MATH 20201 Algebraic Structures I. Exercises.

Sheet 1: Permutations

1. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$.

   (i) Work out $\tau \circ \sigma, \sigma \circ \tau, \tau \circ \tau$ and $\sigma \circ \sigma$ (in standard notation).

   (ii) Write $\sigma, \tau, \tau \circ \sigma, \sigma \circ \tau, \tau \circ \tau$ and $\sigma \circ \sigma$ as a composite of disjoint cycles.

2. Write the following permutations of $\Omega = \{1, 2, 3, 4, 5\}$ in standard notation. (i) $(134) \circ (25)$, (ii) $(134) \circ (134) \circ (134)$, (iii) $(231) \circ (321)$, (iv) $(1324) \circ (531) \circ (24)$.

3. Which of the following equations are true? (i) $(12345) = (34512)$, (ii) $(12345) = (23415)$, (iii) $(54321) = (32154)$, (iv) $(54321) = (15423)$.

4. Write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 8 & 5 & 6 & 1 & 3 & 7 & 2 \end{pmatrix}$$

   and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 9 & 1 & 3 & 4 & 8 & 7 & 2 & 6 \end{pmatrix}$$

   as a composite of disjoint cycles.

5. Write the following permutations of S_5 as a composite of disjoint cycles:

   (i) $(1234) \circ (13) \circ (24) \circ (1432)$

   (ii) $(12345) \circ (1342) \circ (15432)$

   (iii) $(15) \circ (14) \circ (13) \circ (12)$

6. Prove that every non-trivial permutation of $\Omega = \{1, 2, 3, ..., n\}$ can be written as a composite of less than n transpositions.

Sheet 2: Groups

1. Which of the following sets are groups with respect to the binary operation given? Give proofs!

   (i) The set of all 2x2 matrices of the form $\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$ where $a \in \mathbb{R}, a \neq 0$; under matrix multiplication.

   (ii) The set of all 2x2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where $a, b \in \mathbb{R}, a \neq 0 \neq b$, under matrix multiplication.

   (iii) The set $\mathbb{Z} \times \mathbb{Z} = \{(a, b); a, b \in \mathbb{Z}\}$ with multiplication

   $$(a, b)(n, m) = (a + n, b + m);$$

   $a, b, n, m \in \mathbb{Z}$.

   (iv) The set $\mathbb{Z} \times \mathbb{Z}$ with multiplication

   $$(a, b)(n, m) = (a + n, (-1)^n b + m);$$

   $a, b, n, m \in \mathbb{Z}$.

   (v) The power set $P(\Omega)$ of a non empty set Ω under intersection of sets.

   (vi) $P(\Omega)$ as in (v) with the binary operation

   $$AB = (A \cup B) \setminus (A \cap B),$$

   $A, B \subseteq \Omega$. (You may take for granted that this operation is associative.)

2. For the set G and the binary operation $*$ on G as given below, determine if $(G, *)$ is a group. Give reasons if your answer is NO!

   (i) $G = \mathbb{Q} \setminus \{0\}$, $\quad a * b = ab$ (multiplication of numbers),

   (ii) $G = \{1, -1\}$, $\quad a * b = ab$ (multiplication of numbers),

   (iii) $G = \mathbb{Z}$, $\quad a * b = ab$ (multiplication of numbers),

   (iv) G is the set of even integers, $\quad a * b = a + b$ (addition of numbers),

   (v) $G = \{f, g\}$, where f and g are functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(n) = n,\ g(n) = -n\ \forall n \in \mathbb{Z}.$, $\quad * = \circ$ (composition of functions),

(vi) $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$, $\quad * =$ matrix multiplication,

(vii) $G = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$, $\quad * = \circ$ (composition of permutations),

(viii) $G = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$, $\quad * = \circ$ (composition of permutations),

(ix) $G = \{0, 1\}$, $\quad a * b = ab$ (multiplication of numbers),

(x) G is the set of odd integers, $\quad a * b = ab$ (multiplication of numbers).

3. Prove that $\mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$ is a group under the binary operation

$$(a, b)(c, d) = (ac - bd, bc + ad),$$

where $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$.

4. Let $G = \{e, a, b, c\}$, where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let $*$ be matrix multiplication. Write out the multiplication table for $(G, *)$ and prove that it is a group.

5. Let $G = \mathbb{R} \setminus \{1\}$, and $a * b = a + b - ab$. Prove that $(G, *)$ is a group.

6. Let $G = \{e, a, b, c\}$ be the set of permutations of $\Omega = \{1, 2, 3, 4\}$ where

$$e = id, \ a = (12), \ b = (34), \ c = (12) \circ (34)$$

and let $* = \circ$. (composition of permutations). Write out the multiplication table for $(G, *)$ and prove that it is a group.

