ *is effaceable. In this case there is a (unique) natural isomorphism*

$$(S_\bullet^M, e_\bullet) \simeq (\mathcal{L}_\bullet S^M, e_\bullet) \tag{2.2.22}$$

of homological functors where $\mathcal{L}_\bullet S^M$ *denote the left derived functors of* $S^M(_) = S(M,_)$.

Proof. (a) By hypothesis, if $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ is a short exact sequence in $\mathfrak{B}$, then there is a short exact seqeunce of cochain complexes

$$0 \to T(P_\bullet, A) \xrightarrow{\sigma} T(P_\bullet, B) \xrightarrow{\tau} T(P_\bullet, C) \to 0. \tag{2.2.23}$$

The first part of (a) now follows from the exact homology sequence. For the second part, the hypothesis ensures that $(T_M^\bullet, e^\bullet)$ is co-effaceable. The cohomological functor $(\mathcal{R}^\bullet T_M, e^\bullet)$ is also co-effaceable ([18], §XII.9). Thus the claim follows from Theorem 2.2.1(a). Part (b) follows by similar arguments.

2.3 Homology and cohomology for augmented algebras

Let R be a commutative ring with unit 1_R and let (A, ε) be an augmented R-algebra which is also a free R-module, i.e., A has a unit 1_A and $\varepsilon\colon A \to R$ is a morphism of R-algebras with $\varepsilon(1_A) = 1_R$. By $A^+ = ker(\varepsilon)$ we denote the augmentation ideal. Then $(\mathbb{B}_\bullet(A), d_\bullet)$ with

$$\mathbb{B}_k(A) = A \otimes \overbrace{A^+ \otimes \cdots \otimes A^+}^{k \text{ times}}, \tag{2.3.1}$$

and $d_\bullet$ given by

$$\begin{aligned} d_k(x[x_1|..|x_k]) &= xx_1[x_2|..|x_k] \\ &\quad + \sum_{j=1}^{k-1}(-1)^j x[x_1|..|x_jx_{j+1}|..|x_k], \end{aligned} \tag{2.3.2}$$

is a complex of free left A-modules. Moreover, it is easy to check that $s_k\colon \mathbb{B}_k(A) \to \mathbb{B}_{k+1}(A)$ given by

$$s_k(x[x_1|..|x_k]) = 1.[x - \varepsilon(x).1|x_2|..|x_k] \tag{2.3.3}$$

is a homotopy contraction in positive degrees, i.e., for $k \geq 1$ one has

$$s_{k-1} \circ d_k + d_{k+1} \circ s_k = id_{\mathbb{B}_k(A)}. \tag{2.3.4}$$

Hence $\mathbb{B}_\bullet(A) \xrightarrow{\varepsilon} R$ is a projective resolution of the trivial A-module R, which is called the *(reduced) bar resolution*.

The bifunctors

$$\begin{aligned} \mathrm{Hom}_A(_,_)&\colon {}_A\mathrm{Mod} \times {}_A\mathrm{Mod} \longrightarrow {}_R\mathrm{Mod}, \\ _ \otimes_A _&\colon \mathrm{Mod}_A \times {}_A\mathrm{Mod} \longrightarrow {}_R\mathrm{Mod}, \end{aligned} \tag{2.3.5}$$

satisfy the hypothesis of Theorem 2.2.4. We denote by

$$\begin{aligned} \mathrm{Ext}^\bullet_A(_,_)&\colon {}_A\mathrm{Mod} \times {}_A\mathrm{Mod} \to {}_R\mathrm{Mod}, \\ \mathrm{Tor}^A_\bullet(_,_)&\colon \mathrm{Mod}_A \times {}_A\mathrm{Mod} \to {}_R\mathrm{Mod}, \end{aligned} \tag{2.3.6}$$

the (derived) bifunctors obtained by Theorem 2.2.4. For $M \in {}_A\mathrm{Mod}$ we also put

$$\begin{aligned} H^\bullet(A, M) &= \mathrm{Ext}^\bullet_A(R, M) \\ H_\bullet(A, M) &= \mathrm{Tor}^A_\bullet(R, M). \end{aligned} \tag{2.3.7}$$

We also have to consider algebras A with a right action by some group G inducing R-algebra automorphisms and fixing the augmentation. The class of objects of the *twisted module category* ${}_A\mathrm{TMod}^G$ consists of all left A-modules M which are also right G-modules such that

$$a^g.m^g = (a.m)^g \tag{2.3.8}$$

for all $a \in A$, $m \in M$, $g \in G$. As morphisms one takes all A-module homomorphisms commuting with the G-action, i.e.,

$$\mathrm{Hom}_A^G(M, N) = \{\, \phi \in \mathrm{Hom}_A(M, N) \mid \phi(m^g) = \phi(m)^g \,\}. \tag{2.3.9}$$

Giving $\mathbb{B}_\bullet(A)$ the diagonal G-action, it is easy to see that $(\mathbb{B}_\bullet(A), d_\bullet)$ is actually contained in $ob(\mathbf{C}_\bullet({}_A\mathrm{TMod}^G))$. For $M, N \in {}_A\mathrm{TMod}^G$, $\mathrm{Hom}_A(M, N)$ has a canonical right G-action given by

$$\phi^g(m) = \phi(m^{g^{-1}})^g, \tag{2.3.10}$$

$\phi \in \mathrm{Hom}_A(M, N)$. It follows that $H^\bullet(A, M)$ has a right G-action for $M \in ob({}_A\mathrm{TMod}^G)$. In order to show that it is canonical and does not depend on the particular choice of projective resolution one has to work slightly harder.

Let A and B be two augmented R-algebras as described above, and let $M \in ob({}_A\mathrm{Mod})$, $N \in ob({}_B\mathrm{Mod})$. Let $\phi\colon A \to B$ be a morphism of augmented R-algebras. Then every left B-module can be also considered as left A-module via ϕ, so ϕ induces a covariant additive exact functor $\phi^*\colon {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$. Suppose that $\psi \in \mathrm{Hom}_R(N, M)$ is such that

$$a.\psi(n) = \psi(\phi(a).n). \tag{2.3.11}$$

Then ψ induces a natural transformation of contravariant additive functors

$$\psi_*\colon \mathrm{Hom}_B(_, N) \longrightarrow \mathrm{Hom}_A(\phi^*(_), M)) \tag{2.3.12}$$

given by $\psi^*(f)(q) = \psi(f(q))$ for $q \in Q$, $Q \in ob({}_B\mathrm{Mod})$. Let $P_\bullet \to R$ be a projective resolution of R as a left A-module, and let $Q_\bullet \to R$ be a projective resolution of R as a left B-module. By the Comparison Theorem (Thm. 2.2.3(1)), there exists a mapping of chain complexes $\tau_\bullet\colon P_\bullet \to \phi^*(Q_\bullet)$ with $\tau_k \in \mathrm{Hom}_A(P_k, \phi^*(Q_k))$ and which is uniquely determined up to chain homotopy. Hence the pair (ϕ, ψ) induces a mapping $(\phi, \psi)^\bullet\colon H^\bullet(B, N) \to H^\bullet(A, M)$ defined as the composite

$$H^\bullet(\mathrm{Hom}_B(Q_\bullet, N)) \xrightarrow{H^\bullet(\psi^*)} H^\bullet(\mathrm{Hom}_A(\phi^*(Q_\bullet), M) \xrightarrow{H^\bullet(\tau_\bullet)} H^\bullet(\mathrm{Hom}_A(P_\bullet, M)). \tag{2.3.13}$$

It is easily seen that $(\phi, \psi)^\bullet$ is independent of the particular choice of projective resolutions.

In our case the mappings ϕ_g and ψ_g are given by $\phi(a) = a^{g^{-1}}$, $\psi(m) = m^g$. By definition, (2.3.11) is satisfied. For $P_\bullet = Q_\bullet = \mathbb{B}_\bullet(A)$, $\tau_\bullet$ can be chosen to be given by the right action of the element $g^{-1} \in G$ on $\mathbb{B}_\bullet(A)$. For these mappings one can check easily that the action of $g \in G$ on $H^\bullet(A, M)$ is given by the map $(\phi_g, \psi_g)^\bullet$.

A typical example of an algebra with a group action is the group algebra $A = R[N]$ for a normal subgroup $N \triangleleft G$ and the action given by conjugation. Moreover, ${}_{R[G]}\mathrm{Mod}$ is a subcategory of ${}_{R[N]}\mathrm{TMod}^G$, where $M \in ob({}_{R[G]}\mathrm{Mod})$ is given the right action $m^g = g^{-1}.m$, $m \in M$, $g \in G$. If $G = N$, then (2.3.10) implies that the action of G on $H^\bullet(G, M)$ is trivial.

If $A \leq B$ is an augmented subalgebra of B, then the pair $(\iota_{A,B}, res_A^B) = (\phi, \psi)$ induces the restriction mapping

$$(\iota_{A,B}, res_A^B)^\bullet = res_A^B \colon H^\bullet(B, _) \longrightarrow H^\bullet(A, _). \tag{2.3.14}$$

If G acts on B by algebra automorphisms and if $A^g = A$ for all $g \in G$, then res_A^B is also a G-equivariant morphism. This can be seen easily from the above description using the reduced bar resolutions for A and B.

2.4 Spectral sequences

Throughout this section we assume that R is a commutative ring with 1. In order to compute the homology of a chain complex explicitly one often uses a filtration on the complex and considers the associated spectral sequence. A similar technique applies to the cohomology of a cochain complex.

Let $(A_\bullet, d_\bullet)$ be a non-negative chain complex of R-modules. A *filtration* $F_\bullet$ of $(A_\bullet, d_\bullet)$ is a series of sub chain complexes $(F_k A_\bullet, d_\bullet)_{k\in\mathbb{Z}}$, $F_k A_n \leq A_n$, such that

(1) $F_k A_n \leq F_{k+1} A_n$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$,

(2) $d_n(F_k A_n) \leq F_k A_{n-1}$.

The filtration is called *exhaustive* if $\bigcup_{k\in\mathbb{Z}} F_k A_n = A_n$ for all $n \in \mathbb{N}_0$, and *separated* if $\bigcap_{k\in\mathbb{Z}} F_k A_n = 0$. The differential induces a differential d^0 on the associated (bi)graded object

$$E^0_{s,t} = F_s A_{s+t} / F_{s-1} A_{s+t} \tag{2.4.1}$$

of degree $(0, -1)$. A filtration $F_\bullet$ also induces a filtration $F_\bullet$ on $H_\bullet(A_\bullet, d_\bullet)$. Let

$$\begin{aligned} Z^r_{s,t} &= F_s A_{s+t} \cap d^{-1}_{s+t}(F_{s-r} A_{s+t-1}) \\ B^r_{s,t} &= F_s A_{s+t} \cap d_{s+t+1}(F_{s+r} A_{s+t+1}) \\ Z^\infty_{s,t} &= ker(d_{s+t}) \cap F_s A_{s+t} \\ B^\infty_{s,t} &= im(d_{s+t+1}) \cap F_s A_{s+t} \end{aligned} \tag{2.4.2}$$

Here we have set $d_n^{-1}(F_k A_{n-1}) \colon = \{ x \in A_n \mid d_n(x) \in F_k A_{n-1} \}$.

Let $\eta_{s,t} \colon Z^\infty_{s,t} \to H_{s+t}(A_\bullet, d_\bullet)$ be the canonical map. Then we put

$$F_s H_{s+t}(A_\bullet, d_\bullet) = im(\eta_{s,t}) \tag{2.4.3}$$

and it is easy to check that this defines a filtration on $H_\bullet(A_\bullet, d_\bullet)$. Moreover, it is exhaustive if $F_\bullet$ is exhaustive on $(A_\bullet, d_\bullet)$. By definition, $F_\bullet$ is separated on $H_\bullet(A_\bullet, d_\bullet)$ if and only if

$$ker(d_{s+t}) \cap \bigcap_{s\in\mathbb{Z}} (im(d_{s+t+1}) + F_s A_{s+t}) = im(d_{s+t+1}). \tag{2.4.4}$$

If $F_\bullet$ is separated on $(A_\bullet, d_\bullet)$, then $ker(d_{s+t})$ is closed with respect to the topology $\mathfrak{T}_{F_\bullet}$ induced by the filtration $F_\bullet$. In this case (2.4.4) is equivalent to the fact that $im(d_{s+t+1})$

is closed with respect to $\mathfrak{T}_{F_\bullet}$. Therefore we call a filtration $F_\bullet$ on $(A_\bullet, d_\bullet)$ *convergent* if it is separated, exhaustive and $im(d_n)$ is closed with respect to $\mathfrak{T}_{F_\bullet}$.

A *spectral sequence of homological type* is a sequence of R-modules $(E^r_{s,t}, d^r)$, $r, t \in \mathbb{N}_0$, $s \in \mathbb{Z}$, and R-module mappings $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ such that:

(1) $(E^r_{\bullet,\bullet}, d^r)$ is a chain complex with respect to total degree, i.e., $d^r \circ d^r = 0$,

(2) $E^{r+1}_{\bullet,\bullet} = H_{\bullet,\bullet}(E^r_{\bullet,\bullet}, d^r)$.

A spectral sequence determines a bigraded R-module $E^\infty_{\bullet,\bullet}$, which can be constructed as follows. All the R-modules $E^{r+j}_{s,t}$ are subquotients of $E^r_{s,t}$, $j \in \mathbb{N}_0$. Let $(I^r_{s,t})_{r\in\mathbb{N}_0}$, $(K^r_{s,t})_{r\in\mathbb{N}_0}$ be the corresponding sequences of R-submodules of $E^0_{s,t}$ such that $E^r_{s,t} = K^r_{s,t}/I^r_{s,t}$. Then there are the inclusions

$$0 = K^0_{s,t} \leq K^1_{s,t} \leq K^2_{s,t} \leq \cdots \leq I^2_{s,t} \leq I^1_{s,t} \leq I^0_{s,t} = E^0_{s,t}. \tag{2.4.5}$$

Now define $I^\infty_{s,t} = \bigcap_{r\in\mathbb{N}_0} I^r_{s,t}$, $K^\infty_{s,t} = \bigcup_{r\in\mathbb{N}_0} K^r_{s,t}$ and

$$E^\infty_{s,t} = K^\infty_{s,t}/I^\infty_{s,t}, \tag{2.4.6}$$

and say that $(E^r_{\bullet,\bullet}, d^r)$ *converges* to $E^\infty_{\bullet,\bullet}$. The spectral sequence *collapses at* $r = N$ if $d^{N+j} = 0$ for all $j \geq 0$. In this case $E^\infty_{s,t} = E^N_{s,t}$.

Theorem 2.4.1. *Let $(A_\bullet, d_\bullet)$ be a non-negative chain complex and $F_\bullet$ a convergent filtration. Then there exists a first and second quadrant spectral sequence $(E^r_{s,t}, d^r)$ (i.e., $E^r_{s,t} = 0$ for $t < 0$,) of homological type with*

$$E^0_{s,t} = F_s A_{s+t}/F_{s-1}A_{s+t}, \tag{2.4.7}$$

converging to

$$E^\infty_{s,t} = F_s H_{s+t}(A_\bullet, d_\bullet)/F_{s-1}H_{s+t}(A_\bullet, d_\bullet). \tag{2.4.8}$$

Proof. If we set

$$E^r_{s,t} = Z^r_{s,t}/(B^{r-1}_{s,t} + Z^{r-1}_{s-1,t+1}), \tag{2.4.9}$$

then the proof follows from an elementary calculation ([18], Thm. XI.3.1).

One can define filtrations of cochain complexes in a similar way. Let $(A^\bullet, d^\bullet)$ be a cochain complex. A *filtration* $F^\bullet$ of $(A^\bullet, d^\bullet)$ is a series of sub cochain complexes $(F^k A^\bullet, d^\bullet)_{k\in\mathbb{Z}}$, $F^k A^n \leq A^n$ such that

(1) $F^{k+1}A^n \leq F^k A^n$ for all $k \in \mathbb{Z}, n \in \mathbb{N}_0$,

(2) $d^n(F^k A^n) \leq F^k A^{n+1}$.

Note that for cochain complexes the filtrations under consideration are decreasing, while for chain complexes we have considered increasing filtrations. The filtration is called *exhaustive* if $\bigcup_{k\in\mathbb{Z}} F^k A^n = A^n$ for all $n \in \mathbb{N}_0$, and *separated* if $\bigcap_{k\in\mathbb{Z}} F^k A^n = 0$. Let

$$E_0^{s,t} = F^s A^{s+t}/F^{s+1}A^{s+t} \tag{2.4.10}$$

be the associated graded object, and let $d_0 \colon E_0^{s,t} \to E_0^{s,t+1}$ be the induced differential (of bidegree (0, 1)). In this case one defines

$$\begin{aligned} Z_r^{s,t} &= F^s A^{s+t} \cap (d^{s+t})^{-1}(F^{s+r}A^{s+t+1}), \\ B_r^{s,t} &= F^s A^{s+t} \cap d^{s+t-1}(F^{s-r}A^{s+t-1}), \\ Z_\infty^{s,t} &= F^s A^{s+t} \cap ker(d^{s+t}), \\ B_\infty^{s,t} &= F^s A^{s+t} \cap im(d^{s+t-1}). \end{aligned} \tag{2.4.11}$$

Let $\eta^{s,t} \colon Z_\infty^{s,t} \to H^{s+t}(A^\bullet, d^\bullet)$ denote the canonical projection and let

$$F^s H^{s+t}(A^\bullet, d^\bullet) = im(\eta^{s,t}). \tag{2.4.12}$$

The filtration $F^\bullet$ is called *convergent* if $F^\bullet$ is exhaustive, separated and $im(d^n)$ is closed with respect to the topology $\mathfrak{T}_{F^\bullet}$ for all $n \in \mathbb{N}_0$. As in the previous case one can show that the induced filtration $F^\bullet$ on $H^\bullet(A^\bullet, d^\bullet)$ is exhaustive and separated.

A *spectral sequence of cohomological type* is a sequence of R-modules $(E_r^{s,t}, d_r)$, $r, t \in \mathbb{N}_0$, $s \in \mathbb{Z}$, and R-module homomorphisms $d_r \colon E_r^{s,t} \to E_r^{s+r,t-r+1}$ such that:

(1) $(E_r^{\bullet,\bullet}, d_r)$ is a cochain complex (with respect to total degree),

(2) $E_{r+1}^{s,t} = H^{\bullet,\bullet}(E_r^{\bullet,\bullet}, d_r)$.

As before there is a bigraded R-module $E_\infty^{s,t}$ associated to the spectral sequence $(E_r^{\bullet,\bullet}, d_r)$. The dual version of Theorem 2.4.1 is

Theorem 2.4.2. *Let $(A^\bullet, d^\bullet)$ be a non-negative cochain complex and $F^\bullet$ a convergent filtration. Then there exists a first and second quadrant spectral sequence of cohomological type $(E_r^{s,t}, d_r)$ with*

$$E_0^{s,t} = F^s A^{s+t}/F^{s+1}A^{s+t}, \tag{2.4.13}$$

converging to

$$E_\infty^{s,t} = F^s H^{s+t}(A^\bullet, d^\bullet)/F^{s+1}H^{s+t}(A^\bullet, d^\bullet). \tag{2.4.14}$$

Proof. In this case one sets

$$E_r^{s,t} = Z_r^{s,t}/(Z_r^{s+1,t-1} + B_{r-1}^{s,t}) \tag{2.4.15}$$

and proves the required identities by calculation ([21], §2.2.2).

In general it is quite difficult to obtain information about the higher differentials of a spectral sequence, and in most cases one only knows the E_1 or E_2-term of the spectral sequence. However, in some cases the vanishing of the higher differentials d^r (resp. d_r) follows because of degree reasons (cf. (5.1.25.)).

Theorems 2.4.1 and 2.4.2 can be seen also in a more formal way. The construction of the spectral sequences shows that there are covariant functors

$$\begin{aligned} E^\bullet\colon\ & \mathbf{Fil}(\mathbf{C}_\bullet({}_R\mathrm{Mod})^+) \longrightarrow \mathbf{HomSpSeq}_R, \\ E_\bullet\colon\ & \mathbf{Fil}(\mathbf{C}^\bullet({}_R\mathrm{Mod})^+) \longrightarrow \mathbf{CohomSpSeq}_R, \end{aligned} \tag{2.4.16}$$

where $\mathbf{Fil}(\mathbf{C}_\bullet({}_R\mathrm{Mod})^+)$ (resp. $\mathbf{Fil}(\mathbf{C}^\bullet({}_R\mathrm{Mod})^+)$) denote the category of filtered non-negative chain (resp. cochain) complexes. We have also used the abbreviation $\mathbf{HomSpSeq}_R$ (resp. $\mathbf{CohomSpSeq}_R$) for the category of spectral sequences of homological (resp. cohomological) type. For example, this fact is also used to define a multiplicative structure on certain spectral sequences associated to the cohomology rings of cocommutative Hopf algebras.

2.5 Serre's lemma

The existence of certain projective resolutions is essential for the study of homological and cohomological functors. In this subsection we describe a method of lifting projective resolutions from the graded object $gr^\bullet(A)$ associated to a filtration of an algebra to the algebra A itself.

Let R be a commutative ring with a 1 and let (A, ε) be an augmented R-algebra with unit $\eta\colon R \to A$. We call $(P_\bullet, d_\bullet, \varepsilon_\bullet, s_\bullet)$ a *special projective resolution* if:

(a) $(P_0, \varepsilon_0) = (A, \eta \circ \varepsilon)$,

(b) $P_\bullet \overset{\varepsilon}{\to} R$ is a projective resolution over A of the trivial left A-module R,

(c) $s_\bullet$ is R-linear and a homotopy contraction, i.e.,

$$s_{k-1} \circ d_k + d_{k+1} \circ s_k = 1_{P_k} \tag{2.5.1}$$

for $k \geq 1$ and $d_1 \circ s_0 = 1_{P_0} - \varepsilon_0$.

Here we have considered $P_\bullet = \coprod_{k\in\mathbb{N}_0} P_k$ as $\mathbb{N}_0$-graded left A-module and $\varepsilon_\bullet = (\varepsilon_k)_{k\in\mathbb{N}_0}$, where $\varepsilon_k = 0$ for $k \geq 1$ and $\varepsilon_0\colon A \overset{\varepsilon}{\to} R \overset{\eta}{\to} A$. We have

$$\begin{aligned} d_\bullet &\in \mathrm{Hom}_A^{-1}(P_\bullet, P_\bullet), \\ \varepsilon_\bullet &\in \mathrm{Hom}_R^{0}(P_\bullet, P_\bullet), \\ s_\bullet &\in \mathrm{Hom}_R^{1}(P_\bullet, P_\bullet), \end{aligned} \tag{2.5.2}$$

where the upper index denotes the degree of a homomorphisms of graded modules. The fact that $(P_\bullet, d_\bullet, \varepsilon_\bullet, s_\bullet)$ is a special projective resolution can be expressed by the

identities

$$\begin{aligned} d_\bullet \circ d_\bullet &= 0, \\ \varepsilon_\bullet \circ (1_\bullet - \varepsilon_\bullet) &= 0, \\ d_\bullet \circ s_\bullet + s_\bullet \circ d_\bullet &= id_{P_\bullet}, \\ \varepsilon_\bullet \circ s_\bullet = s_\bullet \circ \varepsilon_\bullet &= 0, \\ \varepsilon_\bullet \circ d_\bullet = d_\bullet \circ \varepsilon_\bullet &= 0. \end{aligned} \tag{2.5.3}$$

Note that if R is a field and $P_\bullet \xrightarrow{\varepsilon} R$ is a projective resolution of the trivial module with $P_0 = A$, then one can always find a chain homotopy contraction $s_\bullet$ such that $(P_\bullet, d_\bullet, \varepsilon_\bullet, s_\bullet)$ is special.

