Lecture 6

Finite-difference methods

We now introduce the final numerical scheme which is related to the PDE solution. Finite difference methods are numerical solutions to (in CF, generally) parabolic PDEs. They work by approximating the derivatives at each point in time and then rearranging the equations to solve backward in time. There are three types of methods - the explicit method, which is analogous to the trinomial tree, the implicit method (which you would never use!) and the Crank-Nicolson method which has the best convergence characteristics.

Now the PDE we wish to solve is the Black-Scholes equation (BSM model) where there are continuous dividends:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0$$

Example 6.1 (Domain)**.** Construct the domain for the BSM model.

Solution 6.1.

6.1 Finite-Difference Approximations

Consider a function of two variables $V(S,t)$, if we consider small changes in S and t we can use a Taylor's series to express $V(S+\Delta S,t)$, $V(S-\Delta S,t)$, $V(S,t+\Delta t)$ as follows (all the derivatives are evaluated at (S,t))

$$V(S+\Delta S,t) = V(S,t) + \Delta S\frac{\partial V}{\partial S} + \tfrac{1}{2}(\Delta S)^2\frac{\partial^2 V}{\partial S^2} + O((\Delta S)^3) \tag{6.1}$$

$$V(S-\Delta S,t) = V(S,t) - \Delta S\frac{\partial V}{\partial S} + \tfrac{1}{2}(\Delta S)^2\frac{\partial^2 V}{\partial S^2} + O((\Delta S)^3) \tag{6.2}$$

$$V(S,t+\Delta t) = V(S,t) + \Delta t\frac{\partial V}{\partial t} + \tfrac{1}{2}(\Delta t)^2\frac{\partial^2 V}{\partial t^2} + O((\Delta t)^2) \tag{6.3}$$

In order to use a finite difference scheme we need to use these expansions to approximate the first and second derivatives with respect to S and t.

Example 6.2 (Finite Differences)**.** Use point estimates of the value function V to derive approximations of $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$.

Solution 6.2.

For t use (6.3) and we have

$$\begin{aligned}\frac{\partial V}{\partial t}(S,t) &= \frac{V(S,t+\Delta t)-V(S,t)}{\Delta t}+\frac{1}{2}\Delta t\frac{\partial^2 V}{\partial t^2}+O((\Delta t)^2)\\ &= \frac{V(S,t+\Delta t)-V(S,t)}{\Delta t}+O(\Delta t)\end{aligned}$$

This particular approximation is called forward differencing whilst the preferred method for S is called central differencing. In general central differencing (when appropriate) is the most accurate.

How does this help us?

Reconsider the Black-Scholes equation and in particular the Black-Scholes equation for a European options where there are continuous dividends:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\delta)S\frac{\partial V}{\partial S}-rV=0$$

Example 6.3 (Domain Again...)**.** Sketch the domain and outline how the finite difference formulas can derive a system of equations to obtain the solution in the entire domain.

Solution 6.3.

The boundary conditions are:

- For a call:

$$V(S,T) = \max(S - X, 0)$$
$$V(0,t) = 0$$
$$V(S,t) \to Se^{-\delta(T-t)} - Xe^{-r(T-t)} \quad \text{as} \quad S \to \infty$$

- for a put:

$$V(S,T) = \max(X - S, 0)$$
$$V(0,t) = Xe^{-r(T-t)}$$
$$V(S,t) \to 0 \quad \text{as} \quad S \to \infty$$

We will now form a finite difference grid that describes the $S - t$ space in which we need to solve the Black-Scholes equation. For a numerical method we need to truncate the range of S to $[S^L, S^U]$ where S^L is typically chosen to be zero and S^U needs to be sufficiently large.