7. Using cycle notation, make a list of all elements of the symmetric group S_4.

8. Make a list of all elements of the group $GL(2, \mathbb{Z}_2)$.

Sheet 3: Subroups

1. For the following subsets S of the given group $(G, *)$ determine whether or not S is a subgroup. Give reasons if your answer is NO.

   (i) $G = \mathbb{C},\ * = +,\ S = \{n + mi \mid n, m \in \mathbb{Z}\}$,

   (ii) $G = \mathbb{C},\ * = +,\ S = \{3n + mi \mid n, m \in \mathbb{Z}\}$,

   (iii) $G = \mathbb{Q}^* = \mathbb{Q}\backslash\{0\},\ * = \times$ (multiplication of numbers), $S = \{1, -1\}$,

   (iv) G is the set of all permutations of $\Omega = \{1, 2, ..., n\}$, $* = \circ$, S is the set of all transpositions,

   (v) $G = \mathbb{Z}_6,\ * = \oplus,\ S = \{0, 2, 4\}$,

   (vi) $G = \mathbb{Z}_6,\ * = \oplus,\ S = \{0, 1, 3\}$.

2. Let $n \geqslant 2$ and A_n denote the subset of S_n consisting of all permutations of $\{1, 2, \ldots, n\}$ that can be written as a composite of an even number of transpositions. Prove that A_n is a subgroup of S_n.

3. Work out the orders of the following elements.

   (i) $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{\sqrt{3}}{2} + \frac{1}{2}i \in \mathbb{C}^*$

   (ii) $4, 15, 18, 33 \in \mathbb{Z}_{36}$

   (iii) $\sigma, \tau, \theta \in S_9$, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 & 9 & 8 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 1 \end{pmatrix}$$

   (iv) $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{R})$.

4. What are the largest possible orders of elements in S_3, S_5 and S_7?

5. (i) Compute $\langle 5 \rangle$ in $\mathbb{Z}$.

   (ii) Compute $\langle (1234) \rangle$ in S_4.

   (iii) Compute $\langle 1 \rangle, \langle 2 \rangle$ and $\langle 3 \rangle$ in $\mathbb{Z}_8$.

   (iv) Compute $\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$ in $GL(2, \mathbb{R})$.

6. Prove that in the symmetric group S_n,

$$C((12...n-1)) = \langle (12...n-1) \rangle,$$

   i.e. the centralizer of the cycle $(12...n-1)$ (of length $n-1$) in S_n coincides with the cyclic subgroup generated by that cycle.

7. Using inspection of all the 24 elements of S_4 (or otherwise), work out the centralizer of the transposition (12) in S_4.

8. Prove that the centralizer of the transposition (12) in S_n with $n \geq 4$ is the subgroup $C((12)) = \{\sigma \in S_n \mid 1\sigma, 2\sigma \in \{1, 2\}\}$.

9. Work out the centralizer $C(g)$ in $GL(2, \mathbb{R})$ in each of the following cases:

$$\text{(i)} \quad g = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \text{(ii)} \quad g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{(iii)} \quad g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

10. Work out the centre of

$$T(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; \quad a, b, c \in \mathbb{R}, \ ac \neq 0 \right\}.$$

11. Work out the centre of

$$UT(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}; \quad a, b, c \in \mathbb{R} \right\}.$$

12. Let G be a group and $g \in G$.

    (i) If $C(g) = G$, what can we say about g?

    (ii) If $C(g) = G$ for all $g \in G$, what can we say about G?

Sheet 4: Cyclic Groups

1. Which elements of the cyclic group $(\mathbb{Z}_{30}, \oplus)$ are generators?

2. Prove that $(\mathbb{Q}, +)$ is not a cyclic group.

3. Find all orders of subgroups of (i) $\mathbb{Z}_{31}$, (ii) $\mathbb{Z}_{32}$, (iii) $\mathbb{Z}_{33}$.

4. Find all subgroups of $\mathbb{Z}_{15}$.

5. If G is a cyclic group of order n, and m divides n, show that G contains a subgroup of order m.

Sheet 5: Cosets, Lagrange's Theorem

1. For each pair G, H (where $H \leq G$) determine $[G : H]$ and list all right cosets of H in G.

   (i) $G = \mathbb{Z}_{15},\ H = \langle 12 \rangle$,

   (ii) $G = \{e, (12), (34), (12)(34)\}$ (a subgroup of the symmetric group S_4), $H = \langle (12)(34) \rangle$,

   (iii) $G = \mathbb{R}^* = (\mathbb{R} \setminus \{0\}, \times),\ H = (\mathbb{R}_+, \times)$.