Let R be either $\mathbb{Z}_p$ or $\mathbb{Q}_p$ and let $F^\bullet$ denote the natural filtration associated to the p-adic valuation, i.e., $F^k\mathbb{Z}_p = p^k.\mathbb{Z}_p$, $k \in \mathbb{N}_0$, and $F^k\mathbb{Q}_p = p^k.\mathbb{Z}_p$, $k \in \mathbb{Z}$. We denote the graded object associated to a filtration of a module by $gr^\bullet(_)$. Assume that (A, ε) is an augmented R-algebra with an integral (decreasing) filtration $F^\bullet = (F^k)_{k\in\mathbb{Z}}$ on A such that:

(1) A is a filtered augmented R-algebra, i.e.,

$$\begin{aligned} F^s R.F^t A &\leq F^{s+t}A, \\ F^s A.F^t A &\leq F^{s+t}A, \\ \eta(F^s R) &\leq F^s A, \\ \varepsilon(F^s A) &\leq F^s R; \end{aligned} \tag{2.5.4}$$

(2) the augmentation ideal $I(A)$ is complete with respect to the filtration $F^\bullet A$, and $gr^\bullet(I(A))$ is a free $gr^\bullet(R)$-module.

There is an important tool for constructing special resolutions ([17], (V.2.1)):

Lemma 2.5.1. (J-P. Serre) *Suppose that $(A, \varepsilon, F^\bullet)$ satisfies* (1) *and* (2) *as above. Let $(\bar{P}_\bullet, \bar{d}_\bullet, \bar{\varepsilon}_\bullet, \bar{s}_\bullet)$ be a special projective resolution of $gr^\bullet(A)$, where $\bar{d}_\bullet$ and $\bar{s}_\bullet$, $\bar{\varepsilon}_\bullet$ are mappings of graded left $gr^\bullet(A)$ and $gr^\bullet(R)$-modules, respectively. Assume further that each $\bar{P}_k$ is a free left $gr^\bullet(A)$-module of finite rank.*

Then there exist a special projective resolution $(P_\bullet, d_\bullet, \varepsilon_\bullet, s_\bullet)$ for A and a filtration $F^\bullet$ on every $P^\bullet$ such that:

(i) *P_k is a filtered-free left A-module of rank $rk_A(P_k) = rk_{gr^\bullet(A)}(\bar{P}_k)$,*

(ii) *$d_\bullet$ is a morphism of filtered left A-modules, and $\varepsilon_\bullet$, $s_\bullet$ are morphisms of filtered left R-modules,*

(iii) *$(\bar{P}_\bullet, \bar{d}_\bullet, \bar{\varepsilon}_\bullet, \bar{s}_\bullet) \simeq (gr^\bullet(P_\bullet), gr^\bullet(d_\bullet), gr^\bullet(\varepsilon_\bullet), gr^\bullet(s_\bullet))$ as special projective resolutions of $gr^\bullet(A)$.*

Here a filtered A-module Q of finite rank is called *filtered-free* if there exist a generating set $x_1, .., x_n$ and numbers $r_1, .., r_n$ such that

$$\begin{aligned} Q &= A.x_1 \oplus \cdots \oplus A.x_n, \\ \text{and } F^k Q &= F^{k-r_1}A.x_1 \oplus \cdots \oplus F^{k-r_n}A.x_n. \end{aligned} \tag{2.5.5}$$

Remark 2.5.2. Let R be a commutative ring with a 1 and let $\mathfrak{L}$ be an R-Lie algebra which is also a finitely generated free R-module. One knows that the *Koszul complex* $(\mathbf{Ko}_\bullet(\mathfrak{L}), d_\bullet)$, where

$$\mathbf{Ko}_\bullet(\mathfrak{L}) = \mathfrak{U}_R(\mathfrak{L}) \otimes \Lambda_\bullet(\mathfrak{L}) \tag{2.5.6}$$

and

$$\begin{aligned} d_n(x\,\langle y_1 \wedge \ldots \wedge y_n\rangle) = &\sum_{1\le i\le n} (-1)^{i+1}\, x.y_i\,\langle \check{y}_i\rangle \\ &+ \sum_{1\le i<j\le n} (-1)^{i+j}\, x\,\langle [y_i, y_j] \wedge \check{y}_{i,j}\rangle, \end{aligned} \tag{2.5.7}$$

is a projective resolution of the trivial $\mathfrak{U}_R(\mathfrak{L})$-module ([16]). (Here we have used the abbreviation $\check{y}_i = y_1 \wedge .. \wedge y_{i-1} \wedge y_{i+1} \wedge .. \wedge y_n$.) In [17], (V.1.3.6) it is shown that there exists a homotopy contraction map $s_\bullet$ such that $(\mathbf{Ko}_\bullet(\mathfrak{L}), d_\bullet, \varepsilon_\bullet, s_\bullet)$ is a special projective resolution.

Assume that $R = R_\bullet$ is a $\mathbb{Z}$-graded ring and that $\mathfrak{L} = \mathfrak{L}_\bullet$ is a $\mathbb{Z}$-graded $R_\bullet$-Lie algebra such that $\mathfrak{L}_\bullet$ is a finitely generated free $R_\bullet$-module. Then $\mathfrak{U}_{R_\bullet}(\mathfrak{L}_\bullet)$ is a $\mathbb{Z}$-graded augmented $R_\bullet$-algebra. If $s^0_\bullet$ denotes the 0-component of the chain homotopy contraction described above, then $(\mathbf{Ko}_\bullet(\mathfrak{L}), d_\bullet, \varepsilon_\bullet, s^0_\bullet)$ is a special projective resolution with $d_\bullet, \varepsilon_\bullet, s^0_\bullet$ mappings of $\mathbb{Z}$-graded left $\mathfrak{U}_{R_\bullet}(\mathfrak{L}_\bullet)$-modules or $R_\bullet$-modules, as appropriate.

3 Cohomology and homology for profinite groups

For a profinite group G there are several natural categories of representations. In this section we will give a short description of these categories as well as the cohomological and homological functors they define. Unfortunately there seems to be no nice category of representations which contains them all. This is the reason why we must study different cohomology and homology theories and the relations between them.

When G is a profinite group we write $H \le G$ for a closed subgroup H of G, and $K \le_o G$ for an open subgroup K of G. Similarly we use the notation $H \lhd G$ for a closed normal subgroup H of G, and $K \lhd_o G$ for an open normal subgroup K of G.

3.1 Continuous discrete cohomology

Let G be a profinite group. A left G-module M is called *discrete* if M is a discrete abelian group with continuous G-action $\psi\colon G \times M \to M$. We denote the category of discrete left G-modules by $\mathfrak{D}(G)$[7] It is a full subcategory of ${}_{\mathbb{Z}[G]}\mathrm{Mod}$, where $\mathbb{Z}[G]$ denotes the group algebra of the abstract group G over $\mathbb{Z}$. A left $\mathbb{Z}[G]$-module M is contained in $ob(\mathfrak{D}(G))$ if and only if

$$\mathrm{Stab}_G(m) = \{\, g \in G \mid g.m = m \,\} \le_o G \tag{3.1.1}$$

[7] In [23] this category was denoted by C_G.

is an open subgroup of G for all $m \in M$, i.e.,

$$M = \bigcup_{U \leq_o G} M^U, \tag{3.1.2}$$

where $M^U = \{ m \in M \mid u.m = m \text{ for all } u \in U \}$. For $M \in ob(_{\mathbb{Z}[G]}\mathrm{Mod})$ we define

$$D(M) = \{ m \in M \mid \mathrm{Stab}_G(m) \leq_o G \} \in ob(\mathfrak{D}(G)). \tag{3.1.3}$$

For every morphism $\phi\colon N \to M \in mor(_{\mathbb{Z}[G]}\mathrm{Mod})$ with $N \in ob(\mathfrak{D}(G))$ one has $im(\phi) \leq D(M)$. Hence

$$D\colon {}_{\mathbb{Z}[G]}\mathrm{Mod} \longrightarrow \mathfrak{D}(G) \tag{3.1.4}$$

with $D(\phi) = \phi|_{D(N)}$ for $\phi\colon N \to M \in mor(_{\mathbb{Z}[G]}\mathrm{Mod})$ is a left exact covariant additive functor, and one can verify that it maps injectives to injectives. This yields the following result ([18], Chap. III, Thm. 7.4, Thm. 11.3).

Proposition 3.1.1. *$\mathfrak{D}(G)$ is an abelian module category with enough injectives. Moreover, every module $M \in ob(\mathfrak{D}(G))$ has a minimal injective envelope $i_M\colon M \to J(M)$, i.e., for every injective morphism $\phi\colon M \to J$ into an injective module $J \in ob(\mathfrak{D}(G))$ there exists an injective morphism $\phi^*\colon J(M) \to J$ such that $\phi = \phi^* \circ i_M$.*

For $M \in ob(\mathfrak{D}(G))$, $\mathrm{Hom}_{\mathfrak{D}(G)}(M, _)\colon \mathfrak{D}(G) \to {}_{\mathbb{Z}}\mathrm{Mod}$ is a left exact covariant additive functor with right derived functors $\mathrm{Ext}^n_{\mathfrak{D}(G)}(M, _)$ (Thm. 2.2.4(b), [18], §XII.9). The *continuous discrete cohomology* of G with coefficients in the discrete left G-module N is defined[8] by

$$H^\bullet(G, N) = \mathrm{Ext}^\bullet_{\mathfrak{D}(G)}(\mathbb{Z}, N). \tag{3.1.5}$$

Let $\mathfrak{D}_p(G)$ denote the full subcategory of $\mathfrak{D}(G)$ consisting of discrete left G-modules which are p-torsion modules. It has an important property ([6], Lemma 1.8, [23], Chap. I, Ann. 2, Lemme 3.3).

Proposition 3.1.2. *$\mathfrak{D}_p(G)$ is an abelian category with enough injectives. If $J \in ob(\mathfrak{D}_p(G))$ is injective, it is also injective in $\mathfrak{D}(G)$.*

Proposition 3.1.2 implies that the injective envelope $J(M)$ (Prop. 3.1.1) is contained in $ob(\mathfrak{D}_p(G))$ provided $M \in ob(\mathfrak{D}_p(G))$.

For $M \in ob(\mathfrak{D}_p(G))$ the canonical mapping $\mathrm{Hom}_{\mathfrak{D}(G)}(\mathbb{Z}_p, M) \to \mathrm{Hom}_{\mathfrak{D}(G)}(\mathbb{Z}, M)$ is an isomorphism. Hence

$$H^n(G, M) \simeq \mathrm{Ext}^n_{\mathfrak{D}(G)}(\mathbb{Z}_p, M). \tag{3.1.6}$$

3.2 The compact module category

From now on we fix a prime number p. By $\mathbb{Z}_p[\![G]\!]$ we denote the compact group algebra of G over $\mathbb{Z}_p$, i.e., $\mathbb{Z}_p[\![G]\!] = \varprojlim \mathbb{Z}_p[G/U]$, where U ranges over the inverse

[8] This definition coincides with the one given in [23] (cf. remark in [23], §2.2).

system of open normal subgroups of G (see [10], §7). Hence $\mathfrak{D}_p(G)$ coincides with the category of discrete left p-torsion $\mathbb{Z}_p[\![G]\!]$-modules ($\mathfrak{D}_p(G) = \mathfrak{D}_{\mathbb{Z}_p[\![G]\!]}$ in the notation of [6]).

Let $\mathfrak{F}_p(G) \leq \mathfrak{D}_p(G)$ denote the full subcategory of $\mathfrak{D}_p(G)$ whose objects are finite (discrete) left $\mathbb{Z}_p[\![G]\!]$-modules. By $\mathfrak{C}_p(G)$ we denote the category of topological $\mathbb{Z}_p[\![G]\!]$-modules whose objects consist of inverse limits of objects in $\mathfrak{F}_p(G)$ and whose morphisms are given by

$$\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) = \{\alpha \in \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) \mid \alpha \text{ continuous}\}, \tag{3.2.1}$$

$M, N \in ob(\mathfrak{C}_p(G))$. We consider $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N)$ as a topological space with the compact-open topology. This makes $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N)$ into a complete Hausdorff-topological $\mathbb{Z}_p$-module. Moreover, if M is a finitely generated topological $\mathbb{Z}_p[\![G]\!]$-module, then $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N)$ is compact. Note that every continuous morphism $\alpha\colon M \to N$ is also strict. One important property of $\mathfrak{C}_p(G)$ is ([6], Lemma 1.6):

Proposition 3.2.1. *$\mathfrak{C}_p(G)$ is a topological abelian module category with enough projectives.*

If $M \in ob(\mathfrak{C}_p(G))$ and $N \in ob(\mathfrak{D}_p(G))$, then $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N)$ is a discrete $\mathbb{Z}_p$-module. The bifunctor

$$\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(_, _)\colon \mathfrak{C}_p(G) \times \mathfrak{D}_p(G) \longrightarrow \mathfrak{D}_p, \tag{3.2.2}$$

is contravariant/covariant and left/left exact. Furthermore, $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P, _)$ is exact for $P \in ob(\mathfrak{C}_p(G))$ projective, and $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(_, J)$ is exact for $J \in ob(\mathfrak{D}_p(G))$ injective ([6], Lem. 2.2]). We denote the right-derived bifunctors of $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(_, _)$ as in Theorem 2.2.4(a) by

$$\mathbf{Ext}^n_{\mathbb{Z}_p[\![G]\!]}(_, _)\colon \mathfrak{C}_p(G) \times \mathfrak{D}_p(G) \longrightarrow \mathfrak{D}_p. \tag{3.2.3}$$

Then (3.1.6) implies that

$$H^\bullet(G, M) = \mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M) \tag{3.2.4}$$

for all $M \in ob(\mathfrak{D}_p(G))$.

3.3 Continuous homology

If H is a closed subgroup of the profinite group G and $M \in ob(\mathfrak{C}_p(G))$ (resp. $ob(\mathfrak{D}_p(G))$) is a left $\mathbb{Z}_p[\![G]\!]$-module then we denote by $res^G_H(M) \in ob(\mathfrak{C}_p(H))$ (resp. $ob(\mathfrak{D}_p(H))$) the restriction of M to $\mathbb{Z}_p[\![H]\!]$. The next proposition allows us to define restriction maps in cohomology and homology.

Proposition 3.3.1. *Let H be a closed subgroup of the profinite group G and let $P \in ob(\mathfrak{C}_p(G))$ be projective. Then $res^G_H(P) \in ob(\mathfrak{C}_p(H))$ is projective.*

Proof. This follows from [6], Lemma 1.6 and Corollary 3.3.

On the category $\mathfrak{C}_p$ one has also a tensor product $\widehat{\otimes}_{\mathbb{Z}_p}$ ([6], §2). For $M, N \in ob(\mathfrak{C}_p)$ the tensor product is given by $M\widehat{\otimes}_{\mathbb{Z}_p}N = \varprojlim M/U \otimes_{\mathbb{Z}_p} N/V$ where the inverse system ranges over the pairs (U, V) of open submodules in M and N. Note that the abstract tensor product $M \otimes_{\mathbb{Z}_p} N$ is a dense submodule of $M\widehat{\otimes}_{\mathbb{Z}_p}N$. If $M, N \in ob(\mathfrak{C}_p(G))$ then $M\widehat{\otimes}_{\mathbb{Z}_p}N \in ob(\mathfrak{C}_p(G))$ has canonically the structure of a left (topological) $\mathbb{Z}_p[\![G]\!]$-module. The action of G on $M\widehat{\otimes}_{\mathbb{Z}_p}N$ is given by

$$g.(m\widehat{\otimes}n) = g.m\widehat{\otimes}g.n, \tag{3.3.1}$$

where $m \in M$, $n \in N$, $g \in G$. This tensor product has the following property which is well known for finite groups.

Proposition 3.3.2. *Let $M, P \in ob(\mathfrak{C}_p(G))$, and suppose that P is projective and M is p-torsion-free. Then $M\widehat{\otimes}_{\mathbb{Z}_p}P$ is projective.*

Proof. Note that by hypothesis $res^G_1(M) \in \mathfrak{C}_{\mathbb{Z}_p}$ is free (cf. Cor. 2.1.2). P is a direct summand of a free $\mathbb{Z}_p[\![G]\!]$-module $F \in ob(\mathfrak{C}_p(G))$ generated freely by a set I (cf. [6], Lemma 1.6). Assume that I is finite. The mapping $\phi\colon M\widehat{\otimes}_{\mathbb{Z}_p}F \to M\widehat{\otimes}_{\mathbb{Z}_p}F$, where

$$\phi(\sum_{i\in I} m_i\widehat{\otimes}g_i.i) = \sum_{i\in I} g_i^{-1}.m_i\widehat{\otimes}g_i.i, \tag{3.3.2}$$

$m_i \in M$, $g_i \in G$, is a continuous bijective mapping of $\mathbb{Z}_p[\![G]\!]$-modules. Note that the G-action on the module on the right hand side is given by $g.(m\widehat{\otimes}g_i.i) = m\widehat{\otimes}gg_i.i$.. Thus if J is a basis for M as profinite $\mathbb{Z}_p$-module, then $M\widehat{\otimes}_{\mathbb{Z}_p}F$ is a free $\mathbb{Z}_p[\![G]\!]$-module generated by $J \times I$. If I is infinite, then F is the inverse limit $F = \varprojlim F(X)$ where X runs over the finite subsets of I. By [6], Cor. 3.3, Lemma A.4, $M\widehat{\otimes}_{\mathbb{Z}_p}F$ is free, so $M\widehat{\otimes}_{\mathbb{Z}_p}P$ is projective.

Let $\mathfrak{C}_p(G)^\times$ denote the corresponding category of right $\mathbb{Z}_p[\![G]\!]$-modules. The bifunctor

$$_\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}_ : \mathfrak{C}_p(G)^\times \times \mathfrak{C}_p(G) \longrightarrow \mathfrak{C}_p \tag{3.3.3}$$

is covariant/covariant, right/right exact and biadditive ([6], Lemma 2.1(i)). Let

$$\mathbf{Tor}_\bullet^{\mathbb{Z}_p[\![G]\!]}(_,_)\colon \mathfrak{C}_p(G)^\times \times \mathfrak{C}_p(G) \longrightarrow \mathfrak{C}_{\mathbb{Z}_p} \tag{3.3.4}$$

denote its left derived functors (Thm.2.2.4(b), [18], XII). We define the *continuous homology of G* with coefficients in $M \in ob(\mathfrak{C}_p(G))$ by

$$H_\bullet(G, M) = \mathbf{Tor}_\bullet^{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M). \tag{3.3.5}$$

Let $H \leq_o G$ be an open and $K \leq G$ be a closed subgroup of G. For $M \in ob(\mathfrak{C}_p(K))$, $N \in ob(\mathfrak{C}_p(H))$ we define

$$\begin{aligned} ind_G^K(M) &= \mathbb{Z}_p[\![G]\!]\widehat{\otimes}_{\mathbb{Z}_p[\![K]\!]}M, \\ coind_G^H(N) &= \mathbf{Hom}_{\mathbb{Z}_p[\![H]\!]}(\mathbb{Z}_p[\![G]\!], N) \end{aligned} \tag{3.3.6}$$

the *induced and coinduced* $\mathbb{Z}_p[\![G]\!]$*-modules* with the natural left $\mathbb{Z}_p[\![G]\!]$-action, i.e., for $f \in coind_G^H(M)$ the action is given by $(g.f)(x) = f(x.g)$. For an open subgroup $H \leq_o G$ one can prove easily that

$$ind_G^H(M) \simeq coind_G^H(M) \tag{3.3.7}$$

as $\mathbb{Z}_p[\![G]\!]$-modules (see [5], Chap. III, Prop. 5.9 for the discrete case).

Remark 3.3.3. The proof of Proposition 3.3.2 shows that that for $M \in ob(\mathfrak{C}_p(G))$ one has

$$M\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!] \simeq ind_G^1(res_1^G(M)). \tag{3.3.8}$$

Hence, $M\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!]$ is always induced from the trivial group.

If $H \leq G$ is a closed subgroup and $M \in ob(\mathfrak{C}_p(H))$, then the functor $\widehat{\otimes}_{\mathbb{Z}_p[\![H]\!]}M$ is naturally isomorphic to $\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}\mathbb{Z}_p[\![G]\!]\widehat{\otimes}_{\mathbb{Z}_p[\![H]\!]}M$ considered as a functor from $\mathfrak{C}_p(G)$ to $\mathfrak{C}_p$. So Proposition 3.3.1 yields the following Eckmann-Shapiro type lemma:

Lemma 3.3.4. *Let $H \leq G$ be a closed subgroup of the profinite group G and let $M \in ob(\mathfrak{C}_p(H))$. Then*

$$H_\bullet(G, ind_G^H(M)) \simeq H_\bullet(H, M). \tag{3.3.9}$$

3.4 The Pontrjagin category

Our philosophy is that for "nice" profinite groups it should be possible to define cohomology and homology functors on a category containing $\mathfrak{D}_p(G)$ and $\mathfrak{C}_p(G)$ as full subcategories. This allows one to compare homology and cohomology. We therefore define $\mathfrak{P}_p(G)$ to be the category whose objects are the topological $\mathbb{Z}_p[\![G]\!]$-modules which belong to either $ob(\mathfrak{C}_p(G))$ or in $ob(\mathfrak{D}_p(G))$. The morphisms are given by

$$\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) = \{\alpha \in \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) \mid \alpha \text{ continuous and strict}\}. \tag{3.4.1}$$

Note that if $M \in ob(\mathfrak{D}_p(G))$, $N \in ob(\mathfrak{C}_p(G))$, then a continuous morphism $\alpha\colon M \to N$ is strict if and only if it has finite image.

Proposition 3.4.1. *$\mathfrak{P}_p(G)$ is a topological exact module category containing $\mathfrak{C}_p(G)$ and $\mathfrak{D}_p(G)$ as full subcategories. If $M \in ob(\mathfrak{C}_p(G))$ is projective in $\mathfrak{C}_p(G)$, then it is also projective in $\mathfrak{P}_p(G)$. If $N \in ob(\mathfrak{D}_p(G))$ is injective in $\mathfrak{D}_p(G)$, then it is also injective in $\mathfrak{P}_p(G)$.*

Proof. From the previous remark it follows easily that $\mathfrak{P}_p(G)$ is an exact category ([4], Chap. III, §2.8, remark after Prop. 24). If $P \in ob(\mathfrak{C}_p(G))$ is projective, then $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P, _)\colon \mathfrak{D}_p(G) \to \mathfrak{D}_p$ is exact, while for $J \in ob(\mathfrak{D}_p(G))$ injective, the functor $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(_, J)\colon \mathfrak{C}_p(G) \to \mathfrak{D}_p$ is exact ([6], Lemma 2.2). This implies the assertion.

Note that every short exact sequence in $\mathfrak{P}_p(G)$ is either a short exact sequence in $\mathfrak{C}_p(G)$ or a short exact sequence in $\mathfrak{D}_p(G)$. Again we consider $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N)$,

$M, N \in ob(\mathfrak{P}_p(G))$, as a topological $\mathbb{Z}_p$-module where the topology is the compact-open topology. So for $A, B \in ob(\mathfrak{C}_p(G))$, $M, N \in \mathfrak{D}_p(G)$, $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(A, M)$ is discrete, $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(A, B)$ carries the topology of uniform convergence and $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N)$ carries the topology of pointwise convergence.