6.2 Constructing the grid

We now need to ensure that we have a fine enough grid to allow for most possible movements in S and enough time steps t. As for the binomial and Monte-Carlo method we will discuss later what is a suitable size/number for these steps. We partition the interval $[0, S^U]$ into $jmax$ subintervals each of length $\Delta S = S^U/jmax$, thus the endpoints of the intervals (or grid nodes) are 0, ΔS, $2\Delta S$,..., $(jmax-1)\Delta S$, $jmax\Delta S = S^U$. We also partition the interval $[0,T]$ into $imax$ subintervals each of length $\Delta t = T/imax$, thus the nodes on the grid are 0, Δt,..., $(imax-1)\Delta t$, $imax\Delta t = T$. We will denote the option price at each node $V(j\Delta S, i\Delta t)$ as V_j^i

Example 6.4 (Constructing the grid)**.** Sketch the grid.

Solution 6.4.

Now consider your sketch of the grid or Figure 6.1 and focus attention on i, jth value V_j^i, and a little piece of the grid around that point. We clearly know the information at $t = T$ as this is the payoff from the option, by limiting our focus on

$$\begin{array}{cc} & V_{j+1}^{i+1} \\ V_j^i & V_j^{i+1} \\ & V_{j-1}^{i+1} \end{array}$$

we can approximate the derivatives in the Black-Scholes equation by using our difference equations and from this we can write V_j^i in terms of the other three terms.

6.3 Explicit finite difference method

Recall the BSM equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0$$

The BSM equation approximates to

$$\frac{V_j^{i+1} - V_j^i}{\Delta t} + \tfrac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} - V_{j-1}^{i+1}}{(\Delta S)^2} + (r - \delta) j \Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - rV_j^i = 0$$

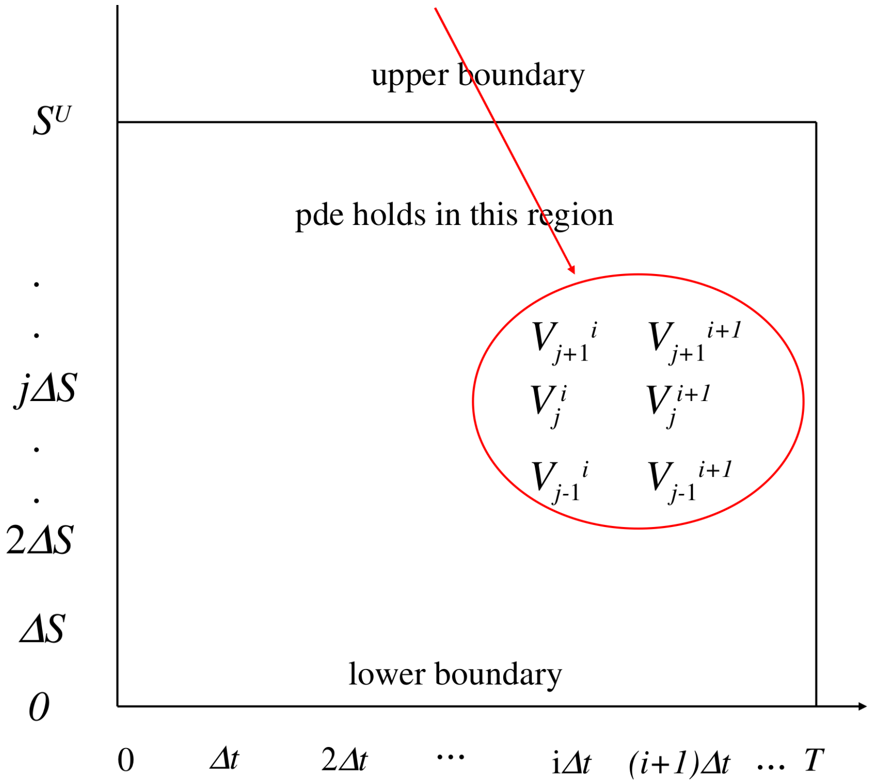

Figure 6.1: A sketch of the finite difference grid, this time highlighting the values around the point (i,j).