2. What are the right cosets of $H = \{z \in \mathbb{C} \setminus \{0\} ; |z| = 1\}$ in $\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \times)$?

3. For $H \leq G$ as specified below, determine the right cosets of H in G.

   (i) $G = \mathbb{R}^*, H = \langle -1 \rangle$

   (ii) $G = \mathbb{C}^*, H = \mathbb{R}^*$

   (iii) $G = \mathbb{C}^*, H = \mathbb{R}_+$

   (iv) $G = \mathbb{Z}_{36}, H = \langle 30 \rangle$

   (v) $G = T(2, \mathbb{R}), H = UT(2, \mathbb{R})$

   (vi) $G = D(2, \mathbb{R}), H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} ; \lambda \in \mathbb{R}, \lambda \neq 0 \right\}$

   (vii) $G = GL(n, \mathbb{R}), H = SL(n, \mathbb{R})$ [Hint: Show that the right coset determined by a matrix $A \in GL(n, \mathbb{R})$ with determinant $\det A = a$ is the set of all nxn matrices B with determinant $\det B = a$.]

4. For G and H as in Ex.3, work out $[G : H]$, the index of H in G.

5. Let G be a group, $H \leq G$, and $x, y \in G$. Prove that $Hx = Hy$ if and only if $xy^{-1} \in H$.

6. Prove that the right cosets of $\mathbb{Z}$ in $\mathbb{R}$ are in 1-1 correspondence with the set $[0, 1) = \{x \in R \mid 0 \leq x < 1\}$.

7. In the symmetric group S_4, work out the cosets

   (i) $H(134)$, (ii) $H(23)$, (iii) $H(1432)$,

   where $H = \langle(1234)\rangle$ is the cyclic subgroup generated by (1234).

8. Let G be a finite group and $K \leq H \leq G$. Prove that $[G : K] = [G : H]\,[H : K]$.

9. Let G be a finite group with $|G| = n$. Prove that $g^n = e$ for all $g \in G$.

10. Let G be a finite group and $K, H \leq G$ with $(|K|\,, |H|) = 1$, i.e. the orders of the subgroups K and H are coprime. Prove that $H \cap K = \{e\}$.

Sheet 6: Homomorphisms and Isomorphisms

1. Which of the following maps are group homomorphisms?

   (a) $\varphi : \mathbb{C}^* \to \mathbb{R},\ \varphi(z) = \log|z|\ \ (z \in \mathbb{C}^*)$

   (b) $\varphi : \mathbb{C} \to \mathbb{R},\ \varphi(a+bi) = b\ (a, b \in \mathbb{R})$

   (c) $\varphi : GL(2,\mathbb{R}) \to \mathbb{R},\ \varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a - d,$
   $a, b, c, d \in \mathbb{R}.$

   (d) $\varphi : T(2,\mathbb{R}) \to D(2,\mathbb{R}), \quad \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$
   $a, b, d \in \mathbb{R}, \quad ad \neq 0$

   (e) $\varphi : S_n \to S_{n+1}$, where for $\sigma = \begin{pmatrix} 1 & 2 & ... & n \\ 1\sigma & 2\sigma & ... & n\sigma \end{pmatrix} \in S_n,$

   $$\varphi(\sigma) = \begin{pmatrix} 1 & 2 & ... & n & n+1 \\ 1\sigma & 2\sigma & ... & n\sigma & n+1 \end{pmatrix} \in S_{n+1}$$

   (f) $\varphi : \mathbb{Z} \to \mathbb{S}_n,\ \varphi(k) = (123...n)^k \in S_n,\ k \in \mathbb{Z}$

2. Let $\varphi : G \to H$ be a homomorphism of groups. Show that if $a \in G$ has order n, then $\varphi(a) \in H$ has order dividing n.

3. Let G be a group. Show that the map $\varphi : G \to G$ defined by $\varphi(a) = a^{-1}$ is a homomorphism if and only if G is abelian.

4. Let G be a group and $x \in G$. Show that the map $\varphi_x : G \to G$ defined by $\varphi_x(a) = x^{-1}ax$ for all $a \in G$ (with x fixed) is an isomorphism.

5. By using group-theoretic properties, show that the following statements are true.

   (a) $S_4 \ncong \mathbb{Z}_{24}$.