Set $\mathfrak{P}_p = \mathfrak{P}_p(1)$ and let $I_p = \mathbb{Q}_p/\mathbb{Z}_p \in ob(\mathfrak{P}_p)$. For $M \in \mathfrak{P}_p(G)$ we let

$$M^* = \mathbf{Hom}_{\mathbb{Z}_p}(M, I_p) \in ob(\mathfrak{P}_p(G)), \tag{3.4.2}$$

where the action is given by $(g.f)(x) = f(g^{-1}.x)$, $x \in M$, $f \in M^*$, $g \in G$. With the obvious assignment on morphisms we have (Prop. 2.1.1, [6], Prop. 2.3):

Proposition 3.4.2. $^*\colon \mathfrak{P}_p(G) \to \mathfrak{P}_p(G)$ *is an exact contravariant functor such that* $^* \circ {}^*$ *is naturally equivalent to the identity functor. Moreover, the canonical mapping*

$$^*\colon \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(N^*, M^*) \tag{3.4.3}$$

given by $\alpha^*(n^*)(m) = n^*(\alpha(m))$, $\alpha \in \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N)$, $n^* \in N^*$, $m \in M$, *is an isomorphism of topological* $\mathbb{Z}_p$*-modules.*

Remark 3.4.3. (i) Since * must take projectives to injectives and injectives to projectives, it follows from Propositions 3.1.1 and 3.1.2 that every compact module $M \in ob(\mathfrak{C}_p(G))$ has a minimal projective cover $\gamma_M\colon P(M) \to M \in mor(\mathfrak{C}_p(G))$.
(ii) If $M \in ob(\mathfrak{C}_p(G))$, $N \in ob(\mathfrak{D}_p(G))$, then general principles of homological algebra ([18], Chap. XII, §9) show that (3.4.3) induces an isomorphism

$$\mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(N^*, M^*) \simeq \mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(M, N) \tag{3.4.4}$$

and that this isomorphism is natural in M and N.
(iii) Let M and N be as in (ii). For $N \in ob(\mathfrak{D}_p(G))$ let $N^\times \in ob(\mathfrak{C}_p(G)^\times)$ be the module defined by (3.4.2) but with the canonical right $\mathbb{Z}_p[\![G]\!]$-module structure. Then there is also an isomorphism

$$\mathbf{Tor}^{\mathbb{Z}_p[\![G]\!]}_\bullet(N^\times, M) \simeq \mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(M, N)^* \tag{3.4.5}$$

and this isomorphism is natural in M and N ([6], Cor. 2.6).
(iv) From (3.4.4) and (3.4.5), it follows that for $M \in ob(\mathfrak{C}_p(G))$,

$$H_\bullet(G, M)^* \simeq H^\bullet(G, M^*) \tag{3.4.6}$$

and that this isomorphism is natural in M.
(v) Let $H \leq_o G$ be a closed subgroup and $J \in ob(\mathfrak{D}_p(G))$ injective. From Pontrjagin duality and Proposition 3.3.1 it follows that $res^H_G(J) \in ob(\mathfrak{D}_p(H))$ is injective. This can be used to show the following Eckmann-Shapiro type lemma: Let $H \leq_o G$ be a closed subgroup of the profinite group G, $M \in ob(\mathfrak{C}_p(H))$, $N \in ob(\mathfrak{D}_p(G))$. Then

$$\mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![H]\!]}(M, N) \simeq \mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(ind^H_G(M), N). \tag{3.4.7}$$

3.5 Some relative homological algebra

When we have used the notion of a derived functor, it has always been with respect to the class $\mathbf{\Sigma}$ of all short exact sequences in $\mathfrak{D}_p(G)$ (resp. $\mathfrak{C}_p(G)$). If one uses the cup product to construct a morphism of connected sequences, then one has to deal with the problem that it behaves nicely only for short exact sequences which split as abelian groups. Therefore we define $\mathbf{\Sigma}^{sp} \subset \mathbf{\Sigma}$ to be the class of short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathfrak{D}_p(G)$ (resp. $\mathfrak{C}_p(G)$) which split under restriction to $\mathfrak{D}_p$ (resp. $\mathfrak{C}_p$). It is easy to check that $\mathbf{\Sigma}^{sp}$ is a proper class of short exact sequences. With the obvious classes $\mathbf{\Sigma}^{sp}_{inj}$ and $\mathbf{\Sigma}^{sp}_{sur}$ of proper injective and surjective morphism, one can show easily:

Proposition 3.5.1.

(1) $\mathfrak{D}_p(G)$ contains enough $\mathbf{\Sigma}^{sp}$-injectives, and every $\mathbf{\Sigma}$-injective is also $\mathbf{\Sigma}^{sp}$-injective.

(2) $\mathfrak{C}_p(G)$ contains enough $\mathbf{\Sigma}^{sp}$-projectives, and every $\mathbf{\Sigma}$-projective is also $\mathbf{\Sigma}^{sp}$-projective.

Proof. (1) Let $M \in ob(\mathfrak{D}_p(G))$. The canonical mapping $i_M\colon M \to J_M = \mathbf{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[\![G]\!], M)$ given by $i_M(m)(x) = x.m$, $m \in M$, $x \in \mathbb{Z}_p[\![G]\!]$, is a morphism of left $\mathbb{Z}_p[\![G]\!]$-modules, where the action on J_M is given by $(g.f)(x) = f(x.g)$. Moreover, $i_M \in \mathbf{\Sigma}^{sp}_{inj}$, since evaluation at $1 \in \mathbb{Z}_p[\![G]\!]$ is an obvious section (in $\mathfrak{D}_p$). It is easy to check that J_M is injective relative to $\mathbf{\Sigma}^{sp}$ ([18], Lemma XII.10.1). Since $\mathbf{\Sigma}^{sp}_{inj} \subset \mathbf{\Sigma}_{inj}$, the second statement is obvious.
(2) follows by similar arguments using induced modules or also by Pontrjagin duality (Prop. 3.4.2).

For short we call a functor $T\colon \mathfrak{A} \to \mathfrak{B}$ an $\mathfrak{A}$-functor. Let $M \in ob(\mathfrak{C}_p(G))$. Proposition 3.5.1 implies that the right derived $\mathfrak{D}_p(G)$-functors $\mathbf{Ext}^k_{\mathbb{Z}_p[\![G]\!]}(M, _)_{sp} = \mathbf{Ext}^k_{\mathbb{Z}_p[\![G]\!],\mathbb{Z}_p}(M, _)$ and $\mathbf{Ext}^k_{\mathbb{Z}_p[\![G]\!]}(M, _)$ are naturally isomorphic. However, considered together with the connecting homomorphisms, $(\mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(M, _)_{sp}, e^\bullet_{sp})$ constitutes a right universal $\mathbf{\Sigma}^{sp}$-connected sequence,[9] while $(\mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(M, _), e^\bullet)$ constitutes a right universal $\mathbf{\Sigma}$-connected sequence of $\mathfrak{D}_p(G)$-functors. Note that $e^\bullet|_{\Sigma^{sp}} = e^\bullet_{sp}$, so the two approaches differ only if one considers these functors as connected sequences. The following lemma shows that natural transformations of these functors are rather rigid with respect to the connecting homomorphisms.

Proposition 3.5.2. *Let $M \in ob(\mathfrak{C}_p(G))$, $T^k(_) = \mathbf{Ext}^k_{\mathbb{Z}_p[\![G]\!]}(M, _)$ and let $\mathbf{U}^\bullet = (U^n, e^n)_{n\in\mathbb{N}_0}$, $U^n\colon \mathfrak{D}_p(G) \to \mathfrak{D}_p$, be a $\mathbf{\Sigma}$-connected sequence of covariant additive functors. Suppose that $\mathbf{f}^\bullet = (f^n)_{n\in\mathbb{N}_0}$, is a sequence of natural transformations such*

[9] For definitions see [18], Chap. XII, §8.

that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^0(A) & \xrightarrow{T^0(\sigma)} & T^0(B) & \xrightarrow{T^0(\tau)} & T^0(C) & \xrightarrow{e^0(\mathbf{s})} & T^1(A)\cdots \\ \downarrow & & {\scriptstyle f^0(A)}\downarrow & & {\scriptstyle f^0(B)}\downarrow & & {\scriptstyle f^0(C)}\downarrow & & {\scriptstyle f^1(A)}\downarrow \\ 0 & \longrightarrow & U^0(A) & \xrightarrow{U^0(\sigma)} & U^0(B) & \xrightarrow{U^0(\tau)} & U^0(C) & \xrightarrow{e^0(\mathbf{s})} & U^1(A)\cdots \end{array} \tag{3.5.1}$$

commutes for all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ *in* $\mathbf{\Sigma}^{sp}$. *Then (3.5.1) commutes for all short exact sequences in* $\mathbf{\Sigma}$.

Proof. Let $\mathbf{T}^\bullet = (T^n, e^n)_{n\in\mathbb{N}_0}$. The sequence $\mathbf{T}_\bullet|_{\mathbf{\Sigma}^{sp}}$ is a right universal $\mathbf{\Sigma}^{sp}$-connected sequence and $\mathbf{T}_\bullet$ is a right universal $\mathbf{\Sigma}$-connnected sequence ([18], Thm. XII.7.6). Hence there exists a unique mapping of $\mathbf{\Sigma}$-connected sequences $\mathbf{g}^\bullet\colon \mathbf{T}^\bullet \to \mathbf{U}^\bullet$ with $g^0 = f^0$. However $\mathbf{g}^\bullet$ is also a mapping of $\mathbf{\Sigma}^{sp}$-connected sequences, so $\mathbf{g}^\bullet = \mathbf{f}^\bullet$.

3.6 Continuous $\mathfrak{P}_p(G)$-functors

In this subsection we study functors between Pontrjagin categories and the connections between them. The necessity of this arises because $\mathfrak{P}_p(G)$ is an exact category which does not really satisfy all the usual requirements of homological algebra, because it does not have enough projectives or injectives. However, one can manage the resulting problems by introducing various continuity concepts. A covariant additive functor $T\colon \mathfrak{P}_p(G) \to \mathfrak{P}_p$ is called *continuous*, if

(1) $T(ob(\mathfrak{D}_p(G)) \subseteq \mathfrak{D}_p$,

(2) $T(ob(\mathfrak{C}_p(G)) \subseteq \mathfrak{C}_p$,

(3) the restriction $T|_{\mathfrak{D}_p(G)}$ commutes with direct limits, i.e.,

$$\varinjlim \circ T|_{\mathfrak{D}_p(G)} = T|_{\mathfrak{D}_p(G)} \circ \varinjlim, \tag{3.6.1}$$

(4) the restriction $T|_{\mathfrak{C}_p(G)}$ commutes with inverse limits, i.e.,

$$\varprojlim \circ T|_{\mathfrak{C}_p(G)} = T|_{\mathfrak{C}_p(G)} \circ \varprojlim. \tag{3.6.2}$$

Here we assume that all morphisms in the inductive system in the right-hand side of (3.6.1) are injective, and all morphisms in the inverse system in the right-hand side of (3.6.2) are surjective. In the dual way one defines a continuous contravariant additive $\mathfrak{P}_p(G)$-functor. The definition implies:

Proposition 3.6.1. *Let* S, $T\colon \mathfrak{P}_p(G) \to \mathfrak{P}_p$ *be continuous* $\mathfrak{P}_p(G)$*-functors and let* $f_\circ\colon S|_{\mathfrak{F}_p(G)} \to T|_{\mathfrak{F}_p(G)}$ *be a natural transformation of* $\mathfrak{F}_p(G)$*-functors. Then* $f_\circ$ *extends uniquely to a natural transformation of continuous* $\mathfrak{P}_p(G)$*-functors* $f\colon S \to T$.

Let $\mathbf{\Sigma} = \mathbf{\Sigma}(\mathfrak{P}_p(G))$ denote the category of all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathfrak{P}_p(G)$. Recall that such a sequence is either a short exact sequence

in $\mathfrak{D}_p(G)$ or one in $\mathfrak{C}_p(G)$. Assume that $\mathbf{s}$ is a short exact sequence in $\mathfrak{C}_p(G)$. Then $B = \varprojlim_i B/U_i$ where U_i ranges over the open $\mathbb{Z}_p[\![G]\!]$-submodules. Let $B_i = B/U_i$, $C_i = C/\tau(U_i)$, $A_i = A/\sigma^{-1}(U_i)$. Then

$$\mathbf{s}_i\colon\ 0 \to A_i \xrightarrow{\sigma_i} B_i \xrightarrow{\tau_i} C_i \to 0, \tag{3.6.3}$$

where σ_i, τ_i are the canonical morphisms, is exact and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\sigma} & B & \xrightarrow{\tau} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim A_i & \xrightarrow{lim(\sigma_i)} & \varprojlim B_i & \xrightarrow{lim(\tau_i)} & \varprojlim C_i & \longrightarrow & 0 \end{array} \tag{3.6.4}$$

with exact rows and with all vertical arrows isomorphisms. A similar statement holds for $\mathbf{s}$ a short exact sequence in $\mathfrak{D}_p(G)$ and direct limits. Thus a $\mathbf{\Sigma}$-connected pair (S, e, T) of continuous covariant additive $\mathfrak{P}_p(G)$-functors has the following continuity properties:

(1) for all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathfrak{C}_p(G)$, the diagram (cf. (3.6.4))

$$\begin{array}{ccc} S(C) & \xrightarrow{e(\mathbf{s})} & T(A) \\ \downarrow & & \downarrow \\ \varprojlim S(C_i) & \xrightarrow{lim(e(\mathbf{s}_i))} & \varprojlim T(A_i) \end{array} \tag{3.6.5}$$

commutes,

(2) for all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathfrak{D}_p(G)$, the diagram

$$\begin{array}{ccc} \varinjlim S(C_i) & \xrightarrow{lim(e(\mathbf{s}_i))} & \varinjlim T(A_i) \\ \downarrow & & \downarrow \\ S(C) & \xrightarrow{e(\mathbf{s})} & T(A) \end{array} \tag{3.6.6}$$

commutes.

We call the $\mathbf{\Sigma}$-connected pair (S, e, T) of continuous covariant additive $\mathfrak{P}_p(G)$-functors *continuously right universal* if, for all $\mathbf{\Sigma}$-connected pairs (S', e', T') of continuous $\mathfrak{P}_p(G)$-functors and natural transformations $f\colon S \to S'$, there is a unique natural transformation $g\colon T \to T'$, such that $(f, g)\colon (S, e, T) \to (S', e', T')$ is a morphism of $\mathbf{\Sigma}$-connected pairs. Dually, (S, e, T) is *continuously left couniversal* if, for all $\mathbf{\Sigma}$-connected pairs (S', e', T') of continuous $\mathfrak{P}_p(G)$-functors and natural transformation $g'\colon T' \to T$, there is a unique natural transformations $f'\colon S' \to S$, such that $(f', g')\colon (S', e', T') \to (S, e, T)$ is a morphism of $\mathbf{\Sigma}$-connected pairs.

Proposition 3.6.2. *Let (S, e, T) be a $\mathbf{\Sigma}$-connected pair of continuous covariant additive $\mathfrak{P}_p(G)$-functors.*

(1) If

$$S(B) \xrightarrow{S(\tau)} S(C) \xrightarrow{e(\mathbf{s})} T(A) \longrightarrow 0 \tag{3.6.7}$$

is right exact for all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathfrak{D}_p(G)$ with B injective, then (S, e, T) is continuously right universal.

(2) If

$$0 \longrightarrow S(C) \xrightarrow{e(\mathbf{s})} T(A) \xrightarrow{T(\sigma)} T(B) \tag{3.6.8}$$

is left exact for all short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathfrak{C}_p(G)$ with B projective, then (S, e, T) is continuously left couniversal.

Proof. (1) Let (S', e', T') be a $\mathbf{\Sigma}$-connected pair of continuous $\mathfrak{P}_p(G)$-functors and $f\colon S \to S'$ a natural transformation. Since $(S|_{\mathfrak{D}_p(G)}, e, T|_{\mathfrak{D}(G)})$ is right universal ([18], Thm. XII.7.6), there exists a unique natural transformation $g|_{\mathfrak{D}_p(G)}$ such that

$$(f|_{\mathfrak{D}_p(G)}, g|_{\mathfrak{D}_p(G)})\colon (S|_{\mathfrak{D}_p(G)}, e, T|_{\mathfrak{D}_p(G)}) \longrightarrow (S'|_{\mathfrak{D}_p(G)}, e', T|'_{\mathfrak{D}_p(G)}) \tag{3.6.9}$$

is a morphism of connected pairs. By Proposition 3.6.1, $g|_{\mathfrak{D}_p(G)}$ can be uniquely extended to a natural transformation $g\colon T \to T'$. From (3.6.5) we conclude that (f, g) is a morphism of connected pairs and that g is unique. Part (2) follows by similar arguments using [18], Thm. XII.7.2 and (3.6.6).

A $\mathbf{\Sigma}$-connected sequence $\mathbf{T}^\bullet = (T^n, e^n)$ of continuous $\mathfrak{P}_p(G)$-functors is called *continuous*. We call the continuous cohomological $\mathfrak{P}_p(G)$-functor $\mathbf{T}^\bullet = (T^n, e^n)$ *co-effaceable* if $T^i(J) = 0$ for all injective modules $J \in ob(\mathfrak{D}_p(G))$ and $i \geq 1$. Dually, we call the continuous homological $\mathfrak{P}_p(G)$-functor $\mathbf{S}^\bullet = (S^n, e^n)$ *effaceable* if $S^i(P) = 0$ for all projective modules $P \in ob(\mathfrak{C}_p(G))$ and $i \leq -1$. A direct consequence of Proposition 3.6.2 and Theorem 2.2.1 is the following:

Corollary 3.6.3. *Let $\mathbf{T}^\bullet$ be a co-effaceable continuous cohomological $\mathfrak{P}_p(G)$-functor, $\mathbf{S}^\bullet$ an effaceable continuous homological $\mathfrak{P}_p(G)$-functor, and $\mathbf{U}^\bullet$ a continuous connected sequence of covariant additive $\mathfrak{P}_p(G)$-functors.*

(1) Suppose that $U^j = 0$ for all $j < 0$. Then for every natural transformation $f\colon T^0 \to U^0$ there exists a unique mapping of connected sequences $\mathbf{f}^\bullet\colon \mathbf{T}^\bullet \to \mathbf{U}^\bullet$ such that $f^0 = f$. If f^0 is a natural isomorphism and $\mathbf{U}^\bullet$ is a co-effaceable continuous cohomological functor, then all the f^i are also natural isomorphisms.

(2) Suppose that $U^j = 0$ for all $j > 0$. Then for every natural transformation $g\colon U^0 \to S^0$ there exists a unique mapping of connected sequences $\mathbf{g}^\bullet\colon \mathbf{U}^\bullet \to \mathbf{S}^\bullet$ such that $g^0 = g$. If g^0 is a natural isomorphism and $\mathbf{U}^\bullet$ is an effaceable continuous homological functor, then all the g^i are also natural isomorphisms.

3.7 Profinite groups of type p-$\mathbf{FP}_\infty$

Let $\mathfrak{C}_p^0(G)$ denote the full subcategory of $\mathfrak{C}_p(G)$ whose objects consist of the topologically finitely generated $\mathbb{Z}_p[\![G]\!]$-modules in $ob(\mathfrak{C}_p(G))$. Note that if $M \in ob(\mathfrak{C}_p^0(G))$, then

(1) $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) \in ob(\mathfrak{C}_p)$ is compact for all $N \in ob(\mathfrak{C}_p(G))$.

(2) $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, _)\colon \mathfrak{C}_p(G) \longrightarrow \mathfrak{C}_p$ commutes with inverse limits, i.e., if $N = \varprojlim N_i$, $N_i \in ob(\mathfrak{C}_p(G))$, and $\alpha_{ij}\colon N_i \to N_j$ are surjective, then the natural morphism

$$\phi\colon \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) \longrightarrow \varprojlim \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N_i) \tag{3.7.1}$$

is an isomorphism of (topological) $\mathbb{Z}_p$-modules.

A compact module $M \in ob(\mathfrak{C}_p(G))$ is called of *type* $\mathbf{FP}_n$, if there exists a partial projective resolution

$$P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \tag{3.7.2}$$

of M in $\mathfrak{C}_p(G)$ with each P_i finitely generated (as a topological $\mathbb{Z}_p[\![G]\!]$-module). Such a partial projective resolution will be called *of finite type*. By $\mathfrak{C}_p^n(G)$ we denote the full subcategory of $\mathfrak{C}_p(G)$ whose objects M are of type $\mathbf{FP}_n$. From the Horseshoe Lemma ([1], Lemma 2.5.1) one sees that $\mathfrak{C}_p^n(G)$ is closed under extensions, i.e., if $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ is a short exact sequence in $\mathfrak{C}_p(G)$ and $A, C \in ob(\mathfrak{C}_p^n(G))$, then $B \in ob(\mathfrak{C}_p^n(G))$. As a direct consequence of Schanuel's Lemma one obtains:

Proposition 3.7.1. *For $M \in \mathfrak{C}_p^0(G)$ and $n \geq 1$ the following conditions are equivalent:*

(1) There is a partial resolution $F_n \to \cdots \to F_0 \to M \to 0$ with each $F_i \in ob(\mathfrak{C}_p(G))$ free of finite rank.

(2) M is of type $\mathbf{FP}_n$.

(3) For every partial projective resolution $P_k \to \cdots \to P_0 \to M \to 0$ of finite type with $k < n$, $ker(P_k \to P_{k-1})$ is finitely generated.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) can be proved by the same arguments as in the discrete case ([5], Chap. VIII, Prop. 4.3.).

A module is called of *type* $\mathbf{FP}_\infty$ if it is of type $\mathbf{FP}_n$ for all $n \in \mathbb{N}_0$. The full subcategory of $\mathfrak{C}_p(G)$ whose objects consist of modules of type $\mathbf{FP}_\infty$ will be denoted by $\mathfrak{C}_p^\infty(G)$. By Proposition 3.7.1, a module $M \in ob(\mathfrak{C}_p^\infty(G))$ of type $\mathbf{FP}_\infty$ has a projective resolution $\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M$ with all $P_i \in ob(\mathfrak{C}_p^0(G))$ finitely generated. Moreover each P_i can be chosen to be free. For $M \in ob(\mathfrak{C}_p^\infty(G))$, $N \in ob(\mathfrak{P}_p(G))$ we define

$$\mathbf{Ext}^n_{\mathbb{Z}_p[\![G]\!]}(M, N) = H^n(\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_\bullet, N), \delta^\bullet) \in ob(\mathfrak{P}_p), \tag{3.7.3}$$

where $\delta^\bullet$ denotes the induced differential.

Theorem 3.7.2. *Let* $M \in ob(\mathfrak{C}_p^\infty(G))$. *Then* $\mathbf{T}^\bullet = (\mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(M, _), e^\bullet)$ *is a continuous co-effaceable cohomological* $\mathfrak{P}_p(G)$*-functor with* $\mathbf{Ext}^0_{\mathbb{Z}_p[\![G]\!]}(M, _) \simeq \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, _)$.

Proof. That $\mathbf{T}^\bullet$ is a cohomological $\mathfrak{P}_p(G)$-functor is a direct consequence of Theorem 2.2.4(a) for $\mathfrak{A}_0 = \mathfrak{C}_p^\infty(G)$, $\mathfrak{A} = \mathfrak{C}_p(G)$, $\mathfrak{B} = \mathfrak{P}_p(G)$, $\mathfrak{Q} = \mathfrak{P}_p$ and $T(_, _) = \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(_, _)$ (using also Prop. 3.4.1). For continuity we must check (3.6.1) and (3.6.2). Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M with all the P_k finitely generated. Let $N = \varprojlim_i N_i \in ob(\mathfrak{C}_p(G))$ with all $\alpha_{ij} : N_i \to N_j$ surjective, and let $\pi_i : N \to N_i$ denote the canonical projections. Since P_k is projective, $\pi^*_{k,i} : \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N) \to \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N_i)$ is surjective. Let

$$\pi^*_k : \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N) \to \varprojlim_i \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N_i) \tag{3.7.4}$$

denote the canonical isomorphism (3.7.1) For the induced differentials

$$\begin{aligned} \delta^k_i &: \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N_i) \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_{k+1}, N_i), \\ \delta^k &: \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N) \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_{k+1}, N), \end{aligned} \tag{3.7.5}$$

one has $im(\delta^{k-1}_i) = \pi^*_{k,i}(im(\delta^{k-1}))$. The compactness of $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N)$ implies that $\pi^*_k(im(\delta^{k-1})) = \varprojlim_i im(\delta^{k-1}_i)$. Since $\pi^*_k(\ker(\delta^k)) = \varprojlim_i ker(\delta^k_i)$, the canonical mapping

$$\pi^*_k : ker(\delta^k)/im(\delta^{k-1}) \longrightarrow \varprojlim ker(\delta^k_i)/im(\delta^{k-1}_i) \tag{3.7.6}$$

is an isomorphism. This proves (3.6.2). A similar argument shows that (3.6.1) also holds. The co-effaceability follows from Proposition 3.4.1.