the unknown here is V_j^i as we have been working backward in time. So we can rearrange in terms of this unknown:

$$V_j^i = \frac{1}{1 + r\Delta t}(AV_{j+1}^{i+1} + BV_j^{i+1} + CV_{j-1}^{i+1}) \tag{6.4}$$

where:

$$A = (\tfrac{1}{2}\sigma^2 j^2 + \tfrac{1}{2}(r-\delta)j)\Delta t$$

$$B = 1 - \sigma^2 j^2 \Delta t$$

$$C = (\tfrac{1}{2}\sigma^2 j^2 - \tfrac{1}{2}(r-\delta)j)\Delta t$$

Thus just as with a binomial tree we have a way of calculating the option value at expiry - the known payoff, and we have scheme for calculating the option value for all values of S at the previous time step. Thus we can use the backward scheme and equation (6.4) for V_i^j to calculate the option value all the way back to $t = 0$.

The differences between the binomial and explicit finite difference method are

- the binomial uses two nodes to the explicit finite difference's three.
- You get to choose the specifications of the grid in the finite difference method
- You also need to specify the behaviour on the upper and lower S boundaries.

6.4 Boundary Conditions

If we attempt to use equation (6.4) to calculate V_0^i then we need to have values of V_{-1}^i which we don't have (e.g. for calls). So for V_0^i and V_{jmax}^i we need to use our boundary conditions.

Example 6.5 (Boundary Conditions)**.** Derive the discretised boundary conditions for a European call option.

Solution 6.5.

Probabilistic interpretation

You will see that it is possible to think of the explicit finite difference scheme as a trinomial tree and A, B and C as probabilities. First note that $A + B + C = 1$, second consider what the expected value of S is at time $i\Delta t$:

$$\begin{aligned} E[S_j^i] &= \frac{1}{1 + r\Delta t}(A(S_j^i + \Delta S)) + B(S_j^i) + C(S_j^i - \Delta S)) \\ &= \frac{1}{1 + r\Delta t}(S_j^i(1 + (r - \delta)\Delta t)) \\ &= \frac{1}{1 + r\Delta t}E[S_j^{i+1}] \end{aligned}$$

the expected future value of S, following GBM, under the risk-neutral probability discounted at the risk-free rate. So A, B and C can also be interpreted as risk-neutral probabilities (you can also check that the variance works).

6.5 Stability and Convergence

Unfortunately, the explicit finite difference scheme is occasionally unstable, in that for particular choices of Δt and ΔS, the scheme will not give an option value even close to the correct answer as small errors magnify during the iterative procedure. There is a mathematical method that can be used to determine what the constraint is, however, we can also appeal to our probabilistic explanation to see what the constraint is for the explicit finite difference method.

Example 6.6 (Stability Restrictions)**.** What can we say about A, B and C if they are probabilities?

Solution 6.6.

The stability often severely restricts choice of Δt, ΔS:

- Δt cannot be too small, or else computation will take too long
- then this puts lower bound on size of ΔS

Nonetheless, we have some flexibility. Two common choices:

- choose Δt, ΔS so that $B = 2/3$ (means A, C approx. 1/6)
- choose Δt, ΔS so that $B = 1/3$ (means A,C approx. 1/3)

Assuming that the scheme is stable then we would like to analyse the accuracy of the method. The errors will arise from only approximating the derivatives, in particular, in the explicit finite difference method:

$$\frac{\partial^2 V}{\partial S^2}(S,t) = \frac{V(S+\Delta S,t) - 2V(S,t) + V(S-\Delta S,t)}{(\Delta S)^2} + O((\Delta S)^2)$$

$$\frac{\partial V}{\partial S}(S,t) = \frac{V(S+\Delta S,t) - V(S-\Delta S,t)}{2\Delta S} + O((\Delta S)^2)$$

$$\frac{\partial V}{\partial t}(S,t) = \frac{V(S,t+\Delta t) - V(S,t)}{\Delta t} + O(\Delta t)$$

And so we would expect the error to decrease linearly with the number of time steps (as with the binomial model) and quadratically with the number of steps in S.