   (b) $\mathbb{Z} \ncong \mathbb{Q}$.

   (c) $\mathbb{Q} \ncong \mathbb{Q}^*$.

   (d) $GL(2,\mathbb{R}) \ncong UT(3,\mathbb{R})$.

Sheet 7 : Conjugacy

1. For $\sigma, \tau \in S_6$ as specified below, decide whether of not τ is a conjugate of σ. If your answer is "yes", find an element $\theta \in S_6$ such that $\tau = \theta^{-1}\sigma\theta$.

   (a) $\sigma = (1234)\,(56)\,,\ \tau = (12)\,(3456)\,,$

   (b) $\sigma = (12345)\,,\ \tau = (123456)\,,$

   (c) $\sigma = (45)\,(36)\,(12)\,,\ \tau = (16)\,(25)\,(34)\,,$

   (d) $\sigma = (12)(346),\ \tau = (123)\,(56)\,.$

2. Work out the number of conjugacy classes in S_5 and S_6.

3. Determine the number of cycles of length n in S_n. [Hint: Use Theorem 7.2, the class formula, and the fact that $C\left((12...n)\right) = \langle(12...n)\rangle$. This will be nicer than a purely combinatorial solution.]

4. Work out the order of the centralizer of $(12)\,(34)$ in S_4 and find all its elements. [Hint: Apart from obvious members of the centralizer, consider (1324) .]

5. Use the formula for conjugates in S_n to prove that $Z(S_n) = \{e\}$ for $n \geq 3$.

6. Show that in any group conjugate elements have the same order.

Sheet 8 : Normal Subgroups

1. For a group G and a subgroup $H \leq G$ as specified below, decide if H is normal in G.

   (a) $G = S_4,\ H = \langle (1234) \rangle\,;$

   (b) $G = T(2, \mathbb{R}),\ H = UT(2, \mathbb{R});$

   (c) $G = GL(2, \mathbb{R}),\ H =$ the subgroup of scalar matrices in $GL(2, \mathbb{R})$;

   (d) $G = S_4$, $H = \{e,\ (12)(34)\,,\ (13)(24)\,,\ (14)(23)\}\,;$

   (e) $G = SL(2, \mathbb{R}),\ H = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$

2. Suppose $H \lhd G$ and $K \lhd G$. Show that $H \cap K \lhd G$.

3. Suppose that $H \lhd G$ and $K \lhd G$. The product of H and K in G is the set $HK = \{hk\ ;\ h \in H, k \in K\}$. Show that $HK \lhd G$.

4. Let $H \leq K \leq G$.

   (a) Suppose $H \lhd G$. Show that $H \lhd K$.

   (b) Suppose that $H \lhd K$ and $K \lhd G$. Does this imply that $H \lhd G$?

5. Let N and M be normal subgroups of G such that $N \cap M = \{e\}$. Show that for all $x \in N$ and all $y \in M$ one has $xy = yx$.[Hint: Show that $x^{-1}y^{-1}xy = e$.]

6. Let G be a group, and let $H \lhd G$ be a cyclic group of order two which is normal in G. Prove that $H \subseteq Z(G)$.

Sheet 9: Factor Groups and the First Isomorphism Theorem

1. For the maps in Ex. 6.1, if they are homomorphisms, find the kernel and the image.

2. Show that $\mathbb{R}^*/\mathbb{R}_+ \cong \{\pm 1\}$, where $\mathbb{R}_+ = \{r \in \mathbb{R}^*;\ r > 0\}$ and $\{\pm 1\}$ is regarded as a group under multiplication.

3. Show that $\mathbb{C}^*/\{z \in \mathbb{C};\ |z| = 1\} \cong \mathbb{R}_+$, where $\mathbb{R}_+ = \{r \in \mathbb{R}^*;\ r > 0\}$.

4. Show that $T(2,\mathbb{R})/UT(2,\mathbb{R}) \cong D(2,\mathbb{R})$.

5. Let
$$N = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix};\ a, b \in \mathbb{R},\ a \neq 0 \right\} \leq T(2,\mathbb{R}).$$
Show that N is normal in $T(2,\mathbb{R})$, and prove that $T(2,\mathbb{R})/N \cong \mathbb{R}^*$. [Hint: Consider the map $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \to ac$.]

6. Let
$$N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ a, b \in \mathbb{R} \right\} \leq UT(3,\mathbb{R}).$$
Show that N is normal in $UT(3,\mathbb{R})$, and prove that $UT(3,\mathbb{R})/N \cong \mathbb{R}$. [Hint: Consider the map
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \to c.]$$