Remark 3.7.3. If $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ is a short exact sequence in $\mathfrak{C}_p^\infty(G)$, then one can use the Horseshoe Lemma and the exact homology sequence for chain complexes (Thm. 2.2.2) to show the existence of a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(C, N) &\xrightarrow{\tau^*} \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(B, N) \xrightarrow{\sigma^*} \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(A, N) \longrightarrow \\ \mathbf{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(C, N) &\xrightarrow{\tau^*} \mathbf{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(B, N) \xrightarrow{\sigma^*} \mathbf{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(A, N) \longrightarrow \cdots \end{aligned} \tag{3.7.7}$$

for $N \in \mathfrak{P}_p(G)$.

For $M \in \mathfrak{C}_p^0(G)^\times$ the usual tensor product $M \otimes_{\mathbb{Z}_p[\![G]\!]} _$ coincides with $M \widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]} _$ if they are considered as functors from $\mathfrak{C}_p(G)$ to $\mathfrak{C}_p$ ([6], Lemma 2.1(ii)). We extend the functor $M \widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]} _$ to $\mathfrak{P}_p(G)$ by setting

$$M \widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]} N = M \otimes_{\mathbb{Z}_p[\![G]\!]} N \tag{3.7.8}$$

in the case that $N \in ob(\mathfrak{D}_p(G))$. Now $M \widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]} _ : \mathfrak{P}_p(G) \to \mathfrak{P}_p$ is a covariant additive right exact functor satisfying $M \widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]} _ (\mathfrak{D}_p(G)) \subseteq \mathfrak{D}_p$,

$M\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}_(\mathfrak{C}_p(G)) \subseteq \mathfrak{C}_p$. By construction ([6], §2), $M\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}_$ satisfies (3.6.2). Since the functor $M\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}_|_{\mathfrak{D}_p(G)}$ coincides with the usual tensor product, it also satisfies (3.6.1). So $M\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}_$ is a continuous covariant right exact $\mathfrak{P}_p(G)$-functor. Moreover, if $P \in ob(\mathfrak{C}_p^0(G)^\times)$ is finitely generated and projective, then it is a direct summand of a finitely generated free (right) $\mathbb{Z}_p[\![G]\!]$-module ([6], Lemma 1.6) which is also free in $\mathrm{Mod}_{\mathbb{Z}_p[\![G]\!]}$. This shows that $P\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}_$ is a continuous covariant exact $\mathfrak{P}_p(G)$-functor. This allows us to define the functors $\mathbf{Tor}_\bullet^{\mathbb{Z}_p[\![G]\!]}(M, _)\colon \mathfrak{P}_p(G) \to \mathfrak{P}_p$ as we did for $\mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}$. Let $M \in ob(\mathfrak{C}_p^\infty(G)^\times)$ and let $\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M$ be a projective resolution of finite type. For $N \in ob(\mathfrak{P}_p(G))$ we set

$$\mathbf{Tor}_\bullet^{\mathbb{Z}_p[\![G]\!]}(M, N) = H_n(P_\bullet\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}N, \partial_\bullet), \tag{3.7.9}$$

where $\partial_\bullet$ also denotes the induced differential. From Theorem 2.2.4(b) one knows that $\mathbf{Tor}^{\mathbb{Z}_p[\![G]\!]}(M, _)$ is a homological functor.

Theorem 3.7.4. *Let $M \in ob(\mathfrak{C}_p^\infty(G))$. Then $\mathbf{T}^{-\bullet} = (\mathbf{Tor}_\bullet^{\mathbb{Z}_p[\![G]\!]}(M, _), e_\bullet)$ is a continuous effaceable homological $\mathfrak{P}_p(G)$-functor with $T^0 = M\widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]}_$.*

Proof. $T^{-n} = \mathbf{Tor}_n^{\mathbb{Z}_p[\![G]\!]}(M, _)\colon \mathfrak{P}_p(G) \to \mathfrak{P}_p$ are covariant additive functors satisfying (1) and (2) of the definition of a continuous $\mathfrak{P}_p(G)$-functor. The previous remark has shown that for $N \in ob(\mathfrak{D}_p(G))$, $T^{-n}(N)$ coincides with $\mathrm{Tor}_n^{\mathbb{Z}_p[\![G]\!]}(M, N)$. Hence it satisfies (3.6.1) (cf. [25], 4.1.7, 5.1.9). (3.6.2) can be shown by a similar argument as in the proof of Theorem 3.7.2. The effaceability follows from [6], Lemma 2.1(iii).

By analogy with the case of discrete groups we say that the profinite group G is of *type* p-$\mathbf{FP}_n$, $n \in \mathbb{N} \cup \{\infty\}$, if the trivial $\mathbb{Z}_p[\![G]\!]$-module $\mathbb{Z}_p$ is contained in $ob(\mathfrak{C}_p^n(G))$. For a profinite group G of type p-$\mathbf{FP}_\infty$ it is possible to define continuous cohomological and homological $\mathfrak{P}_p(G)$-functors

$$\begin{aligned} \mathbf{H}^\bullet(G, _) &= \mathbf{Ext}^\bullet_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, _)\colon \mathfrak{P}_p(G) \longrightarrow \mathfrak{P}_p \\ \mathbf{H}_\bullet(G, _) &= \mathbf{Tor}_\bullet^{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, _)\colon \mathfrak{P}_p(G) \longrightarrow \mathfrak{P}_p \end{aligned} \tag{3.7.10}$$

which extend the previous cohomology and homology functors and which are coeffaceable and effaceable respectively. Moreover Pontrjagin duality induces a natural isomorphism (3.4.6)

$$\mathbf{H}_\bullet(G, _)^* \simeq \mathbf{H}^\bullet(G, _^*). \tag{3.7.11}$$

3.8 Rational cohomology

Let $\mathfrak{fd}_p(G)$ denote the category whose objects are finite-dimensional $\mathbb{Q}_p$-vectorspaces which are also topological left $\mathbb{Z}_p[\![G]\!]$-modules and whose morphisms are given by

$$\mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![G]\!]}(M, N) = \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N), \tag{3.8.1}$$

$M, N \in ob(\mathfrak{fd}_p(G))$.

Proposition 3.8.1.

(1) $\mathfrak{fd}_p(G)$ *is an abelian category.*

(2) Every module $M \in ob(\mathfrak{fd}_p(G))$ *contains a compact and open* $\mathbb{Z}_p[\![G]\!]$*-submodule* $C \in ob(\mathfrak{C}_p(G))$ *which is a finitely generated p-torsion-free* $\mathbb{Z}_p$*-module with* $rk_{\mathbb{Z}_p}(C) = dim_{\mathbb{Q}_p}(M)$.

(3) Let $\phi\colon M \to N$ *be a surjective morphism in* $\mathfrak{fd}_p(G)$ *and* $C \leq N$ *be a compact* $\mathbb{Z}_p[\![G]\!]$*-submodule of* N. *Then there exists a compact* $\mathbb{Z}_p[\![G]\!]$*-submodule* C' *of* M *such that* $\phi(C') = C$.

Proof. (1) is obvious. For (2) let x_1,..,x_d be a $\mathbb{Q}_p$-basis of M. Then $C = \mathbb{Z}_p[\![G]\!].x_1 + \cdots + \mathbb{Z}_p[\![G]\!].x_d$ has the required properties. For (3) let $x_1, \ldots, x_k$ be a basis of C as a $\mathbb{Z}_p$-module. Let $y_1, \ldots, y_k \in M$ with $\phi(y_i) = x_i$. Then $C' = \mathbb{Z}_p[\![G]\!].y_1 + \cdots + \mathbb{Z}_p[\![G]\!].y_k$ satisfies $\phi(C') = C$ and is compact.

For $M \in ob(\mathfrak{C}_p(G))$, $N \in ob(\mathfrak{fd}_p(G))$ we set

$$\mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![G]\!]}(M, N) = \{\alpha \in \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N) \mid \alpha \text{ continuous}\}, \tag{3.8.2}$$

where N is considered as a topological $\mathbb{Z}_p$-module. Hence

$$\mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![G]\!]}(_, _)\colon \mathfrak{C}_p(G) \times \mathfrak{fd}_p(G) \longrightarrow {}_{\mathbb{Q}_p}\mathrm{Mod} \tag{3.8.3}$$

is a left/left exact contravariant/covariant biadditive bifunctor. For $P \in ob(\mathfrak{C}_p(G))$ projective, Proposition 3.8.1(3) implies that

$$\mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![G]\!]}(P, _)\colon \mathfrak{fd}_p(G) \longrightarrow {}_{\mathbb{Q}_p}\mathrm{Mod} \tag{3.8.4}$$

is exact. Thus we can apply Theorem 2.2.4(a) for $\mathfrak{A} = \mathfrak{A}_0 = \mathfrak{C}_p(G)$, $\mathfrak{B} = \mathfrak{fd}_p(G)$, $\mathfrak{Q} = {}_{\mathbb{Q}_p}\mathrm{Mod}$ and $T(_, _) = \mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![G]\!]}(_, _)$. For $M \in ob(\mathfrak{C}_p(G))$ we denote by $(\mathbb{E}\mathrm{xt}^\bullet_{\mathbb{Z}_p[\![G]\!]}(M, _), e^\bullet)$ the corresponding cohomological functor. We define the *rational cohomology* of G with coefficients in the category $\mathfrak{fd}_p(G)$ by

$$\mathbb{H}^\bullet(G, _) = \mathbb{E}\mathrm{xt}^\bullet_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, _). \tag{3.8.5}$$

Theorem 3.8.2. *Let* G *be a profinite group of type* p-$\mathbf{FP}_\infty$. *Let* $N \in ob(\mathfrak{fd}_p(G))$ *and let* $C \leq N$ *be a compact and open* $\mathbb{Z}_p[\![G]\!]$*-submodule of* N. *Then the canonical mapping*

$$\mathbf{H}^\bullet(G, C) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{H}^\bullet(G, N) \tag{3.8.6}$$

is an isomorphism. In particular,

$$dim_{\mathbb{Q}_p}(\mathbb{H}^k(G, N)) < +\infty \tag{3.8.7}$$

for all $k \in \mathbb{N}_0$, $N \in ob(\mathfrak{fd}_p(G))$. *Furthermore, the exact sequence* $0 \to C \xrightarrow{\sigma} N \xrightarrow{\tau} N/C \to 0$ *induces a long exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathbf{H}^0(G, C) \xrightarrow{\sigma^*} \mathbb{H}^0(G, N) \xrightarrow{\tau^*} \mathbf{H}^0(G, N/C) \longrightarrow \\ \mathbf{H}^1(G, C) \xrightarrow{\sigma^*} \mathbb{H}^1(G, N) \xrightarrow{\tau^*} \mathbf{H}^1(G, N/C) \longrightarrow \cdots \end{aligned} \tag{3.8.8}$$

of $\mathbb{Z}_p$-modules.

Proof. Let $P_\bullet \xrightarrow{\varepsilon} \mathbb{Z}_p$ be a projective resolution of finite type. In this case $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, C)$ is a finitely generated $\mathbb{Z}_p$-module and the canonical morphism

$$\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, C) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![G]\!]}(P_k, N) \tag{3.8.9}$$

is an isomorphism. Also $_ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an exact covariant additive functor on the abelian category of finitely generated $\mathbb{Z}_p$-modules. This proves (3.8.6) and (3.8.7). Note that $N/C \in ob(\mathfrak{D}_p(G))$, so for $\phi \in \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N/C)$ the $\mathbb{Z}_p$-module $\phi(P_k)$ must be finite. This shows that

$$\begin{aligned} 0 \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, C) &\xrightarrow{\sigma^*} \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N) \\ &\xrightarrow{\tau^*} \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_k, N/C) \longrightarrow 0 \end{aligned} \tag{3.8.10}$$

is exact for $P_k \in ob(\mathfrak{C}_p^\infty(G))$ projective and finitely generated. Now (3.8.8) follows from the exact homology sequence (Thm. 2.2.2).

4 Finiteness conditions and Poincaré duality

Again let G be a profinite group and p a prime number. By $\mathfrak{S}_p(G) \subseteq ob(\mathfrak{D}_p(G))$ we denote the set of (isomorphism types of) simple discrete left G-modules which are p-torsion modules. In particular, every module $M \in \mathfrak{S}_p(G)$ is annihilated by $p\mathbb{Z}_p$. It is well-known that if G is a pro-p group, then $\mathfrak{S}_p(G)$ just consists of the trivial G-module $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$ ([23], Chap. I, Prop. 20). On the other hand if $|\mathfrak{S}_p(G)| = 1$, then G must be a pro-p group. More generally, one can show easily using Brauer character theory ([1], Cor. 5.3.5) that the set $\mathfrak{S}_p(G)$ is finite if and only if G is virtually pro-p.

4.1 The cohomological p-dimension

Recall that for a profinite group G the *cohomological p-dimension* $cd_p(G)$ is given by

$$cd_p(G) = sup\{\, n \in \mathbb{N}_0 \mid H^n(G, M) \neq 0 \text{ for some } M \in ob(\mathfrak{D}_p(G))\,\}. \tag{4.1.1}$$

It has the following property.

Proposition 4.1.1. *The following are equivalent:*

(1) $cd_p(G) \leq n$,

(2) $H^{n+1}(G, F) = 0$ *for all* $F \in ob(\mathfrak{F}_p(G))$,

(3) $H^{n+1}(G, S) = 0$ *for all* $S \in \mathfrak{S}_p(G)$,

(4) There exists a projective resolution of $\mathbb{Z}_p$ *in* $\mathfrak{C}_p(G)$ *of length* n.

Proof. The implications (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. (3) $\Rightarrow$ (4) has been proved in [6], Cor. 3.2.

Let H be a closed subgroup of the profinite group G. Then it follows from Proposition 4.1.1 and Proposition 3.1.1 that

$$cd_p(H) \leq cd_p(G) \tag{4.1.2}$$

([[23], Prop. I.14). In the case that $|G\colon H|$ is coprime to p, or if H is open in G and $cd_p(G) < +\infty$, then equality holds in (4.1.2). This fact implies that profinite groups with finite cohomological p-dimension are necessarily p-torsion-free. If $N \lhd G$ is a closed normal subgroup of G, then the Lyndon-Hochschild-Serre spectral sequence for continuous discrete cohomology ([23], §I.2.6.(b)) implies that

$$cd_p(G) \leq cd_p(N) + cd_p(G/N). \tag{4.1.3}$$

4.2 Profinite groups of type p-$\mathbf{FP}_\infty$

If $H \leq_o G$ is an open subgroup of G, then $\mathbb{Z}_p[\![G]\!]$ is a free $\mathbb{Z}_p[\![H]\!]$-module of rank $|G\colon H|$. Now Proposition 3.7.1 implies (cf. proof of [5], Chap. VIII, Prop. 5.1):

Proposition 4.2.1. *Let $H \leq_o G$ be an open subgroup of G and $n \in \mathbb{N}_0 \cup \{\infty\}$. Then H is of type p-$\mathbf{FP}_n$ if and only if G is of type p-$\mathbf{FP}_n$.*

Let G be a pro-p group. Then $cd_p(G)$ is less or equal to 1 if and only if G is a free pro-p group [23], §I.4.2, Cor. 2). Hence a free pro-p group is of type p-$\mathbf{FP}_\infty$ if and only if it is finitely generated ([23], §I.4.2). This shows that a closed subgroup H of a profinite group G of type p-$\mathbf{FP}_\infty$ is not necessarily of type p-$\mathbf{FP}_\infty$. From the definition of the p-$\mathbf{FP}_\infty$ property one obtains also the following:

Proposition 4.2.2. *Let G be a profinite group of type p-$\mathbf{FP}_\infty$. Then for all $F \in ob(\mathfrak{F}_p(G))$, $n \in \mathbb{N}_0$,*

$$|H^n(G, F)| < +\infty. \tag{4.2.1}$$

One would like to have the p-$\mathbf{FP}_\infty$ condition as the conclusion of a statement rather than in the hypothesis. For virtual pro-p groups the p-$\mathbf{FP}_n$ condition can be characterized by (4.2.1).

Proposition 4.2.3. *Let G be a profinite group which is virtually pro-p and $n \in \mathbb{N}_0$. Then the following are equivalent:*

(1) G is of type p-$\mathbf{FP}_n$.

(2) $H^k(G, F)$ is finite for all $F \in ob(\mathfrak{F}_p(G))$, $k \leq n$,

(3) $\dim_{\mathbb{F}_p}(H^k(G, S)) < +\infty$ for all $S \in \mathfrak{S}_p(G)$, $k \leq n$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. Assume that (3) holds and let $S \in \mathfrak{S}_p(G)$. Let $\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z}_p \longrightarrow 0$ denote the minimal projective

resolution of $\mathbb{Z}_p$ in $\mathfrak{C}_p(G)$ (Remark 3.4.3(i)). Let $\mathbf{\Omega}^k(\mathbb{Z}_p) = ker(\partial_{k-1})$ $(\partial_0 = \varepsilon)$, $\mathbf{\Omega}^0(\mathbb{Z}_p) = \mathbb{Z}_p$. Since $P_\bullet \xrightarrow{\varepsilon} \mathbb{Z}_p$ is the minimal projective resolution, the complex

$$\begin{aligned} 0 \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_0, S) &\xrightarrow{\delta^0} \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_1, S) \\ &\xrightarrow{\delta^1} \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_2, S) \xrightarrow{\delta^2} \cdots \end{aligned} \tag{4.2.2}$$

has to have trivial differentials (cf. [1], Cor. 2.5.4). Thus

$$\mathbf{Ext}^k_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, S) \simeq \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(\mathbf{\Omega}^k(\mathbb{Z}_p), S). \tag{4.2.3}$$

Since P_k is finitely generated if and only if $\mathbf{\Omega}^k(\mathbb{Z}_p)$ is, the assertion follows from the following fact applied to $\mathbf{\Omega}^k(\mathbb{Z}_p)$.

Lemma 4.2.4. *Let G be a profinite group which is virtually pro-p and $M \in ob(\mathfrak{C}_p(G))$. Then the following conditions are equivalent:*

(1) M is finitely generated,

(2) $\dim_{\mathbb{F}_p}(\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, S)) < +\infty$ for all $S \in \mathfrak{S}_p(G)$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. From (2) and the remark at the beginning of §4 one concludes that the number of maximal open submodules in M is finite. Hence

$$Rad(M) = \bigcap_{\substack{Q \leq_o M \\ Q \text{ maximal}}} Q \tag{4.2.4}$$

is open in M. So $M/Rad(M)$ is finite. This yields the assertion.

For pro-p groups one obtains:

Corollary 4.2.5. *A pro-p group G is of type p-$\mathbf{FP}_\infty$ if and only if $H^\bullet(G, \mathbb{F}_p)$ is an $\mathbb{F}_p$-algebra of finite type, i.e., $dim_{\mathbb{F}_p}(H^k(G, \mathbb{F}_p)) < +\infty$ for all $k \in \mathbb{N}_0$.*

If $\mathbb{Z}_p[\![G]\!]$ is noetherian[10], then G has to be of type p-$\mathbf{FP}_\infty$. Note that $\mathbb{Z}_p[\![G]\!]$ is noetherian if and only if $\mathfrak{C}^0_p(G) = \mathfrak{C}^\infty_p(G)$. In this case $\mathfrak{C}^0(G)$ is an abelian category with enough projectives.

For profinite groups of type p-$\mathbf{FP}_\infty$ one has also:

Theorem 4.2.6 (Lyndon–Hochschild–Serre spectral sequence)**.** *Let G be a profinite group of type p-$\mathbf{FP}_\infty$, $N \lhd_o G$ an open normal subgroup.*

(a) *Let $M \in ob(\mathfrak{P}_p(G))$. Then there exists a first quadrant spectral sequence $(E_r^{s,t}, d_r)$ with*

$$E_2^{s,t} = \mathbf{H}^s(G/N, \mathbf{H}^t(N, M)) \Longrightarrow \mathbf{H}^{s+t}(G, M). \tag{4.2.5}$$

[10] i.e., every closed submodule of a (topologically) finitely generated $\mathbb{Z}_p[\![G]\!]$-module is finitely generated.

(b) *Let $M \in ob(\mathfrak{fd}_p(G))$. Then*

$$H^0(G/N, _) \circ res^G_N \colon \mathbb{H}^\bullet(G, M) \longrightarrow \mathbb{H}^\bullet(N, M)^{G/N} \tag{4.2.6}$$

is an isomorphism.

Proof. (a) Let

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z}_p \tag{4.2.7}$$

be a projective resolution of finite type over $\mathbb{Z}_p[G/N]$ and

$$\cdots \xrightarrow{\partial'_3} Q_2 \xrightarrow{\partial'_2} Q_1 \xrightarrow{\partial'_1} Q_0 \xrightarrow{\varepsilon'} \mathbb{Z}_p \tag{4.2.8}$$

a projective resolution of finite type over $\mathbb{Z}_p[\![G]\!]$. Consider the double complex

$$E_0^{s,t} = \mathbf{Hom}_{\mathbb{Z}_p[G/N]}(P_s, \mathbf{Hom}_{\mathbb{Z}_p[\![N]\!]}(Q_t, M)). \tag{4.2.9}$$

Since N is open in G, the restriction of $Q_\bullet \xrightarrow{\varepsilon'} \mathbb{Z}_p$ to N is also a projective resolution of finite type (Prop. 3.3.1). Note that $\mathbf{Hom}_{\mathbb{Z}_p[\![N]\!]}(Q_t, M)$ is a left $\mathbb{Z}_p[\![G]\!]$-module whose restriction to $\mathbb{Z}_p[\![N]\!]$ is trivial, so $\mathbf{Hom}_{\mathbb{Z}_p[\![N]\!]}(Q_t, M) \in ob(\mathfrak{P}_p(G/N))$. Let d_0 be the differential induced by $\partial_\bullet$ and d_1 be $(-1)^s$-times the differential induced by $\partial'_\bullet$. This sign convention ensures that

$$d_1 \circ d_0 + d_0 \circ d_1 = 0. \tag{4.2.10}$$

Now $(E_0^{s,t}, d_0, d_1)$ is a double complex ([2], §3.4). Since $\mathbf{Hom}_{\mathbb{Z}_p[G/N]}(P_s, _)$ is exact, one obtains for spectral sequence associated to this double complex and applying d_1 first,

$$\begin{aligned} E_1^{s,t} &= \mathbf{Hom}_{\mathbb{Z}_p[G/N]}(P_s, \mathbf{H}^t(N, M)), \\ E_2^{s,t} &= \mathbf{H}^s(G/N, \mathbf{H}^t(N, M)). \end{aligned} \tag{4.2.11}$$

It remains to calculate $H^\bullet(\mathbf{tot}^\bullet(E_0^{\bullet,\bullet}), d_0 + d_1)$.