One should be careful when assuming that this is how the scheme converges, if we write out the Taylor expansion in full we see that this convergence depends upon all of the derivatives being well behaved (e.g. not infinite). However, we know that in the case of European options, the payoff at expiry is discontinuous leading to an infinite first derivative - and so it seems likely that our approximation may not work as well here. Additionally, in the case of an American option, across the early exercise boundary, the second derivative is infinite, this again will lead to difficulty in the approximations - in particular, the assumption as to the error from finite-difference methods. There are ways of overcoming this as we shall see.

6.6 Extensions

Discrete dividends

It is straightforward to factor in the payment of discrete dividends into the explicit finite difference method. Assume that the dividend is paid at t_d and that there is a node on t_d in our finite difference grid. The dividend for the underlying asset is a function of the price at t_d, $D(S)$. Denote the time just before the dividend as t_d^- and the time just after as t_d^+ Then we use the fact that $V(S,t_d^-) = V(S-D(S),t_d^+)$ So calculate $V(S,t_d^+)$ as usual and then calculate $V(S+D(S),t_d^-)$ by using interpolation. If the grid values of S above and below $S+D(S)$ are $V(S_{up})$ and $V(S_{down})$ then

$$V(S,t_d-) = (1-h)V(S_{down}) + hV(S_{up})$$

where $h = (S+D(S)) - S_{down}$

Intro to a better method...

The problems with the explicit finite difference method are twofold:

- It is has stability issues;
- The convergence is only linear in Δt.

There is a related finite difference method that overcomes both of these problems. Consider approximating the derivatives of a function at $t + \Delta t/2$. Then we can write the following Taylor series expansions:

$$V(S, t + \Delta t) = V(S, t + \Delta t/2) + \tfrac{1}{2}\Delta t \frac{\partial V}{\partial t} + \tfrac{1}{4}(\Delta t)^2 \frac{\partial^2 V}{\partial t^2} + O((\Delta t)^3)$$

$$V(S, t) = V(S, t + \Delta t/2) - \tfrac{1}{2}\Delta t \frac{\partial V}{\partial t} + \tfrac{1}{4}(\Delta t)^2 \frac{\partial^2 V}{\partial t^2} + O((\Delta t)^3)$$

Thus we can estimate the first derivative at $S, t + \Delta t/2$ as

$$\frac{\partial V}{\partial t} = \frac{V(S, t + \Delta t) - V(S, t)}{\Delta t} + O((\Delta t)^2)$$

The S derivatives must also be evaluated at $S, t + \Delta t/2$ to give

$$\frac{\partial V}{\partial S} = \frac{1}{2}\left(\frac{V(S + \Delta S, t) - V(S - \Delta S, t)}{2\Delta S} + \frac{V(S + \Delta S, t + \Delta t) - V(S - \Delta S, t + \Delta t)}{2\Delta S}\right) + O((\Delta S)^2, (\Delta t)^2)$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V(S + \Delta S, t) - 2V(S, t) + V(S + \Delta S, t)}{2(\Delta S)^2} + \frac{V(S + \Delta S, t + \Delta t) - 2V(S, t + \Delta t) + V(S + \Delta S, t + \Delta t)}{2(\Delta S)^2}) + O((\Delta S)^2, (\Delta t)^2)$$

We have introduced the finite-difference method which is a way of solving partial differential equations by estimating the first and second derivatives and then substituting the estimates in the PDE. This scheme was explained for the Black Scholes PDE and in particular we derived the explicit finite difference scheme to solve the European call and put option problems. The convergence of the method is similar to the binomial tree and, in fact, the method can be considered a trinomial tree. Unfortunately, however, the method can be unstable which puts constraints on our grid size. We finished by introducing a method with improved convergence and stability..we will see more on this later....