Lemma 4.2.7. *Let $A \in ob(\mathfrak{C}_p^0(G/N))$, $B \in ob(\mathfrak{C}_p^0(G))$, $M \in ob(\mathfrak{P}_p(G))$. Then*

$$\mathbf{Hom}_{\mathbb{Z}_p[G/N]}(A, \mathbf{Hom}_{\mathbb{Z}_p[\![N]\!]}(B, M)) \simeq \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(A \widehat{\otimes}_{\mathbb{Z}_p} B, M) \tag{4.2.12}$$

and the isomorphism is given by a natural transformation (of bifunctors), where A is considered as $\mathbb{Z}_p[\![G]\!]$-module with $\mathbb{Z}_p[\![N]\!]$ acting trivially.

Proof. ([2], Lemma 3.5.1) It follows that $(E_0^{\bullet,\bullet}, d_0, d_1)$ and $(\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_\bullet \widehat{\otimes}_{\mathbb{Z}_p} Q_\bullet, M), \partial, (-1)^\bullet \partial')$ are isomorphic double complexes. Since P_s and Q_t are projective in $\mathfrak{C}_p$ (Prop. 3.3.1), the complex $\mathbf{tot}_\bullet(P_\bullet \widehat{\otimes}_{\mathbb{Z}_p} Q_\bullet, \partial + (-1)^\bullet \partial')$ is exact in positive degrees

([6], Lemma 2.1). By Proposition 3.3.2, $P_s \widehat{\otimes}_{\mathbb{Z}_p} Q_t \in ob(\mathfrak{C}_p(G))$ is projective. This yields

$$H^\bullet(\mathbf{tot}^\bullet(E_0^{\bullet,\bullet}), d_0 + d_1) = \mathbf{H}^\bullet(G, M)$$

and the proof is complete.
(b) The same argument shows the existence of a first quadrant spectral sequence with

$$E_2^{s,t} = H^s(G/N, \mathbb{H}^t(N, M)) \Longrightarrow \mathbb{H}^{s+t}(G, M). \tag{4.2.13}$$

As G/N is finite and $\mathbb{H}^\bullet(N, M)$ divisible, one has

$$E_\infty^{s,t} = E_2^{s,t} = \begin{cases} H^0(G/N, \mathbb{H}^t(N, M)) & \text{for } s = 0, \\ 0 & \text{for } s \neq 0. \end{cases} \tag{4.2.14}$$

Since the vertical edge homomorphism coincides with the stable restriction mapping, this yields the claim.

There is also the following Eckmann-Shapiro type lemma:

Lemma 4.2.8. *Let G be a profinite group of type p-$\mathbf{FP}_\infty$, $H \leq_o G$ an open subgroup of G and $M \in ob(\mathfrak{C}_p(H))$. Then*

$$\mathbf{H}^\bullet(H, M) \simeq \mathbf{H}^\bullet(G, ind_G^H(M)). \tag{4.2.15}$$

Proof. Let $Q \in ob(\mathfrak{C}_p^0(G))$. The mappings

$$\begin{aligned} \alpha &: \mathbf{Hom}_{\mathbb{Z}_p[\![H]\!]}(Q, M) \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(Q, \mathbf{Hom}_{\mathbb{Z}_p[\![H]\!]}(\mathbb{Z}_p[\![G]\!], M)), \\ \beta &: \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(Q, \mathbf{Hom}_{\mathbb{Z}_p[\![H]\!]}(\mathbb{Z}_p[\![G]\!], M)) \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p[\![H]\!]}(Q, M), \end{aligned} \tag{4.2.16}$$

given by

$$\begin{aligned} \alpha(f)(q)(x) &= f(x.q), \\ \beta(F)(q) &= F(q)(1), \end{aligned} \tag{4.2.17}$$

$q \in Q$, $x \in \mathbb{Z}_p[\![G]\!]$, are inverse natural transformations of covariant functors. Since the restriction of a projective resolution of $\mathbb{Z}_p$ in $\mathfrak{C}_p(G)$ of finite type is a projective resolution of $\mathbb{Z}_p$ in $\mathfrak{C}_p(H)$ of finite type, one obtains a natural isomorphism

$$\mathbf{H}^\bullet(H, M) \simeq \mathbf{H}^\bullet(G, coind_G^H(M)). \tag{4.2.18}$$

Now the assertion follows from (3.3.7).

Remark 4.2.9. Let G be of type p-$\mathbf{FP}_\infty$. Lemma 4.2.8 shows that the compact $\mathbb{Z}_p$-modules $\mathbf{H}^k(G, \mathbb{Z}_p[\![G]\!])$ are invariants of the commensurability class of G, i.e., one has

$$\mathbf{H}^\bullet(G, \mathbb{Z}_p[\![G]\!]) \simeq \mathbf{H}^\bullet(H, \mathbb{Z}_p[\![H]\!]) \tag{4.2.19}$$

for all open subgroups $H \leq_o G$.

4.3 Profinite groups of type p-FP

The profinite group G is called *of type* p-**FP** if there exists a projective resolution of the trivial $\mathbb{Z}_p[\![G]\!]$-module $\mathbb{Z}_p$ which is of finite type and of finite length. If G is of type p-$\mathbf{FP}_\infty$ and of finite cohomological p-dimension, the minimal projective resolution of $\mathbb{Z}_p$ has to be of finite length (4.2.3) and of finite type (cf. Prop. 3.7.1). Thus a profinite group is of type p-**FP** if and only if it is of finite cohomological p-dimension and of type p-$\mathbf{FP}_\infty$. For groups of type p-**FP**, there is always equality in (4.1.3). More precisely:

Proposition 4.3.1. *Let G be a profinite group and $N \lhd G$ a closed normal subgroup of type p-**FP** such that $cd_p(G/N) < +\infty$. Then*

$$cd_p(G) = cd_p(G/N) + cd_p(N). \tag{4.3.1}$$

Proof. Since $cd_p(G) = cd_p(H)$ for every closed subgroup $H \leq G$ which contains a pro-p-Sylow subgroup of G, we may assume that G/N is a pro-p group. Let $m = cd_p(N)$ and $n = cd_p(G/N)$. Then $H^m(N, S)$ is finite (Prop. 4.2.1) and non-trivial (Prop. 4.1.1) for some $S \in \mathfrak{S}_p(G)$. Hence the assertion follows from the following lemma and the Lyndon–Hochschild–Serre spectral sequence.

Lemma 4.3.2. *Let P be a pro-p group with $cd_p(P) = n$. Then $H^n(P, F) \neq 0$ for any $F \in ob(\mathfrak{F}_p(G))$, $F \neq 0$.*

Proof. Assume that the assertion is false and let $F \in ob(\mathfrak{F}_p(G))$, $F \neq 0$ and $H^n(P, F) = 0$. Assume further that F is minimal with respect to cardinality. Then $F \not\simeq \mathbb{F}_p$ (Prop. 4.1.1). Hence F has a non-trivial $\mathbb{Z}_p[\![P]\!]$-submodule F' with $F/F' \neq 0$. From the long exact sequence one concludes that $H^n(P, F/F') = 0$, a contradiction.

Remark 4.3.3. Let G be a pro-p group. Then every projective module in $\mathfrak{C}_p(G)$ is actually free. The minimal projective cover of the semi-simple module $\mathbb{F}_p^r$ coincides with the canonical map $\gamma\colon \mathbb{Z}_p[\![G]\!]^r \to \mathbb{F}_p^r$. Let G be a pro-p group of type p-$\mathbf{FP}_\infty$ and let $P_\bullet \xrightarrow{\varepsilon} \mathbb{Z}_p$ denote the minimal projective resolution of the trivial $\mathbb{Z}_p[\![G]\!]$-module $\mathbb{Z}_p$. Using (4.2.3) one concludes that $P_k \simeq \mathbb{Z}_p[\![G]\!]^{r_k}$, where $r_k = \dim_{\mathbb{F}_p} H^k(G, \mathbb{F}_p)$.

For profinite groups G of type p-**FP** with $cd_p(G) = n$ there are actually two ways to show the existence of the dualizing module $I_p(G)$ (or $D_p(G)$). A rather constructive way is to proceed as in the discrete case ([5], Chap. VIII, Prop. 6.8). Alternatively one can use the representability of the functor $H^n(G, _)^*\colon \mathfrak{F}_p(G) \to \mathfrak{F}_p$ ([13], Chap. II, no. 4). One can prove ([23], Prop. I.17):

Proposition 4.3.4.

*(1) Let G be of type p-**FP** with $cd_p(G) = n$. Then there exists a discrete $\mathbb{Z}_p[\![G]\!]$-module $I_p(G) \in ob(\mathfrak{D}_p(G))$ and a homomorphism $i\colon H^n(G, I_p(G)) \to I_p$ (of abelian groups) such that given any $M \in \mathfrak{F}_p(G)$, $f \in \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, I_p(G))$, the assignment*

$$T(f)\colon H^n(G, M) \xrightarrow{f} H^n(G, I_p(G)) \xrightarrow{i} I_p \tag{4.3.2}$$

defines a natural isomorphism of contravariant $\mathfrak{F}_p(G)$-functors

$$T : \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(_, I_p(G)) \longrightarrow H^n(G, _)^*. \tag{4.3.3}$$

(2) The pair $(I_p(G), i)$ is uniquely defined up to isomorphism.

(3) If $H \leq_o G$ is an open subgroup of G, then

$$I_p(H) = res^G_H(I_p(G)). \tag{4.3.4}$$

Pontrjagin duality (Prop. 3.4.2) implies that there is a natural isomorphism of contravariant $\mathfrak{F}_p(G)$-functors

$$T^* : \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(I_p(G)^*, _^*) \longrightarrow H^n(G, _)^*. \tag{4.3.5}$$

Moreover, both functors are right exact, so we may extend this natural isomorphism to a natural isomorphism of $\mathfrak{C}_p(G)$-functors

$$T^* : \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(I_p(G)^*, _^*) \longrightarrow \mathbf{H}^n(G, _)^*. \tag{4.3.6}$$

Note that (4.3.6) and Pontrjagin duality also imply also other identities, e.g., from (3.4.3) and (3.4.5) one can show that for any $M \in ob(\mathfrak{C}_p(G))$,

$$\begin{aligned} \mathbf{H}^n(G, M) &\simeq \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(I_p(G)^*, M^*)^* \\ &\simeq I_p(G)^{\times} \widehat{\otimes}_{\mathbb{Z}_p[\![G]\!]} M \simeq \mathbf{H}_0(G, I_p(G)^* \widehat{\otimes}_{\mathbb{Z}_p} M), \end{aligned} \tag{4.3.7}$$

and the isomorphism is natural in M ([6], Lem. 2.4). This is a well known identity for discrete groups of type FP ([5], (VIII.6.9)). Since $coind^1_G(I_p) = \mathbb{Z}_p[\![G]\!]^*$, we have

$$\mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!])^* = \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(I_p(G)^*, coind^1_G(I_p)) = \mathbf{Hom}_{\mathbb{Z}_p}(I_p(G)^*, I_p). \tag{4.3.8}$$

Thus $\mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!]) \simeq I_p(G)^*$ as compact $\mathbb{Z}_p$-modules. One can use (4.3.8) and the uniqueness of the dualizing module to show that $I_p(G)^{\times} \simeq \mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!])$ as right $\mathbb{Z}_p[\![G]\!]$-modules ([5], Chap. VIII, §6). We will call $D_p(G) = I_p(G)^*$ the *compact dualizing module*.

Remark 4.3.5. Assume that G is of type p-**FP** and virtually pro-p. From (4.2.1) and (4.3.6) it follows that $\mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(D_p(G), F)$ is finite for all $F \in ob(\mathfrak{F}_p(G))$. This implies that $D_p(G)$ is finitely generated (Lemma 4.2.4). One might expect this to be true without the assumption that G be virtually pro-p.

4.4 Duality groups

In accordance with the definition for discrete groups ([5], Chap. VIII, §10) we call G a *duality group of dimension n* (at the prime p), if G is of type p-**FP** with $cd_p(G) = n$, and

$$\mathbf{H}^k(G, \mathbb{Z}_p[\![G]\!]) = 0 \text{ for } k \neq n, \tag{4.4.1}$$

and $\mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!])$ is p-torsionfree. If $\mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!]) \simeq \mathbb{Z}_p$ (as abelian groups), G is called a *Poincaré duality group of dimension n*.

Proposition 4.4.1. *Let G be a profinite group with $cd_p(G) = n < +\infty$ and $H \leq_o G$ an open subgroup of G. Then G is a duality group (resp. Poincaré duality group), if and only if H is a duality group (resp. Poincaré duality group).*

Proof. G is of type p-**FP** if and only H is of type p-**FP** (4.1.2), Prop. 4.2.1). Now the assertion follows from (4.2.19).

Just as for discrete groups ([5], Chap. VIII, Prop. 5.2) we have:

Lemma 4.4.2. *Let G be a profinite group of type p-$\mathbf{FP}_\infty$. Let $k \in \mathbb{N}_0$ and let $F \in ob(\mathfrak{C}_p(G))$ be a free $\mathbb{Z}_p[\![G]\!]$-module such that $\mathbf{H}^k(G, F) = 0$. Then $\mathbf{H}^k(G, P) = 0$ for all projective $\mathbb{Z}_p[\![G]\!]$-modules $P \in ob(\mathfrak{C}_p(G))$.*

Proof. $\mathbb{Z}_p[\![G]\!]$ is a direct summand of F. Since $\mathbf{H}^k(G, _)$ is additive, it follows that $\mathbf{H}^k(G, \mathbb{Z}_p[\![G]\!]) = 0$, and so $\mathbf{H}^k(G, M) = 0$ for all finitely generated free $\mathbb{Z}_p[\![G]\!]$-modules. As $\mathbf{H}^k(G, _)$ commutes with inverse limits in $\mathfrak{C}_p(G)$ (Thm. 3.7.2), $\mathbf{H}^k(G, _)$ vanishes on all free $\mathbb{Z}_p[\![G]\!]$-modules ([6], §1). Since any projective $\mathbb{Z}_p[\![G]\!]$-module is a direct summand of a free $\mathbb{Z}_p[\![G]\!]$-module ([6], Lemma 1.6), the additivity of $\mathbf{H}^k(G, _)$ implies the assertion.

As in the discrete case, profinite duality groups can be characterized by the following properties ([5], Chap. VIII, Thm. 10.1).

Theorem 4.4.3. *Let G be a profinite group of type p-**FP** with $cd_p(G) = n$. Then the following are equivalent:*

(1) G is a duality group.

(2) There is a natural isomorphism of homological $\mathfrak{C}_p(G)$-functors

$$\mathbf{H}^{n-\bullet}(G, _) \longrightarrow \mathbf{H}_\bullet(G, D_p(G)\widehat{\otimes}_{\mathbb{Z}_p}_). \tag{4.4.2}$$

(3) There is a natural isomorphism of cohomological $\mathfrak{D}_p(G)$-functors

$$\mathbf{H}_{n-\bullet}(G, _) \longrightarrow \mathbf{H}^\bullet(G, \mathbf{Hom}_{\mathbb{Z}_p}(D_p(G), _)). \tag{4.4.3}$$

Proof. (1) $\Rightarrow$ (2). Assume that G is a duality group. Then $\mathbf{T}^\bullet = (T^\bullet, e^\bullet)$ with $T^{-k} = \mathbf{H}^{n-k}(G, _)$ and the connecting homomorphisms given as in Theorem 3.7.2 is an effaceable homological $\mathfrak{C}_p(G)$-functor (Lemma 4.4.2). Since $D_p(G)$ is a free $\mathbb{Z}_p$-module (4.3.8), $D_p(G)\widehat{\otimes}_{\mathbb{Z}_p}_ \colon \mathfrak{C}_p(G) \to \mathfrak{C}_p(G)$ is an exact covariant additive $\mathfrak{C}_p(G)$-functor taking projectives to projectives (Prop. 3.3.2). Hence $\mathbf{S}^\bullet = (S^\bullet, e^\bullet_S)$ with $S^{-k} = \mathbf{H}_k(G, D_p(G)\widehat{\otimes}_{\mathbb{Z}_p}_)$ and the obvious connecting homomorphisms is also an effaceable homological functor. The claim follows from Theorem 2.2.1(b) and (4.3.7).

(2) $\Rightarrow$ (1). From (4.4.2) it follows that $\mathbf{H}^k(G, \mathbb{Z}_p[\![G]\!]) = 0$ for $k \neq n$. It remains to show that $D_p(G)$ is is p-torsion-free. By (3.3.8) and (3.3.9),

$$\begin{aligned}
\mathbf{H}^{n-1}(G, \mathbb{F}_p\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!]) &\simeq \mathbf{H}_1(G, \mathbb{F}_p\widehat{\otimes}_{\mathbb{Z}_p}D_p(G)\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!]) \\
&\simeq \mathbf{H}_1(G, ind_G^1(res_1^G(\mathbb{F}_p\widehat{\otimes}_{\mathbb{Z}_p}D_p(G))) \\
&\simeq \mathbf{H}_1(1, res_1^G(\mathbb{F}_p\widehat{\otimes}_{\mathbb{Z}_p}D_p(G))) = 0.
\end{aligned} \tag{4.4.4}$$

Hence

$$0 \longrightarrow \mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!]) \xrightarrow{.p} \mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!]) \longrightarrow \mathbf{H}^n(G, \mathbb{F}_p \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![G]\!]) \longrightarrow 0 \tag{4.4.5}$$

is exact. This shows that $D_p(G) \simeq \mathbf{H}^n(G, \mathbb{Z}_p[\![G]\!])$ is p-torsion-free. (4.4.2) and (4.4.3) are obviously equivalent by Pontrjagin duality (3.7.11).

Example 4.4.4. Let G be a free pro-p group over the finite set $Y = \{ y_1, \ldots, y_r \}$. Then $cd_p(G) = 1$. If we put $x_i = y_i - 1$, then $\mathbb{Z}_p[\![G]\!]$ is a free $\mathbb{Z}_p$-algebra (in $\mathfrak{C}_{\mathbb{Z}_p}$) over the set $X = \{ x_1, .., x_r \}$ ([17], Prop. II.3.1.4). There is a projective resolution

$$0 \longrightarrow \oplus_{1 \le i \le r} \mathbb{Z}_p[\![G]\!].x_i \xrightarrow{\partial} \mathbb{Z}_p[\![G]\!] \xrightarrow{\varepsilon} \mathbb{Z}_p \longrightarrow 0 \tag{4.4.6}$$

with ∂ the obvious map. If we put $X^* = \mathrm{Hom}(X, \mathbb{Z}_p)$ and $x_i \in X^*$, $x_i^*(x_j) = \delta_{ij}$, we obtain

$$\begin{aligned} \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p[\![G]\!], \mathbb{Z}_p[\![G]\!]) &= \mathbb{Z}_p[\![G]\!] \\ \mathbf{Hom}_{\mathbb{Z}_p[\![G]\!]}(\oplus_{1 \le i \le r} \mathbb{Z}_p[\![G]\!].x_i, \mathbb{Z}_p[\![G]\!]) &= X^* \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![G]\!] \end{aligned} \tag{4.4.7}$$

with ∂^* given by

$$\partial^*(1) = \sum_{1 \le i \le r} x_i^* \widehat{\otimes} x_i =: \Delta(X). \tag{4.4.8}$$

Hence

$$\begin{aligned} \mathbf{H}^0(G, \mathbb{Z}_p[\![G]\!]) &= 0 \\ \mathbf{H}^1(G, \mathbb{Z}_p[\![G]\!]) &= X^* \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![G]\!] / \Delta(X).\mathbb{Z}_p[\![G]\!] \end{aligned} \tag{4.4.9}$$

So G is a duality group, but a Poincaré duality group if and only if $|Y| = 1$.

4.5 Poincaré duality groups

If $res_1^G(D_p(G)) \simeq \mathbb{Z}_p$, then

$$\begin{aligned} D_p(G) \widehat{\otimes}_{\mathbb{Z}_p} _ &: \mathfrak{P}_p(G) \longrightarrow \mathfrak{P}_p(G) \\ \mathbf{Hom}_{\mathbb{Z}_p}(D_p(G), _) &: \mathfrak{P}_p(G) \longrightarrow \mathfrak{P}_p(G) \end{aligned} \tag{4.5.1}$$

are covariant exact functors. Furthermore, $D_p(G) \widehat{\otimes}_{\mathbb{Z}_p} _$ maps projectives in $\mathfrak{C}_p(G)$ to projectives (Prop. 3.3.2) and $\mathbf{Hom}_{\mathbb{Z}_p}(D_p(G), _)$ maps injectives in $\mathfrak{D}_p(G)$ to injectives. From Theorem 4.4.3 and Proposition 3.6.1 we can conclude

Proposition 4.5.1. *Let G be a profinite Poincaré duality group (at the prime p) of dimension n and $D_p(G)$ its compact dualizing module. Then there are natural isomorphisms*

$$\begin{aligned} \mathbf{H}^{n-k}(G, _) &\simeq \mathbf{H}_k(G, D_p(G) \widehat{\otimes}_{\mathbb{Z}_p} _), \\ \mathbf{H}_{n-k}(G, _) &\simeq \mathbf{H}^k(G, \mathbf{Hom}_{\mathbb{Z}_p}(D_p(G), _)), \end{aligned} \tag{4.5.2}$$

of continuous covariant additive $\mathfrak{P}_p(G)$-functors.

This yields

Corollary 4.5.2. *Let G be a profinite Poincaré duality group (at the prime p) of dimension n. Let M be a $\mathbb{Z}_p[\![G]\!]$-module which is isomorphic to $\mathbb{Z}_p$ as an abelian group. Then the following are equivalent:*

(1) $\mathbf{H}^n(G, M) \neq 0$.

(2) $D_p(G)\widehat{\otimes}_{\mathbb{Z}_p} M \simeq \mathbb{Z}_p$ *as* $\mathbb{Z}_p[\![G]\!]$*-modules, where* $\mathbb{Z}_p$ *denotes the trivial* $\mathbb{Z}_p[\![G]\!]$*-module.*

From the way that we have constructed the natural transformations in (4.5.2), we can see that they actually constitute a mapping of $\mathbf{\Sigma}$-connected sequences. These natural transformations can be described in a more explicit way using the cup product.

Let G be a profinite group. The cup product is a family of group homomorphisms

$$f^{s,t}\colon \mathbf{H}^s(G, M) \otimes_{\mathbb{Z}_p} \mathbf{H}^t(G, N) \longrightarrow \mathbf{H}^{s+t}(G, M \otimes_{\mathbb{Z}_p} N) \tag{4.5.3}$$

defined for all $s, t \in \mathbb{N}_0$ and $M, N \in \mathfrak{D}_p(G)$, such that

(1) $f^{0,0}\colon \mathbf{H}^0(G, M) \otimes_{\mathbb{Z}_p} \mathbf{H}^0(G, N) \to \mathbf{H}^0(G, M \otimes_{\mathbb{Z}_p} N)$ is the canonical map,

(2) $f^{s,t}$ is natural in M and N,

(3) $f^{s+1,t}(e^s_{sp}(\mathbf{s}) \otimes id) = e^{s+t}_{sp}(\mathbf{s} \otimes N) f^{s,t}$, for all proper short exact sequences $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ in $\mathbf{\Sigma}^{sp}(\mathfrak{D}_p(G))$,

(4) $f^{s,t+1}(id \otimes e^t_{sp}(\mathbf{s})) = e^{s+t}_{sp}(M \otimes \mathbf{s}) f^{s,t}$, for all proper short exact sequences $\mathbf{s} \in \mathbf{\Sigma}^{sp}(\mathfrak{D}_p(G))$.

Its construction and uniqueness can be proved as for discrete groups (see [18], Chap. XII, §10, [5], Chap. V.3). Note that the connecting homomorphism on the left hand side is defined with the usual sign convention ([18], Chap. XII, (10.1), [5], Chap. V, (3.3)).

Let G be a profinite group of type p-**FP** with $cd_p(G) = n$. For $M \in ob(\mathfrak{F}_p(G))$ we set

$$M^\vee = \mathbf{Hom}_{\mathbb{Z}_p}(M, I_p(G)) \in ob(\mathfrak{D}_p(G)). \tag{4.5.4}$$

Consider the mapping

$$\Psi^k_M\colon \mathbf{H}^k(G, M) \otimes \mathbf{H}^{n-k}(G, M^\vee) \longrightarrow I_p, \tag{4.5.5}$$

given by $\Psi^k_M = i \circ \mathbf{H}^n(ev) \circ f^{k,n-k}$, where i is given as in Proposition 4.3.4 and ev is the obvious evaluation mapping. This induces a map

$$\Phi^k_M\colon \mathbf{H}^k(G, M) \longrightarrow \mathbf{H}^{n-k}(G, M^\vee)^* \tag{4.5.6}$$

where $\Phi^k_M(x_k)(y_{n-k}) = \Psi^k_M(x_k \otimes y_{n-k})$, $x_k \in \mathbf{H}^k(G, M)$, $y_{n-k} \in \mathbf{H}^{n-k}(G, M^\vee)$.

Let $(\gamma\colon M \to N) \in mor(\mathfrak{F}_p(G))$ and let $\gamma^\vee\colon N^\vee \to M^\vee$ denote the induced mapping. The mappings

$$\alpha^k, \beta^k\colon \mathbf{H}^k(G, M) \otimes_{\mathbb{Z}_p} \mathbf{H}^{n-k}(G, N^\vee) \longrightarrow I_p \tag{4.5.7}$$

given by

$$\begin{aligned}\alpha^k &= i \circ \mathbf{H}^n(ev \circ (id \otimes \gamma^\vee)) \circ f^{k,n-k}\\ \beta^k &= i \circ \mathbf{H}^n(ev \circ (\gamma \otimes id)) \circ f^{k,n-k}\end{aligned} \tag{4.5.8}$$

coincide. From this and property (2) of the cup product it follows that

$$\Phi^k_{_}\colon \mathbf{H}^k(G, _)_{\mathfrak{F}} \longrightarrow \mathbf{H}^{n-k}(G, _^\vee)^* \tag{4.5.9}$$

is a natural transformation of covariant $\mathfrak{F}_p(G)$-functors. One has to spend some more effort in order to show that it is a mapping of $\mathbf{\Sigma}^{sp}(\mathfrak{F}_p(G))$-connected sequences. Let $\mathbf{s}\colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ be a short exact sequence in $\mathbf{\Sigma}^{sp}(\mathfrak{F}_p(G))$, i.e., considered as a short exact sequence of abelian groups it splits. The dual sequence $\mathbf{s}^\vee\colon 0 \to C^\vee \xrightarrow{\tau^\vee} B^\vee \xrightarrow{\sigma^\vee} A^\vee \to 0$ is a short exact sequence in $\mathbf{\Sigma}^{sp}(\mathfrak{D}_p(G))$. Consider $\mathbf{s}$ and $\mathbf{s}^\vee$ as exact chain complexes with B and $B^\vee$ in degree 0. The total complex $(Q_\bullet, d_\bullet)$ of the associated double complex

$$\begin{array}{ccccc} A\otimes C^\vee & \xrightarrow{\sigma\otimes C^\vee} & B\otimes C^\vee & \xrightarrow{\tau\otimes C^\vee} & C\otimes C^\vee \\ \downarrow{\scriptstyle -A\otimes\tau^\vee} & & \downarrow{\scriptstyle B\otimes\tau^\vee} & & \downarrow{\scriptstyle -C\otimes\tau^\vee} \\ A\otimes B^\vee & \xrightarrow{\sigma\otimes B^\vee} & B\otimes B^\vee & \xrightarrow{\tau\otimes B^\vee} & C\otimes B^\vee \\ \downarrow{\scriptstyle -A\otimes\sigma^\vee} & & \downarrow{\scriptstyle B\otimes\sigma^\vee} & & \downarrow{\scriptstyle -C\otimes\sigma^\vee} \\ A\otimes A^\vee & \xrightarrow{\sigma\otimes A^\vee} & B\otimes A^\vee & \xrightarrow{\tau\otimes A^\vee} & C\otimes A^\vee \end{array} \tag{4.5.10}$$

is also exact. Note that $ker(d_{-1})$ is the pull-back of the diagram

$$\begin{array}{ccc} & & C\otimes B^\vee \\ & & \downarrow{\scriptstyle C\otimes\sigma^\vee} \\ B\otimes A^\vee & \xrightarrow{\tau\otimes A^\vee} & C\otimes A^\vee \end{array} \tag{4.5.11}$$

Now the short exact sequence $\mathbf{t} = (\mathbf{s}\otimes A^\vee \oplus (-1)C\otimes \mathbf{s}^\vee)\circ \Delta_{C\otimes A^\vee}$ is given by

$$\mathbf{t}\colon\ 0 \longrightarrow A\otimes A^\vee \oplus C\otimes C^\vee \xrightarrow{\lambda} ker(d_{-1}) \xrightarrow{\eta} C\otimes A^\vee \longrightarrow 0, \tag{4.5.12}$$

where η is given by the pull-back diagram and $\lambda = \sigma\otimes A^\vee \oplus (-1)C\otimes\tau^\vee$. The sequence $\mathbf{t}$ is congruent to

$$\begin{aligned}\mathbf{t}'\colon\ & A\otimes A^\vee \oplus C\otimes C^\vee \xrightarrow{\gamma}\\ & A\otimes A^\vee \oplus B\otimes B^\vee \oplus C\otimes C^\vee/im(d_1) \xrightarrow{\eta\circ\bar{d}_0} C\otimes A^\vee\end{aligned} \tag{4.5.13}$$

where γ denotes the canonical mapping and $\bar{d}_0$ the induced mapping. This shows that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes A^\vee \oplus C \otimes C^\vee & \xrightarrow{\lambda} & ker(d_{-1}) & \xrightarrow{\eta} & C \otimes A^\vee & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ev_A + ev_C} & & \downarrow{\scriptstyle \chi} & & & & \\ 0 & \longrightarrow & I_p(G) & = & I_p(G) & \longrightarrow & 0. & & \end{array} \tag{4.5.14}$$

Lemma 4.5.3. *Let G be a profinite group and $\mathbf{s} \in \mathbf{\Sigma}^{sp}(\mathfrak{F}_p(G))$, $\mathbf{s}^\vee \in \mathbf{\Sigma}^{sp}(\mathfrak{D}_p(G))$ as above. Let (S, e, T) be a $\mathbf{\Sigma}^{sp}$-connected pair of $(\mathfrak{D}_p(G))$-functors. Then the diagram*

$$\begin{array}{ccc} S(C \otimes A^\vee) & \xrightarrow{e(\mathbf{s} \otimes A^\vee)} & T(A \otimes A^\vee) \\ \downarrow{\scriptstyle e(C \otimes \mathbf{s}^\vee)} & & \downarrow{\scriptstyle T(ev)} \\ T(C \otimes C^\vee) & \xrightarrow{T(ev)} & T(I_p(G)) \end{array} \tag{4.5.15}$$

commutes.

Proof. From (4.5.14) we see that the composition

$$S(C \otimes A^\vee) \xrightarrow{e(\mathbf{t})} T(A \otimes A^\vee \oplus C \otimes C^\vee) \xrightarrow{T(ev_A + ev_C)} T(I_p(G)) \tag{4.5.16}$$

is trivial. Hence the additivity and functoriality of the connecting homomorphism ([18], Chap. XII, (7.2a–c)) yield the desired result.

Assume that $D_p(G)$ is a finitely generated free $\mathbb{Z}_p$-module. Then Φ^k can be extended to a natural transformation of continuous covariant $\mathfrak{P}_p(G)$-functors (Prop. 3.6.1), which we will denote by the same letter. Moreover, it has the following property.

Proposition 4.5.4. *Let G be a profinite group of type p-**FP**, $cd_p(G) = n$. Suppose that $D_p(G)$ is a finitely generated free $\mathbb{Z}_p$-module. Then*

$$\Phi^\bullet \colon (\mathbf{H}^\bullet(G, _), e^\bullet) \longrightarrow (\mathbf{H}^{n-\bullet}(G, _^\vee)^*, (-1)^{\bullet+1}(e^{n-\bullet} \circ {}^\vee)^*) \tag{4.5.17}$$

is a mapping of continuous $\mathbf{\Sigma}$-connected sequences of covariant $\mathfrak{P}_p(G)$-functors. Furthermore, Φ^n coincides with the Pontrjagin dual of the mapping T (4.3.3) and hence is a natural isomorphism. If G is a Poincaré duality group, then Φ^k is a natural isomorphism for all $k = 0, \ldots, n$.

Proof. By hypothesis, $^\vee$ is an exact contravariant additive $\mathfrak{P}_p(G)$-functor. So the right hand side of (4.5.17) is a continuous $\mathbf{\Sigma}$-connected sequence of covariant $\mathfrak{P}_p(G)$-functors.

Let $\mathbf{s} \colon 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ be a short exact sequence in $\mathbf{\Sigma}^{sp}(\mathfrak{F}_p(G))$ and let $\mathbf{s}^\vee$ denote the dual sequence. Using property (3) and (4) of the cup product and Lemma

4.5.3, one can check easily that the diagram

$$\begin{array}{ccc} \mathbf{H}^k(G, C) & \xrightarrow{\Phi^k} & \mathbf{H}^{n-k}(G, C^\vee)^* \\ {\scriptstyle e_k(\mathbf{s})}\downarrow & & \downarrow{\scriptstyle (-1)^{k+1}e^{n-k-1}(\mathbf{s}^\vee)^*} \\ \mathbf{H}^{k+1}(G, A) & \xrightarrow{\Phi^{k+1}} & \mathbf{H}^{n-k-1}(G, A^\vee)^* \end{array} \tag{4.5.18}$$

is commutative. Hence $\Phi^\bullet$ is a morphism of $\mathbf{\Sigma}^{sp}(\mathfrak{F}_p(G))$-connected sequences of $\mathfrak{F}_p(G)$-functors. From (3.6.6) one concludes that $\Phi^\bullet$ is also a morphism of $\mathbf{\Sigma}^{sp}(\mathfrak{D}_p(G))$-connected sequences of $\mathfrak{D}_p(G)$-functors. Then Proposition 3.5.2 implies that $\Phi^\bullet$ is a morphism of $\mathbf{\Sigma}(\mathfrak{D}_p(G))$-connected sequence of $\mathfrak{D}_p(G)$-functors. Now (3.6.5) implies that $\Phi^\bullet$ is a morphism of $\mathbf{\Sigma}(\mathfrak{P}_p(G))$-connected sequences of $\mathfrak{P}_p(G)$-functors.

Let $M \in ob(\mathfrak{F}_p(G))$. The map

$$\mathbf{H}^n(ev) \circ f^{n,0} \colon \mathbf{H}^n(G, M) \otimes_{\mathbb{Z}_p} \mathbf{H}^0(G, M^\vee) \longrightarrow \mathbf{H}^n(G, I_p(G)) \tag{4.5.19}$$

coincides with the coefficient morphism, i.e., $\mathbf{H}^n(ev) \circ f^{n,0}(x_n)(y_0) = y_0(x_n)$ where $x_n \in \mathbf{H}^n(G, M)$, $y_0 \in \mathbf{H}^0(G, M^\vee)$ ([5], Chap. 3, Exer. 1). This shows that $\Phi^n = T^*$. If G is a Poincaré duality group, then the final remark follows from Corollary 3.6.3.

Remark 4.5.5. Proposition 4.5.4 allows us to make the natural isomorphisms occuring in Proposition 4.5.1 much more precise. A more elegant way would have been to use the cap product which we do not introduce in this article. The Pontrjagin dual $i^* \colon \mathbb{Z}_p \to \mathbf{H}^n(G, I_p(G))$ of the dualizing morphism i (Prop. 4.3.4) defines a cohomology class $z = i^*(1)$. For discrete Poincaré duality groups the cap product with the equivalent of this cohomology class z yields the first of the natural isomorphisms occuring in (4.5.2) (cf. [5], Thm. 10.1, second proof).

The cup product can be used to prove the following Poincaré duality criterion for pro-p groups which is due to J-P. Serre.[11]

Theorem 4.5.6. *Let G be a pro-p group of type p-**FP** with $cd_p(G) = n$. Then the following are equivalent:*

(1) G is a Poincaré duality group.

(2) $dim_{\mathbb{F}_p}(H^n(G, \mathbb{F}_p)) = 1$ and the canonical pairing

$$H^k(G, \mathbb{F}_p) \otimes H^{n-k}(G, \mathbb{F}_p) \longrightarrow H^n(G, \mathbb{F}_p) \tag{4.5.20}$$

induced by the cup product is non-degenerate.

Proof.[12] If G is a Poincaré duality group, then

$$H^n(G, \mathbb{F}_p) = H^n(G, \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{F}_p, I_p(G))) = H^0(G, \mathbb{F}_p) = \mathbb{F}_p. \tag{4.5.21}$$

[11] In [23], §I.4.5 Poincaré duality groups were defined by the property (2). In [23], Prop. I.30 it was shown that they satisfy property (1).

[12] The proof of the proposition is basically the same as in [23].

The non-degeneracy of (4.5.20) follows from Proposition 4.5.3. Assume that G satisfies (2). Let $\mathbb{F}_p\mathfrak{P}_p(G)$ denote the proper subcategory of $\mathfrak{P}_p(G)$ consisting of all modules in $ob(\mathfrak{P}_p(G))$ which are annihilated by $p \in \mathbb{Z}_p$ and let

$$\alpha_k : \mathbf{H}^k(G, _) \longrightarrow \mathbf{H}^{n-k}(G, _^*)^* \tag{4.5.22}$$

be the natural transformations of $\mathbb{F}_p\mathfrak{P}_p(G)$-functors induced by the cup product (Prop. 4.5.4). Since $\mathbf{H}^k(G, _)$ and $\mathbf{H}^{n-k}(G, _^*)^*$ are cohomological $\mathbb{F}_p\mathfrak{P}_p(G)$-functors, it follows from the Five Lemma and induction that the α_k are natural isomorphisms of $\mathbb{F}_p\mathfrak{P}_p(G)$-functors. In particular, by Lemma 3.3.4,

$$\begin{aligned} \mathbf{H}^{n-1}(G, \mathbb{F}_p[\![G]\!]) &\simeq \mathbf{H}^1(G, \mathbb{F}_p[\![G]\!]^*)^* \\ &\simeq \mathbf{H}_1(G, \mathbb{F}_p[\![G]\!]) \simeq \mathbf{H}_1(G, ind_G^1(\mathbb{F}_p)) = 0. \end{aligned} \tag{4.5.23}$$

The long exact sequence implies that $0 \to D_p(G) \xrightarrow{\cdot p} D_p(G)$ is exact. Thus $D_p(G)$ is p-torsion-free and therefore a free $\mathbb{Z}_p$-module (cf. Cor. 2.5.2). On the other hand the uniqueness of the dualizing module for the category $\mathbb{F}_p\mathfrak{P}_p(G)$ implies that $D_p(G)/p.D_p(G) \simeq \mathbb{F}_p$. Hence $D_p(G)$ is isomorphic to $\mathbb{Z}_p$ as a $\mathbb{Z}_p$-module and $\mathbf{H}^\bullet(G, \mathbf{Hom}_{\mathbb{Z}_p}(_, I_p(G)))^* \simeq \mathbf{H}_\bullet(G, D_p(G)\widehat{\otimes}_{\mathbb{Z}_p}_)$ is a continuous homological $\mathfrak{P}_p(G)$-functor (3.7.11). By the Five Lemma, induction and continuity, one concludes that Φ^k (4.5.9) is a natural isomorphisms of $\mathfrak{P}_p(G)$-functors for $k = 0, \ldots, n$. Hence G is a Poincaré duality group (Thm. 4.4.3).

Let G be a Poincaré duality group and let

$$\phi_{D_p(G)} : G \longrightarrow \mathbb{Z}_p^* = GL_{\mathbb{Z}_p}(D_p(G)) \tag{4.5.24}$$

denote its representation on its compact dualizing module. One can distinguish three types of Poincaré duality groups.

(1) $D_p(G) \simeq \mathbb{Z}_p$ is the trivial $\mathbb{Z}_p[\![G]\!]$-module. In this case G is called *orientable*.

(2) $im(\phi_{D_p(G)})$ is finite. In this case we call G *virtually orientable*.

(3) $im(\phi_{D_p(G)})$ is infinite.

We will see later that, for a p-torsion-free compact p-adic analytic group virtual orientability is related to the unimodularity of its associated $\mathbb{Q}_p$-Lie algebra (Cor. 5.2.6).

4.6 The mod p cohomology ring

Let $\mathfrak{X}$ be a group theoretic property for profinite groups which is closed with respect to open subgroups, i.e., if G has the property $\mathfrak{X}$ and $H \leq_o G$, then H has also the property $\mathfrak{X}$. We say that G has *virtually* the property $\mathfrak{X}$ (or is virtually $\mathfrak{X}$), if G contains an open subgroup with property $\mathfrak{X}$. So one has the notions of profinite virtual Poincaré duality groups, profinite groups which are virtually of type p-**FP**, profinite groups with finite virtual cohomological p-dimension, etc.

Some of the group theoretic properties $\mathfrak{X}$ we have studied in this section have strong implications on the structure of the mod p cohomology ring $H^\bullet(G, \mathbb{F}_p)$, where

G has property $\mathfrak{X}$. For example, this applies to groups of finite cohomological p-dimension, groups of type p-$\mathbf{FP}_\infty$ and Poincaré duality groups (Prop. 4.1.1, Prop. 4.2.2, Prop. 4.5.1). Moreover, the class of pro-p groups with one of these properties is characterized by the structure of the mod p cohomology ring (Prop. 4.1.1(3), Cor. 4.2.5, Thm. 4.5.6). For profinite groups of type p-**VFP** (i.e., profinite groups which are virtually of type p-**FP**) one has the following:

Proposition 4.6.1. *Let G be profinite group of type p-**VFP**. Then $H^\bullet(G, \mathbb{F}_p)$ (with cup product) is a finitely generated $\mathbb{F}_p$-algebra.*

Proof. See [22], Prop. 13.5. To us it is not known whether this particular structure of the mod p cohomology ring characterizes pro-p groups with this property.

For groups of type p-$\mathbf{FP}_\infty$ it is possible to define a cup product on $H^\bullet(G, \mathbb{Z}_p)$. Using similar arguments as in [12], §7.4, one can also prove that for profinite groups of type p-**VFP**, $H^\bullet(G, \mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p$-algebra.

5 Cohomology of p-adic analytic groups

5.1 Cohomological finiteness conditions

One of the central results in the theory of p-adic analytic groups is that they contain always an open subgroup which is uniformly powerful ([17, Chap. V, 2.2.7.1]). More precisely, one can show

Theorem 5.1.1 (M. Lazard)**.** *Let G be a compact p-adic analytic group. Then G contains an open characteristic uniformly powerful pro-p group.*

Proof. ([10, Cor. 8.34]). This result is actually the key to the proofs of many theorems concerning compact p-adic analytic groups. Uniformly powerful subgroups of compact p-adic analytic groups are play the role of the connected component for real analytic groups. One consequence of Theorem 5.1.1 is also the following:

Theorem 5.1.2 (M. Lazard)**.** *Let G be a compact p-adic analytic group. Then $\mathbb{Z}_p[\![G]\!]$ is noetherian. In particular, compact p-adic analytic groups are of type p-$\mathbf{FP}_\infty$.*

Proof. ([17, Prop. V.2.2.4], [10, Cor. 7.25], and §4.2).

We do not know whether there exist virtual pro-p- groups G for which $\mathbb{Z}_p[\![G]\!]$ is noetherian and which are not p-adic analytic. We call a profinite group G *noetherian* if every closed subgroup $H \leq G$ is finitely generated. For solvable groups one has the following characterization:

Theorem 5.1.3. *Let G be a solvable virtual pro-p group. Then the following are equivalent.*

(1) $\mathbb{Z}_p[\![G]\!]$ is noetherian.

(2) G is noetherian.

(3) G is p-adic analytic.

The proof of Theorem 5.1.3 makes use of the following

Lemma 5.1.4. *Let G be a virtual pro-p group such that $\mathbb{Z}_p[\![G]\!]$ is noetherian. Then every closed subgroup $A \leq G$ is of type p-$\mathbf{FP}_\infty$. In particular, G is noetherian.*

Proof. In order to prove the lemma we may assume that G a pro-p group (Prop. 4.2.1). Let $A \leq G$ be a closed subgroup. By Corollary 4.2.5, it suffices to show that $H^k(A, \mathbb{F}_p)$ is finite for all $K \in \mathbb{N}_0$. From the Eckmann-Shapiro lemma (3.4.7) one obtains

$$H^k(A, \mathbb{F}_p) = \mathbf{Ext}^k_{\mathbb{Z}_p[\![G]\!]}(ind_G^A(\mathbb{Z}_p), \mathbb{F}_p). \tag{5.1.1}$$

As $\mathbb{Z}_p[\![G]\!]$ is noetherian, one has $\mathfrak{C}^0(G) = \mathfrak{C}^\infty(G)$. Since $ind_G^A(\mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p[\![G]\!]$-module, it follows that it is a $\mathbb{Z}_p[\![G]\!]$-module of type p-$\mathbf{FP}_\infty$. Thus $H^k(A, \mathbb{F}_p)$ is finite.

Proof of Theorem 5.1.3. It remains to prove the implication (2) $\Rightarrow$ (3) for G a pro-p group. We proceed by induction on the derived length $dl(G)$ of G. Every finitely generated abelian pro-p group is p-adic analytic. Hence the assertion holds for $dl(G) = 1$. Let G be a noetherian solvable pro-p group with $dl(G) = n$. Then $A = G^{(n-1)}$, the $(n-1)^{th}$ closed derived subgroup of G, is a closed abelian subgroup, and by assumption, A is finitely generated. Thus A is p-adic analytic. Moreover, G/A is a noetherian with $dl(G/A) = n-1$. So the induction hypothesis yields that G/A is p-adic analytic. By a theorem of Serre, extensions of p-adic analytic groups are p-adic analytic ([17, Thm. III.3.4.5], [10, Thm. 9.7]). This completes the proof.

For uniformly powerful pro-p groups the mod p cohomology ring can be calculated explicitly. One obtains the following striking result:

Theorem 5.1.5 (Lazard)**.** *Let G be a uniformly powerful pro-p group and let $P^{el.ab.} = P/P^p$. Then*

$$H^\bullet(P, \mathbb{F}_p) \simeq \Lambda_\bullet((P^{el.ab.})^*), \tag{5.1.2}$$

where $\Lambda_\bullet(_)$ denotes the exterior $\mathbb{F}_p$-algebra. In particular, P is a Poincaré duality group in dimension $n = rk_p(P)$ (at the prime p).

The proof is given at the end of this subsection. For a different proof close to Lazard's original proof see [27, Chap. II].

For p odd powerful pro-p groups can be characterized by their mod p cohomology ring.

Theorem 5.1.6. *Let p be odd and let P be a finitely generated pro-p group. Then the following are equivalent.*

(1) The canonical mapping

$$\gamma_P \colon H^1(P, \mathbb{F}_p) \wedge H^1(P, \mathbb{F}_p) \to H^2(P, \mathbb{F}_p) \tag{5.1.3}$$

is injective.

(2) P is powerful.

Proof. Let $Q = P/\Phi(P)$ where $\Phi(P) := \overline{P^p[P, P]}$ denotes the Frattini group, and let $\pi : P \to Q$ denote the canonical projection. One has a commutative diagram

$$\begin{array}{ccc} H^1(P, \mathbb{F}_p) \wedge H^1(P, \mathbb{F}_p) & \xrightarrow{\gamma_P} & H^2(P, \mathbb{F}_p) \\ {\scriptstyle \pi_1^* \wedge \pi_1^*}\uparrow & & {\scriptstyle \pi_2^*}\uparrow \\ H^1(Q, \mathbb{F}_p) \wedge H^1(Q, \mathbb{F}_p) & \xrightarrow{\gamma_Q} & H^2(Q, \mathbb{F}_p) \end{array} \tag{5.1.4}$$

where the horizontal arrows are induced by cup product, and the vertical arrows are given by inflation. By construction,

$$\pi_1^* : H^1(Q, \mathbb{F}_p) \to H^1(P, \mathbb{F}_p) \tag{5.1.5}$$

is an isomorphism [23, I.4.2]. So $\pi_1^* \wedge \pi_1^*$ is an isomorphism.

For $d \in H^2(Q, \mathbb{F}_p)$ let

$$1 \longrightarrow \mathbb{F}_p \longrightarrow Q_d \longrightarrow Q \longrightarrow 1 \tag{5.1.6}$$

denote the equivalence class of group extensions classified by the element d. As $H^2(Q, \mathbb{F}_p) = im(\gamma_Q) \oplus im(\beta)$, where β denotes the first Bockstein operator ([12, §3.4]), one has

Fact 5.1.6.1. *$Q_d^p = 1$ if and only if $d \in im(\gamma_Q)$.*

For short we set $Q_c = Q_{\gamma_Q(c)}$ and denote by $\pi_c : Q_c \to Q$ the canonical projection. Let

$$P_c = Q_c \times_{(\pi_c, \pi)} P := \{ (q, p) \in Q_c \times P \mid \pi_c(q) = \pi(p) \} \tag{5.1.7}$$

Now $\gamma_P(c_P) = 0$ if and only if the extension $\mathbb{F}_p \to P_c \to P$ splits. So $\gamma_P(c_P) = 0$, if and only if there exists a group homomorphism τ which makes the diagram

$$\begin{array}{ccc} P & = & P \\ {\scriptstyle \tau}\downarrow & & {\scriptstyle \pi}\downarrow \\ Q_c & \xrightarrow{\pi_C} & Q \end{array} \tag{5.1.8}$$

commute. If $c \neq 0$ such a map τ induces a surjection $\tau^* : P/\Phi(P) \to Q_c/\Phi(Q_c)$, and thus τ must be surjective.

Let P be powerful and $c \in H^1(Q, \mathbb{F}_p) \wedge H^1(Q, \mathbb{F}_p)$, $c \neq 0$. As $P/\overline{P^p}$ is elementary abelian and $Q_c^p = 1$, there exist no surjective group homomorphisms $\tau : P \to Q_c$. Thus $\gamma_P(c_P) \neq 0$ for all $c \neq 0$ and γ_P is injective.

Let γ_P be injective and assume that P is not powerful. Then $\overline{P^p} \lneq \Phi(P)$. Hence there exists $N \lhd_o P$ such that $\overline{P^p} \leq N \lneq \Phi(P)$ and $|\Phi(P)/N| = p$. Then $P^* := P/N$ is a non-abelian p-group of exponent p and $P^* \simeq Q_d$ for some $d \in H^2(Q, \mathbb{F}_p)$, $d \neq 0$. As P^* is of exponent p, $d \in im(\gamma_Q)$ (Fact 5.1.6.1). Let $d = \gamma_Q(c)$, $c \in H^1(Q, \mathbb{F}_p) \wedge H^1(Q, \mathbb{F}_p)$. In particular, $c \neq 0$. Let $c_P := \pi_1^* \wedge \pi_1^*(c)$. By hypothesis, one has $\gamma_P(c_P) \neq 0$. However, from the previous remark (5.1.8) one concludes that $\gamma_P(c_P) = 0$, a contradiction, and the theorem is proved.

Profinite groups of finite cohomological p-dimension are necessarily p-torsionfree. Note that for a finitely generated powerful pro-p group, being uniformly powerful is equivalent to being torsionfree ([10, Chap. 4]). Hence one obtains the following straight forward consequence of Theorem 5.1.6.

Corollary 5.1.7. *Let p be odd and let G be a finitely generated pro-p group with*

$$H^\bullet(P, \mathbb{F}_p) \simeq \Lambda_\bullet((P^{el.ab.})^*). \tag{5.1.9}$$

Then P is uniformly powerful.

In order to prove Poincaré duality for p-torsion-free compact p-adic analytic groups we have to make use of the following result on the cohomological p-dimension due to Serre ([24]):

Theorem 5.1.8 (Serre)**.** *Let G be a p-torsionfree profinite group and $H \leq_o G$ an open subgroup of G. Then $cd_p(H) = cd_p(G)$.*

From Proposition 4.4.1, Theorem 5.1.5 and Theorem 5.1.8 we conclude:

Theorem 5.1.9 (Lazard)**.** *Let G be a compact p-adic analytic group. Then G is a virtual Poincaré duality group in dimension $n = dim(G)$ at the prime p, where $dim(G)$ denotes the dimension of G as p-adic analytic manifold.*

Note that Theorem 5.1.9 also implies that a compact p-adic analytic group G is a Poincaré duality group provided G is p-torsion-free.

For profinite groups which are virtually of type p-**FP** it is possible to define the p-Euler-Poincaré charcteristic $\chi_p(_)$ as in the discrete case. If G is of type p-**FP** the p-Euler-Poincaré charcteristic is given by

$$\chi(G) = \sum_{i=0}^{cd_p(G)} (-1)^i dim_{\mathbb{F}_p}(H^i(G, \mathbb{F}_p)). \tag{5.1.10}$$

If G is virtually of type p-**FP** one defines $\chi(G) = \frac{1}{|G/U|} \cdot \chi(U)$, where $U \leq_o G$ is an open subgroup of G of finite cohomological p-dimension. It turns out that this definition is independent of the choice of the subgroup $U \leq_o G$ ([23, §I.4.1]).

Corollary 5.1.10. *Let G be an infinite compact p-adic analytic group. Then one has*

$$\chi_p(G) = 0. \tag{5.1.11}$$

Remark 5.1.11. The Demuškin pro-p groups furnish another class of Poincaré duality groups ([23, Chap. I, §4.5, Ex. 2]). Their p-Euler-Poincaré characteristic can differ from 0. However, if G is a Demuškin pro-p group with $\chi_p(G) = 0$, then G is p-adic analytic.

Lazard's proof of Theorem 5.1.5 ([17, Chap. V, §2]) can be seen as a constructive version of a more general principle which was introduced by May ([19], [20]) and which is known as the *May spectral sequence*. The advantage of this technique is that it yields approximations for the cohomology and homology of a finitely generated pro-p group with not necessarily trivial coefficients.

Let G be a finitely generated pro-p group and let

$$\mathbb{F}_p[\![G]\!] = \varprojlim_{U \lhd_o G} \mathbb{F}_p[G/U], \tag{5.1.12}$$

be the compact $\mathbb{F}_p$-group algebra. It has an obvious augmentation $\varepsilon\colon \mathbb{F}_p[\![G]\!] \to \mathbb{F}_p$. We set $I(G) := ker(\varepsilon)$. We denote by $gr_\bullet(G)$ the restricted $\mathbb{F}_p$-Lie algebra associated to the Jennings filtration of G and which we assume to be graded by negative[13] integers in the obvious way. Let $\mathbf{\Delta}_\bullet(G) = \mathfrak{U}^{[p]}(gr_\bullet(G))$ denote its $-\mathbb{N}_0$-graded restricted universal enveloping algebra which coincides with the $-\mathbb{N}_0$-graded $\mathbb{F}_p$-group algebra associated to the augmentation filtration of $\mathbb{F}_p[\![G]\!]$ by the Jennings–Quillen theorem ([10, Chap. 12]).

For $M \in ob(\mathfrak{C}^0_{\mathbb{F}_p[\![G]\!]})$ there is an increasing filtration given by

$$F_{-k}M = I(\mathbb{F}_p[\![G]\!])^k.M \tag{5.1.13}$$

for $k \in \mathbb{N}_0$ and $F_kM = M$ for $k \in \mathbb{N}$. Since G is assumed to be finitely generated, it is obvious that $F_\bullet$ is exhaustive and separated. Let $gr_\bullet(M)$ denote the associated $\mathbb{Z}$-graded $\mathbf{\Delta}_\bullet(G)$-module. Since we assumed M to be finitely generated as an $\mathbb{F}_p[\![G]\!]$-module, $gr_\bullet(M)$ is discrete as abelian group.

A discrete module $N \in ob(\mathfrak{D}_{\mathbb{F}_p[\![G]\!]})$ is filtered in the opposite way. There is an increasing filtration $F_\bullet$ given by $F_kN = 0$ for $k < 0$ and

$$F_kN = \{\, n \in N \mid I(\mathbb{F}_p[\![G]\!])^k.n = 0 \,\} \tag{5.1.14}$$

Again, it is easy to check that $F_\bullet$ is exhaustive and separated. We denote by $gr_\bullet(N)$ the associated $\mathbb{Z}$-graded left $\mathbf{\Delta}_\bullet(G)$-module.

Theorem 5.1.12 (May spectral sequence)**.** *Let G be a finitely generated pro-p group.*

(1) Let $M \in ob(\mathfrak{C}^0_{\mathbb{F}_p[\![G]\!]})$. Then there exists a second quadrant spectral sequence $(E^r_{\bullet,\bullet}, d^r)$, $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ of homological type with

$$E^1_{s,t} = \mathrm{Tor}^{\mathbf{\Delta}_\bullet(G)}_{s,t}(\mathbb{F}_p, gr_\bullet(M)), \tag{5.1.15}$$

(where s denotes the filtration degree and t denotes the complementary[14] degree) converging to

$$E^\infty_{s,t} = F_sH_{s+t}(G,M)/F_{s-1}H_{s+t}(G,M). \tag{5.1.16}$$

For $M = \mathbb{F}_p$ the spectral sequence converges as a spectral sequence of $\mathbb{F}_p$-coalgebras. Moreover, the induced filtration $F_\bullet$ on $H_\bullet(G,M)$ is exhaustive and separated.

(2) Let $N \in ob(\mathfrak{D}_{\mathbb{F}_p[\![G]\!]})$. Then there exists a second quadrant spectral sequence $(E_r^{\bullet,\bullet}, d_r)$, $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$ of cohomological type with

$$E_1^{s,t} = \mathrm{Ext}^{s,t}_{\mathbf{\Delta}_\bullet(G)}(\mathbb{F}_p, gr_\bullet(N)), \tag{5.1.17}$$

[13] This is equivalent to considering the Jennings filtration as an increasing filtration.

[14] This means that $s + t$ equals the homological degree.

(where s denotes the filtration degree and t the complentary degree) converging to

$$E_\infty^{s,t} = F^s H^{s+t}(G,N)/F^{s+1}H^{s+t}(G,N). \tag{5.1.18}$$

For $N = \mathbb{F}_p$ the spectral sequence converges as a spectral sequence of $\mathbb{F}_p$-algebras. Moreover, the induced filtration $F^\bullet$ on $H^\bullet(G,N)$ is exhaustive and separated.

Proof. For finite p-groups this theorem is well-known and was discussed in great detail in [9]. In order to handle the convergency problems we have to recall and extend the proof given in [9].
(1) Let $(\hat{\mathbb{B}}_\bullet, d_\bullet)$ denote the topological unreduced bar resolution of the trivial right $\mathbb{F}_p[\![G]\!]$-module $\mathbb{F}_p$, i.e.,

$$\hat{\mathbb{B}}_k = \overbrace{\mathbb{F}_p[\![G]\!]\widehat{\otimes}_{\mathbb{F}_p}\cdots\widehat{\otimes}_{\mathbb{F}_p}\mathbb{F}_p[\![G]\!]}^{k\text{-times}}\widehat{\otimes}_{\mathbb{F}_p}\mathbb{F}_p[\![G]\!] \tag{5.1.19}$$

and d_k is given by the formula

$$\begin{aligned} d_k(\langle x_1|\cdots|x_k\rangle x) = {} & \langle x_2|\cdots|x_k\rangle.x \\ & + \sum_{1\le j\le k-1}(-1)^i\langle x_1|\cdots|x_jx_{j+1}|\cdots|x_k\rangle.x \\ & + (-1)^k\langle x_1|\cdots|x_{k-1}\rangle x_k x. \end{aligned} \tag{5.1.20}$$

It is well known that $(\hat{\mathbb{B}}_\bullet, d_\bullet)$ is a projective resolution of $\mathbb{F}_p$ as right $\mathbb{F}_p[\![G]\!]$-module ([6, Cor. 3.3]). Let $(C_\bullet, \partial_\bullet) = (\hat{\mathbb{B}}_\bullet\widehat{\otimes}_{\mathbb{F}_p[\![G]\!]}M, d_\bullet\widehat{\otimes}M)$, and let $F_\bullet$ denote the canonical increasing filtration on $C_\bullet$, i.e.,

$$F_sC_k = \sum_{s=s_1+\cdots+s_k+s_{k+1}} F_{s_1}\mathbb{F}_p[\![G]\!]\widehat{\otimes}_{\mathbb{F}_p}\cdots\widehat{\otimes}_{\mathbb{F}_p}F_{s_k}\mathbb{F}_p[\![G]\!]\widehat{\otimes}_{\mathbb{F}_p[\![G]\!]}F_{s_{k+1}}M, \tag{5.1.21}$$

where $F_\bullet\mathbb{F}_p[\![G]\!]$ is given by (5.1.13). Then $F_\bullet$ is an exhaustive and separated filtration on C_k for all $k \in \mathbb{N}_0$. Moreover, the topology induced by the filtration $F_\bullet$ coincides with the natural topology on C_k. From (5.1.20) it follows that ∂_k is continuous. Hence $F_\bullet$ is convergent (§2.4). It is easy to see that the E^1-term of the associated spectral sequence coincides with (5.1.15). The final remark follows since the Alexander-Whitney map ([1, p. 59]) is a mapping of filtered chain complexes ((2.4.16)).
(2) Let $(\hat{\mathbb{B}}_\bullet, d_\bullet)$ denote the unreduced bar resolution of the trivial left $\mathbb{F}_p$-module and let $F_\bullet$ be the canonical increasing filtration on $\hat{\mathbb{B}}_\bullet$ (5.1.21). For $N \in ob(\mathfrak{D}_{\mathbb{F}_p[\![G]\!]})$, let $F^\bullet$ be the decreasing filtration on

$$(C^\bullet, \delta^\bullet) = (\mathbf{Hom}_{\mathbb{F}_p[\![G]\!]}(\hat{\mathbb{B}}_\bullet, N), d_\bullet^*) \tag{5.1.22}$$

given by

$$F^sC^k = \{\,\phi \in \mathbf{Hom}_{\mathbb{F}_p[\![G]\!]}(\hat{\mathbb{B}}_k, N) \mid \phi(F_j\hat{\mathbb{B}}_k) \le F_{j-s}N \text{ for all } j \in \mathbb{Z}\,\}. \tag{5.1.23}$$

Then $F^1C^k = 0$. A morphism $\psi \in \mathbf{Hom}_{\mathbb{F}_p[\![G]\!]}(\mathbb{B}_k, N)$ must have finite image. This implies that $\psi \in \bigcup_{s\in-\mathbb{N}_0} F^sC^k$ and that the filtration $F^\bullet$ is exhaustive and separated on $C^\bullet$. For the spectral sequence associated to the filtered cochain complex $(C^\bullet, \delta^\bullet, F^\bullet)$ one obtains

$$\begin{aligned} E_0^{s,t} &= \mathrm{Hom}^s_{\mathbf{\Delta}_\bullet(G)}(gr_\bullet(\mathbb{B}_{s+t}), gr_\bullet(N)) \\ E_1^{s,t} &= \mathrm{Ext}^{s,t}_{\mathbf{\Delta}_\bullet(G)}(\mathbb{F}_p, gr_\bullet(N)) \end{aligned} \tag{5.1.24}$$

([9, §2.3]). The topology induced by the filtration coincides with the discrete topology. So $F^\bullet$ is convergent. The final remark can be proved as in (1).

Unfortunately there is nothing known about the higher differentials in these spectral sequences. Nevertheless, Theorem 5.1.12(2) can be used to prove Lazard's theorem as follows.

Proof of Theorem 5.1.5. Let P be a uniformly powerful group and $P^{el.ab.} = \mathbb{F}_p.x_1 \oplus \cdots \oplus \mathbb{F}_p.x_n$. Then $\mathbf{\Delta}_\bullet(P) = S_\bullet[x_1, \ldots, x_n]$ is a polynomial $\mathbb{F}_p$-algebra generated in degree -1. Using the Koszul resolution (Rem. 2.5.2]) one concludes that

$$\mathrm{Ext}^{s,t}_{\mathbf{\Delta}_\bullet(G)}(\mathbb{F}_p, \mathbb{F}_p) = \Lambda_{\bullet,\bullet}[y_1, \ldots, y_n], \tag{5.1.25}$$

where each generator y_i is of bidegree $(-1, 2)$. Since $(E_r^{\bullet,\bullet}, d_r)$ is a spectral sequence of differential graded algebras ([9, Lemma 14]), one concludes that $E_\infty^{\bullet,\bullet} \simeq E_1^{\bullet,\bullet}$ as bigraded $\mathbb{F}_p$-algebras. From the filtration on $H^n(G, \mathbb{F}_p)$, it is obvious that $H^\bullet(G, \mathbb{F}_p)$ is isomorphic to $\mathbf{tot}_\bullet(\Lambda_{\bullet,\bullet}[y_1, \ldots, y_n])$ as an $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebra. The final claim follows from Serre's Poincaré duality criterion (Thm. 4.5.6).

5.2 Rational cohomology

In this subsection we consider the rational cohomology of a p-adic analytic group G with coefficients in $\mathfrak{fd}_p(G)$. For simplicity let $\mathbf{p} = p$ if p is odd and $\mathbf{p} = 4$ if $p = 2$.

In [17] Lazard defined several classes of finitely generated pro-p groups by the existence of a certain filtration on the group and its properties, e.g., the class of p-valuable groups and p-saturable groups. We will restrict our considerations to the class of uniformly powerful groups (=u.p.f. groups) and denote the associated category by $\mathfrak{P}f^{t.f.}$. It is well known that every finitely generated uniformly powerful group is p-saturable and therefore p-valuable. As corresponding p-filtration ([17, Chap. III, (2.1)]) one can take the lower p-central series (suitably re-indexed for $p = 2$). For $P \in ob(\mathfrak{P}f^{t.f.})$ we put

$$\begin{aligned} F^kP &= P^{p^{k-1}}, \qquad p \text{ odd} \\ F^{k+1}P &= P^{p^{k-1}}, \qquad p = 2. \end{aligned} \tag{5.2.1}$$

We define the associated valuation $\omega_P : P \to \mathbb{N}_0 \cup \{\infty\}$ by

$$\omega_P(g) = \max\{\, k \in \mathbb{N}_0 \cup \{\infty\} \mid g \in F^kP \,\}. \tag{5.2.2}$$

The associated graded $\mathbb{F}_p$-module will be denoted by $gr^\bullet(P)$. Note that $gr^\bullet(P)$ is an $\mathbb{F}_p[t]$-Lie algebra, $deg(t) = 1$, with operations given by

$$\begin{aligned} &= (g,h).F^{k+l+1}P, \\ t.(g.F^{k+1}P) &= g^p, \end{aligned} \tag{5.2.3}$$

where $g \in F^kP$, $h \in F^lP$ and $(g,h) = g^{-1}h^{-1}gh$ denotes the group commutator. As P is a finitely generated uniformly powerful group, $gr^\bullet(P)$ is a finitely generated free $\mathbb{F}_p[t]$-module. The compact group algebra $\mathbb{Z}_p[\![P]\!]$ has a canonical (decreasing) filtration given by

$$\begin{aligned} F^k\mathbb{Z}_p[\![P]\!] &= J^k, \quad J = p.\mathbb{Z}_p.1 + I(\mathbb{Z}_p[\![P]\!]), \qquad p \text{ odd} \\ F^k\mathbb{Z}_p[\![P]\!] &= 2^k.\mathbb{Z}_p + 2^{k-2}I(\mathbb{Z}_p[\![P]\!]) \\ &\quad +2^{k-4}I(\mathbb{Z}_p[\![P]\!])^2 + \cdots + I(\mathbb{Z}_p[\![P]\!])^{\lceil \frac{k}{2} \rceil} p = 2, \end{aligned} \tag{5.2.4}$$

where $I(\,)$ denotes the augmentation ideal and $\lceil x \rceil$ stands for the smallest integer larger that x. Again we set

$$\omega(x) = \max\{\, k \in \mathbb{N}_0 \cup \{\infty\} \mid x \in F^k\mathbb{Z}_p[\![P]\!] \,\}. \tag{5.2.5}$$

By $gr^\bullet(\mathbb{Z}_p[\![P]\!])$ we denote the associated graded $\mathbb{F}_p$-module, which is canonically and $\mathbb{F}_p[t]$-algebra, where $t = p + F^2\mathbb{Z}_p[\![P]\!]$ is of degree 1.

Theorem 5.2.1. *Let $P \in ob(\mathfrak{P}f^{t.f.})$ and let $F^\bullet$ be given as in (5.2.1), (5.2.4). Then:*

(1) There is a canonical isomorphism

$$\mathfrak{U}_{\mathbb{F}_p[t]}(gr^\bullet(P)) \simeq gr^\bullet(\mathbb{Z}_p[\![P]\!]), \tag{5.2.6}$$

$$\begin{aligned} \omega(1) &= 0, \\ \omega(x+y) &\geq \min\{\omega(x), \omega(y)\}, \\ \omega(x \cdot y) &\geq \omega(x) + \omega(y), \\ \omega(\lambda \cdot x) &= v(\lambda) + \omega(x) \end{aligned} \tag{5.2.7}$$

(3) For $g \in P$ one has

$$\omega_P(g) = \omega(g-1). \tag{5.2.8}$$

In particular,

$$P \cap (1 + F^k\mathbb{Z}_p[\![P]\!]) = F^kP. \tag{5.2.9}$$

Proof. The theorem is just a special case of [17, Thm. III.2.3.3]. Let $i\colon P \to \mathbb{Z}_p[\![P]\!]$ denote the canonical mapping given by $i(g) = g - 1$. For $g - 1 \in F^k\mathbb{Z}_p[\![P]\!]$ ($k \geq 2$

for $p = 2$) one has $g^p - 1 \equiv p.(g-1) \mod F^{k+2}\mathbb{Z}_p[\![P]\!]$. Hence one proves easily by induction that $\omega_P(g) \leq \omega(g-1)$ for all $g \in P$. This implies that ω coincides with the canonical filtration ω on $\mathbb{Z}_p[\![P]\!]$ induced by ω_P ([17, Def. III.2.3.1.2]). Hence the induced morphism

$$gr^\bullet(i)\colon gr^\bullet(P) \longrightarrow gr^\bullet(\mathbb{Z}_p[\![P]\!]) \tag{5.2.10}$$

is an injective morphism of $\mathbb{F}_p[t]$-Lie algebras and

$$gr^\bullet(i)^*\colon \mathfrak{U}_{\mathbb{F}_p[t]}(gr^\bullet(P)) \longrightarrow gr^\bullet(\mathbb{Z}_p[\![P]\!]) \tag{5.2.11}$$

is an isomorphism of associative $\mathbb{F}_p[t]$-algebras ([17, Thm. III.2.3.3]). This proves (1). Since $gr^\bullet(\mathbb{Z}_p[\![P]\!])$ is a free $\mathbb{F}_p[t]$-module, the filtration is actually a valuation and (5.2.7) is satisfied. The filtration is separated and all ideals $F^k\mathbb{Z}_p[\![P]\!]$ are open and of finite index in $\mathbb{Z}_p[\![P]\!]$. Since $\mathbb{Z}_p[\![P]\!]$ is compact with respect to the topology induced by the filtration $F^\bullet$, this topology has to coincide with the natural topology. This shows (2). (3) is a direct conseqeunce of (1).

The valuation ω can be extended to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![P]\!]$ by the assignment

$$\omega(q \otimes x) = v(q) + \omega(x), \tag{5.2.12}$$

where $v\colon \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ denotes the p-adic valuation with $v(p) = 1$. Obviously, all statements of (5.2.7) are also satisfied for $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![P]\!], \omega)$. We denote by $\hat{A}[\![P]\!]$ the completion of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![P]\!]$ and denote by ω again also the extension of ω to $\hat{A}[\![P]\!]$. The associated filtration on $\hat{A}[\![P]\!]$ is given by

$$F^k\hat{A}[\![P]\!] = \{\, x \in \hat{A}[\![P]\!] \mid \omega(x) \geq k \,\}. \tag{5.2.13}$$

We call $\hat{A}[\![P]\!]$ the *completed group algebra*[15] of P. From the construction it is clear that every morphism $\phi\colon P \to Q$ in $\mathfrak{P}f^{t.f.}$ extends uniquely to a continuous morphism

$$\phi^*\colon \hat{A}[\![P]\!] \longrightarrow \hat{A}[\![Q]\!]. \tag{5.2.14}$$

So $\hat{A}[\![_]\!]$ can be considered as a covariant functor from $\mathfrak{P}f^{t.f.}$ to the category of associative $\mathbb{Q}_p$-algebras (with valuation). The $\mathbb{Z}_p$-Lie algebra of P is defined by

$$\mathrm{Log}(P) = \{\, log(g) \in \hat{A}[\![P]\!] \mid g \in P \,\}, \tag{5.2.15}$$

where the bracket is given by $[x, y] = x.y - y.x$, and log is given by the power series $log(x) = \sum_{k\in\mathbb{N}} \frac{-1^k}{k}.(x-1)^k$. Note that for $g \in F^kP$,

$$log(g) \equiv (g-1) \mod F^k\hat{A}[\![P]\!]. \tag{5.2.16}$$

The filtration $F^\bullet$ on $\hat{A}[\![P]\!]$ induces a filtration on $\mathrm{Log}(P)$, which we also denote by $F^\bullet$.

[15] This definition coincides with the definition in [10, §7] only for p odd. For $p = 2$ see [10, Exer. 7.10].

Proposition 5.2.2.

(1) Let $\mathrm{Lie}_{\mathbb{Z}_p}^{p.f.}$ *denote the category of* $\mathbb{Z}_p$*-Lie algebras whose objects are* $\mathbb{Z}_p$*-Lie algebras* $\mathfrak{L}$ *which are finitely generated free* $\mathbb{Z}_p$*-modules satisfying* $[\mathfrak{L}, \mathfrak{L}] \leq \mathbf{p}.\mathfrak{L}$*. Then* $\mathrm{Log}\colon \mathfrak{P}f^{t.f.} \to \mathrm{Lie}_{\mathbb{Z}_p}^{p.f.}$ *is a covariant (exact) functor which induces an equivalence of categories.*

(2) For all $P \in ob(\mathfrak{P}f^{t.f.})$ *there exists a canonical isomorphism*

$$gr^{\bullet}(P) \simeq gr^{\bullet}(\mathrm{Log}(P)) \tag{5.2.17}$$

of graded $\mathbb{F}_p[t]$*-Lie algebras.*

(3) The associative $\mathbb{Q}_p$*-algebra generated by* $\mathrm{Log}(P) \leq \hat{A}[\![P]\!]$ *is isomorphic to* $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{A}_{\mathbb{Z}_p}(\mathrm{Log}(P))$ *and it is dense in* $\hat{A}[\![P]\!]$*.*

Proof. (1) The fact that $\mathrm{Log}(_)$ is a covariant functor follows directly from the construction. The fact that it induces an equivalence of categories follows from the existence of the functor $Exp\colon \mathrm{Lie}_{\mathbb{Z}_p}^{p.f.} \to \mathfrak{P}f^{t.f.}$ ([10, Thm. 9.10], [17, Chap. IV]). (2) follows from (5.2.16) and the fact that $gr^{\bullet}(i)$ (5.2.10) is an injective homomorphism of graded $\mathbb{F}_p[t]$-Lie algebras. (3) follows from (2) and Theorem 5.2.1(1).

Let G be a compact p-adic analytic group and $P \lhd_o G$ an open normal uniformly powerful subgroup of G. Then P has a continuous right G-action by conjugation, $h^g = g^{-1}hg$, $g \in G$, $h \in P$. This implies that there are also a canonical right G-actions on $\hat{A}[\![P]\!]$ and $\mathrm{Log}(P)$ where G acts by $\mathbb{Q}_p$-algebra and $\mathbb{Z}_p$-Lie algebra homomorphisms respectively.

If G is a compact p-adic analytic group then the $\mathbb{Q}_p$-Lie algebra $L(G)$ associated to G is defined by

$$L(G) = \varinjlim_{\substack{U \lhd_o G \\ U \text{ u.p.f.}}} \mathbb{Q}_p \otimes \mathrm{Log}(U). \tag{5.2.18}$$

All the morphisms are given by inclusion and are therefore compatible with the right G-action, so $L(G)$ has a canonical right G-action which we will denote by

$$Ad\colon G \longrightarrow Aut(L(G)). \tag{5.2.19}$$

It is easily seen that $L(G) \simeq \mathbb{Q}_p \otimes \mathrm{Log}(U)$ for every open normal uniformly powerful subgroup $U \lhd_o G$ ([17, Chap. V, (2.4.2.4)]) and that the isomorphism is also an isomorphism of right G-modules.

It is easy to check that

$$Pf(W) = \{\, id_W + \alpha \mid \alpha \in \mathbf{p}.\mathrm{End}_{\mathbb{Z}_p}(W) \,\} \subset \mathrm{End}_{\mathbb{Z}_p}(W) \tag{5.2.20}$$

is a finitely generated uniformly powerful pro-p group, and that

$$\mathfrak{pf}(W) = \mathbf{p}.\mathrm{End}_{\mathbb{Z}_p}(W) \subset \mathrm{End}_{\mathbb{Z}_p}(W) \tag{5.2.21}$$

is a powerful $\mathbb{Z}_p$-Lie lattice, i.e., $\mathfrak{pf}(W) \in ob(\mathrm{Lie}_{\mathbb{Z}_p}^{u.p.f.})$. We also set

$$Al(W) = \mathbb{Z}_p.id_W + \mathbf{p}.\mathrm{End}_{\mathbb{Z}_p}(W) \subset \mathrm{End}_{\mathbb{Z}_p}(W) \tag{5.2.22}$$

which is a compact $\mathbb{Z}_p$-algebra. Using $log\colon Pf(W) \to \mathfrak{pf}(W)$ and $exp\colon \mathfrak{pf}(W) \to Pf(W)$ given by $exp(x) = \sum_{k\in\mathbb{N}_0} \frac{1}{k!}x^k$, we deduce that

$$\mathrm{Log}(Pf(W)) = \mathfrak{pf}(W). \tag{5.2.23}$$

Let $P \in ob(\mathfrak{P}f^{t.f.})$ and let $M \in ob(\mathfrak{C}_p(P))$ be a finitely generated free $\mathbb{Z}_p$-module. Let $\phi_M\colon P \to GL_{\mathbb{Z}_p}(M)$ denote the associated representation. We say that the representation ϕ_M is *powerful*, if $im(\phi_M) \leq Pf(M)$. A powerful representation ϕ_M induces a representation of $\mathbb{Z}_p$-Lie algebras

$$\mathrm{Log}(\phi_M)\colon \mathrm{Log}(P) \longrightarrow \mathfrak{pf}(M) \tag{5.2.24}$$

and $\mathbb{Q}_p$-algebras

$$\hat{A}(\phi_M)\colon \hat{A}[\![P]\!] \longrightarrow \mathrm{End}_{\mathbb{Q}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M). \tag{5.2.25}$$

Let G be a compact p-adic analytic group, $V \in ob(\mathfrak{fd}_p(G))$ and let $\phi\colon G \to GL_{\mathbb{Q}_p}(V)$ denote the associated representation. Then V contains a $\phi(G)$-stable open $\mathbb{Z}_p$-lattice M (Prop. 3.8.1(2)), and $H = \phi^{-1}(Pf(M)) \lhd_o G$ is an open normal pro-p-subgroup of G. Let $P \lhd_o H$ be a uniformly powerful characteristic subgroup of H (Thm. 5.1.1). By construction $P \lhd_o G$ is a uniformly powerful open normal subgroup of G and, $\phi|_P\colon P \to Pf(M)$ is a powerful representation. Let

$$L_{M,P}(\phi) = \mathbb{Q}_p \otimes \mathrm{Log}(\phi|_P)\colon L(G) \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{End}_{\mathbb{Z}_p}(M) \to \mathrm{End}_{\mathbb{Q}_p}(V) \tag{5.2.26}$$

denote the associated representation for the $\mathbb{Q}_p$-Lie algebra $L(G)$. It is independent of the chosen uniformly powerful characteristic subgroup $P \lhd_o H$ (5.2.18), and of the chosen $\phi(G)$-stable $\mathbb{Z}_p$-lattice M. So we have proved ([17, Thm. V.2.4.9]):

Proposition 5.2.3. *Let G be a compact p-adic analytic group. Then every module $V \in ob(\mathfrak{fd}_p(G))$ is canonically a left $L(G)$-module. In particular, there is an exact functor*

$$L(_)\colon \mathfrak{fd}_p(G) \longrightarrow {}_{L(G)}\mathrm{TMod}^G. \tag{5.2.27}$$

Proof. It suffices to show that the action of $L(G)$ on $V \in ob(\mathfrak{fd}_p(G))$ is G-equivariant.

Let $M \leq V$ be a G-stable $\mathbb{Z}_p$-lattice, $\phi\colon G \to \mathrm{End}_{\mathbb{Q}_p}(V)$ the associated representation and let $P \lhd_o G$ be uniformly powerful such that $\phi(P) \leq Pf(M)$. The action of P on $M \in ob(\mathfrak{C}_p(G))$ is G-equivariant (§2.3). Therefore the action of $\hat{A}[\![P]\!]$ on $V = \mathbb{Q}_p \otimes M$ is G-equivariant and thus also the action of $\mathrm{Log}(P)$ on M. This proves the claim.

For a finitely generated uniformly powerful pro-p group P we call $V \in ob(\mathfrak{fd}_p(P))$ *powerful* if it contains a P-stable open $\mathbb{Z}_p$-lattice M such that the associated representation $\phi_M\colon P \to GL_{\mathbb{Z}_p}(M)$ is powerful. The main goal of this subsection is to prove the following theorem which can be seen as an extended version of Theorem V. 2.4.10 in [17].

Theorem 5.2.4. (a) *Let P be a finitely generated uniformly powerful group and let $V \in ob(\mathfrak{fd}_p(P))$ be a powerful $\mathbb{Z}_p[\![P]\!]$-module. Then there is a canonical isomorphism*

$$\mathbb{H}^\bullet(P, V) \simeq H^\bullet(L(P), L(V)). \tag{5.2.28}$$

(b) *Let G be a compact p-adic analytic group and $W \in ob(\mathfrak{fd}_p(G))$. Then there is a canonical isomorphism*

$$\mathbb{H}^\bullet(G, W) \simeq H^\bullet(L(G), L(W))^G. \tag{5.2.29}$$

If $P \lhd_o G$ is an open normal uniformly powerful group for which the restriction of W to P is a powerful representation, then P acts trivially on $H^\bullet(L(G), L(W))$.

Proof.(a) Let $F^\bullet$ be the filtration on $\mathbb{Z}_p[\![P]\!]$ given by (5.2.4). The $\mathbb{F}_p[t]$-algebra $gr^\bullet(\mathbb{Z}_p[\![P]\!])$ is isomorphic to the universal enveloping algebra of $gr^\bullet(P)$. Since $gr^\bullet(P)$ is a finitely generated free $\mathbb{F}_p[t]$-module, $gr^\bullet(\mathbb{Z}_p[\![P]\!])$ has a finite special projective resolution $(\bar{P}_\bullet, \bar{d}_\bullet, \bar{\varepsilon}_\bullet, \bar{s}_\bullet)$ with $\bar{d}_\bullet$ a morphism of graded $gr^\bullet(\mathbb{Z}_p[\![P]\!])$-modules and $\bar{\varepsilon}_\bullet$, $\bar{s}_\bullet$ morphisms of graded $\mathbb{F}_p[t]$-modules, and all the $\bar{P}_k$ are free $gr^\bullet(\mathbb{Z}_p[\![P]\!])$-modules of finite rank (Rem. 2.5.2).

By Serre's Lemma (Lemma 2.5.1), this implies that $\mathbb{Z}_p[\![P]\!]$ has a finite special projective resolution $(P_\bullet, d_\bullet, \varepsilon_\bullet, s_\bullet)$, and all morphisms are morphism of filtered $\mathbb{Z}_p[\![P]\!]$-modules or $\mathbb{Z}_p$-modules, as appropriate.

All the filtered modules $(P_k, F^\bullet)$ are finitely generated filtered free, and the underlying filtration is a valuation (Lemma 2.5.1(iii)). Let $P_k^* = \mathbb{Q}_p \otimes P_k$ and $d_\bullet^*$, $\varepsilon_\bullet^*$, $s_\bullet^*$ denote the canonical extensions of the mappings $d_\bullet$, $\varepsilon_\bullet$, $s_\bullet$ to $P_\bullet^*$, and let $F^\bullet$ denote the extension of the filtration as in (5.2.12). Let Q_k denote the completion of P_k^* with respect to the filtration $F^\bullet$ and let $d_\bullet^*$, $\varepsilon_\bullet^*$, $s_\bullet^*$ denote the extension of the mappings to $Q_\bullet$. By construction $(Q_\bullet, d_\bullet^*, \varepsilon_\bullet^*, s_\bullet^*)$ is a finite special projective resolution of the $\mathbb{Q}_p$-algebra $\hat{A}[\![P]\!]$ (2.4.3), and all Q_k are free $\hat{A}[\![P]\!]$-modules of finite rank.

Note that $\mathrm{Log}(P) \leq \hat{A}[\![P]\!]$ is a filtered $\mathbb{Z}_p$-Lie algebra and the filtration (induced by the filtration on $\hat{A}[\![P]\!]$) coincides with the p-adic filtration (suitably re-indexed for $p = 2$). Hence $(\mathbf{Ko}_\bullet(\mathrm{Log}(P)), d_\bullet, \varepsilon_\bullet, s_\bullet)$ is a special projective resolution and all the mappings are mappings of filtered $\mathfrak{U}_{\mathbb{Z}_p}(\mathrm{Log}(P))$-modules or $\mathbb{Z}_p$-modules (Rem. 2.5.2). Since all filtrations are actually valuations (5.2.7), one can construct a special projective resolution $(Q_\bullet^\circ, d_\bullet^\circ, \varepsilon_\bullet^\circ, s_\bullet^\circ)$ of the $\mathbb{Q}_p$-algebra $\hat{A}[\![P]\!]$ as in the previous case

(Prop. 5.2.2(3)). Note that Q_k° coincides with the completion of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbf{Ko}_k(\mathrm{Log}(P))$, and hence is a free $\hat{A}[\![P]\!]$-module of finite rank.

Since P_k is a finitely generated free $\mathbb{Z}_p[\![P]\!]$-module, $(P_\bullet, d_\bullet, \varepsilon_\bullet, s_\bullet)$ is a special projective resolution in $\mathfrak{C}_p(P)$ and also ${}_{\mathbb{Z}_p[\![P]\!]}\mathrm{Mod}$. As $\mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![P]\!]}(P_k, M) = \mathrm{Hom}_{\mathbb{Z}_p[\![P]\!]}(P_k, M)$ for all $M \in ob(\mathfrak{fd}_p(P))$, there is a canonical isomorphism

$$\gamma : \mathbb{H}^\bullet(P, V) \longrightarrow H^\bullet(\mathbb{Z}_p[\![P]\!], V), \tag{5.2.30}$$

where $\mathbb{H}^\bullet(P, _)$ stands for rational cohomology ((3.8.5.)), and $H^\bullet(\mathbb{Z}_p[\![P]\!], _)$ coincides with the usual cohomology functor for an augmented $\mathbb{Z}_p$-algebra (2.3.7).

Since $\mathbb{Z}_p[\![P]\!] \leq \hat{A}[\![P]\!]$ is a $\mathbb{Z}_p$-subalgebra, there is a canonical restriction mapping

$$\begin{aligned} \mathrm{res}^\bullet_{gr} : H^\bullet(\hat{A}[\![P]\!], V) &= \mathrm{Ext}^\bullet_{\hat{A}[\![P]\!]}(\mathbb{Q}_p, V) \\ &\longrightarrow \mathrm{Ext}^\bullet_{\mathbb{Z}_p[\![P]\!]}(\mathbb{Z}_p, V) = H^\bullet(\mathbb{Z}_p[\![P]\!], V) \end{aligned} \tag{5.2.31}$$

defined for all $V \in ob({}_{\hat{A}[\![P]\!]}\mathrm{Mod})$. If $V \in ob(\mathfrak{fd}_p(P))$ is powerful, every morphism $\phi \in \mathrm{Hom}_{\mathbb{Z}_p[\![P]\!]}(P_k, V)$ is continuous (with respect to the filtration $F^\bullet$ on P_k), and so can be lifted to a morphism $\phi^* \in \mathrm{Hom}_{\hat{A}[\![P]\!]}(Q_k, V)$. This shows that $\mathrm{res}_{gr} : \mathrm{Hom}_{\hat{A}[\![P]\!]}(Q_k, V) \to \mathrm{Hom}_{\mathbb{Z}_p[\![P]\!]}(P_k, V)$ is an isomorphism, and so $\mathrm{res}^\bullet_{gr}$ is an isomorphism.

Similarly, we can construct the morphism

$$\mathrm{res}^\bullet_{lie} : H^\bullet(\hat{A}[\![P]\!], V) \longrightarrow H^\bullet(L(P), V) \tag{5.2.32}$$

for all $V \in ob({}_{\hat{A}[\![P]\!]}\mathrm{Mod})$. Basically, the same argument as in the previous case shows that for a powerful $\mathbb{Z}_p[\![P]\!]$-module $V \in ob(\mathfrak{fd}_p(P))$, $\mathrm{res}^\bullet_{lie}$ is an isomorphism. We have produced the following canonical isomorphisms

$$\mathbb{H}^\bullet(P, V) \xrightarrow{\gamma^\bullet} H^\bullet(\mathbb{Z}_p[\![P]\!], V) \xleftarrow{\mathrm{res}^\bullet_{gr}} H^\bullet(\hat{A}[\![P]\!], V) \xrightarrow{\mathrm{res}^\bullet_{lie}} H^\bullet(L(P), V). \tag{5.2.33}$$

This proves the claim.

(b) All morphisms in (5.2.33) are G-equivariant, so the claim follows from Theorem 4.2.6(b).

This theorem has some immediate and important consequences:

Corollary 5.2.5. *Let G be a p-torsion-free compact p-adic analytic group, let $D_p(G) \in ob(\mathfrak{C}_p(G))$ be its compact dualizing module and let $\phi_{D_p(G)} : G \to \mathbb{Z}_p^*$ denote the associated representation of G. Then*

$$\phi_{D_p(G)}(g) = det(Ad(g)), \tag{5.2.34}$$

where $Ad : G \to Aut(L(G))$ is defined in (5.2.19).

Proof. Let $dim(G) = n$. By considering the Koszul complex we conclude that for $W \in ob({}_{\mathfrak{U}(L(G))}\mathrm{Mod})$,

$$H^n(L(G), W) = Hom_{L(G)}(\Lambda_n(L(G)), W). \tag{5.2.35}$$

Note that $x \in L(G)$ acts (from the left) on $\Lambda_n(L(G))$ by multiplication with $tr(ad(x))$, where $ad(x)(y) = [x, y]$, $x, y \in L(G)$. Moreover, $tr(ad(x^g)) = tr(ad(x))$ for all $g \in G$. The natural (right) G-action on $\Lambda_n(L(G))$ is given by multiplication with $det(Ad(g))$, $g \in G$. The Koszul resolution is a G-equivariant resolution of $\mathfrak{U}(L(G))$, so for $W \in ob({}_{L(G)}\mathrm{TMod}^G)$, (5.2.35) is a G-morphism. For $W = \Lambda_n(L(G))$, $id \in \mathrm{Hom}_{L(G)}(\Lambda_n(L(G)), \Lambda_n(L(G)))$ is a fixed point of the G-action, so $H^n(L(G), \Lambda_n(L(G)))^G = \mathbb{Q}_p$.

If $M \in ob(\mathfrak{C}_p(G))$ and $M \simeq \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module, with the action of G given by multiplication with $det(Ad(g^{-1}))$, $g \in G$, it is easy to check that $L(\mathbb{Q}_p \otimes M) \simeq \Lambda_n(L(G))$ and the isomorphism is an isomorphism in ${}_{L(G)}\mathrm{TMod}^G$. Hence $\mathbf{H}^n(G, M) \neq 0$ and the claim follows by Corollary 4.5.2.

Corollary 5.2.6. *Let G be a p-torsion-free compact p-adic analytic group with $dim(G) = n$. Then the following are equivalent:*

(i) *$L(G)$ is a unimodular $\mathbb{Q}_p$-Lie algebra, i.e., $H^n(L(G), \mathbb{Q}_p) \neq 0$.*

(ii) *G is a virtually orientable Poincaré duality group.*

(iii) *G has strict cohomological p-dimension $n + 1$.*

Proof. (i)⇔(ii): For $P \in ob(\mathfrak{P}f^{t.f.})$, $Ad^\times \colon P \to Aut(\mathrm{Log}(P))$ is a powerful representation and $exp(tr(log(Ad^\times(g) - 1))) = det(Ad^\times(g))$. Now, by Corollary 5.2.5, P is orientable if and only if $\mathbb{Q}_p \otimes \mathrm{Log}(P)$ is unimodular.
(ii)⇔(iii): This follows from [23, Prop. I.19].

We call a p-adic analytic group *semi-simple* if $L(G)$ is a semi-simple $\mathbb{Q}_p$-Lie algebra. For these groups Lazard's theorem and a well known vanishing result for cohomology of semi-simple Lie algebras in characteristic 0 imply the following result:

Corollary 5.2.7. *Let G be a semi-simple p-adic analytic group.*

(a) *Let $W \in ob(\mathfrak{fd}_p(G))$ be such that for every open subgroup $H \leq_o G$, $res^G_H(W)$ has no trivial composition factor (i.e., $\mathbb{H}\mathrm{om}_{\mathbb{Z}_p[\![H]\!]}(\mathbb{Q}_p, res^G_H(W)) = 0$). Then $\mathbb{H}^\bullet(G, W) = 0$.*

(b) *Let $M \in ob(\mathfrak{C}_p(G))$ be a finitely generated $\mathbb{Z}_p$-module such that $W = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ satisfies the hypothesis of* (a)*. Then $\mathbf{H}^k(G, M)$ is finite for all $k \in \mathbb{N}_0$.*

Proof. (a) The hypothesis ensures that $Hom_{L(G)}(\mathbb{Q}_p, W) = 0$. Since $L(G)$ is semisimple, $W = W_1 \oplus \cdots W_r$ is is a direct sum of simple $L(G)$-modules W_i by Weyl's theorem. Hence the claim follows from Theorem 5.2.4 and a well known vanishing result for the cohomology of semi-simple Lie algebras in characteristic 0 ([15, Prop. XVIII.3.1]).
(b) Let $T(M) \leq M$ denote the p-torsion part of M. Then $T(M) \in ob(\mathfrak{F}_p(G))$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M/T(M) \simeq W$. The claim follows from the long exact sequence, part (a), (3.8.6) and (4.2.1).

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