Lecture 5

Analysis of binomial option pricing

Having introduced how to value European and American options on dividend paying underlying assets we now look at the accuracy of the binomial method. In particular we consider the difference between 'distribution error' and 'non-linearity' error and the difficulty in ensuring monotonic convergence. We extend the analysis to look at the computational effort when valuing an option on two underlying assets and a simple method for reducing the computational effort here.

5.1 Convergence

When analysing convergence we need to consider the error from a numerical scheme, if V_{exact} is the correct option value and V_n is the value from a binomial tree with n steps then:

$$Error_n = V_{exact} - V_n$$

To formally define convergence, there exists a constant, κ, such that for all time steps, n, where c is the order of convergence. This can also be written as

$$Error_n = O(\frac{1}{n^c})$$

As long as $c > 0$ then then V_n will converge to V_{exact}.

Unfortunately there is not a simple proof to show the convergence of the binomial lattice to the correct option price. When considering European options it is simple to look at this empirically because we have an analytic expression for V_{exact} (the Black-Scholes price). By the Central Limit Theorem we also know that the prices will eventually converge as the binomial

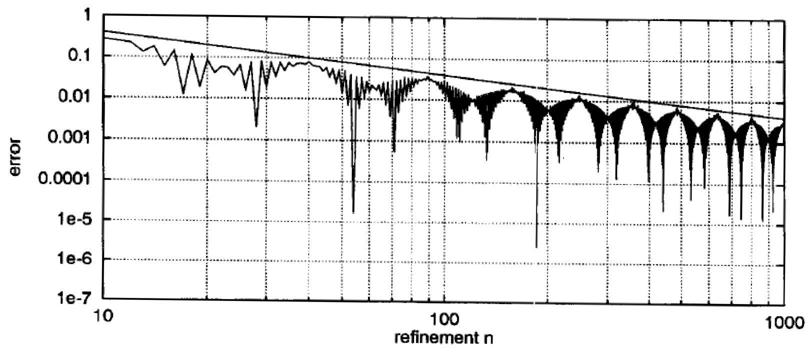

Figure 5.1: Errors from the CRR model reproduced from Leisen and Reimer (1996).

distribution converges to the lognormal distribution. All empirical evidence indicates that for all basic binomial models (CRR, RB etc) $c = 1$, or that V_n converges to the Black Scholes at a rate of $1/n$. So, in general to halve the error you must double the number of time steps. For a rigorous proof see Leisen and Reimer (Applied Mathematical Finance, 1996).

The diagram in figure 5.1 (from Leisen and Reimer, 1996) shows the error from the CRR model relative to $1/n$, where the upper line denotes how the error would reduce with $1/n$ convergence and the sawtooth pattern is the actual error.

We would like this convergence to be monotonic for two reasons.

1. First, we would like to know that as we construct a lattice with more steps we will get closer to the exact answer. This is especially important when we have no analytic value for the exact answer.

2. Second, if the problem is computationally intensive, we can save effort by using extrapolation procedures.

As we see from the graph on the previous figure the convergence to the exact option value looks far from monotonic for the binomial lattice. We would like to investigate why this is the case...

Extrapolation

If convergence to the exact option value is monotonic and at a known rate then there is a simple extrapolation technique. Consider the following equations for lattices with different numbers of time steps

$$V_{exact} = V_n + \frac{\kappa}{n} + o(\frac{1}{n})$$

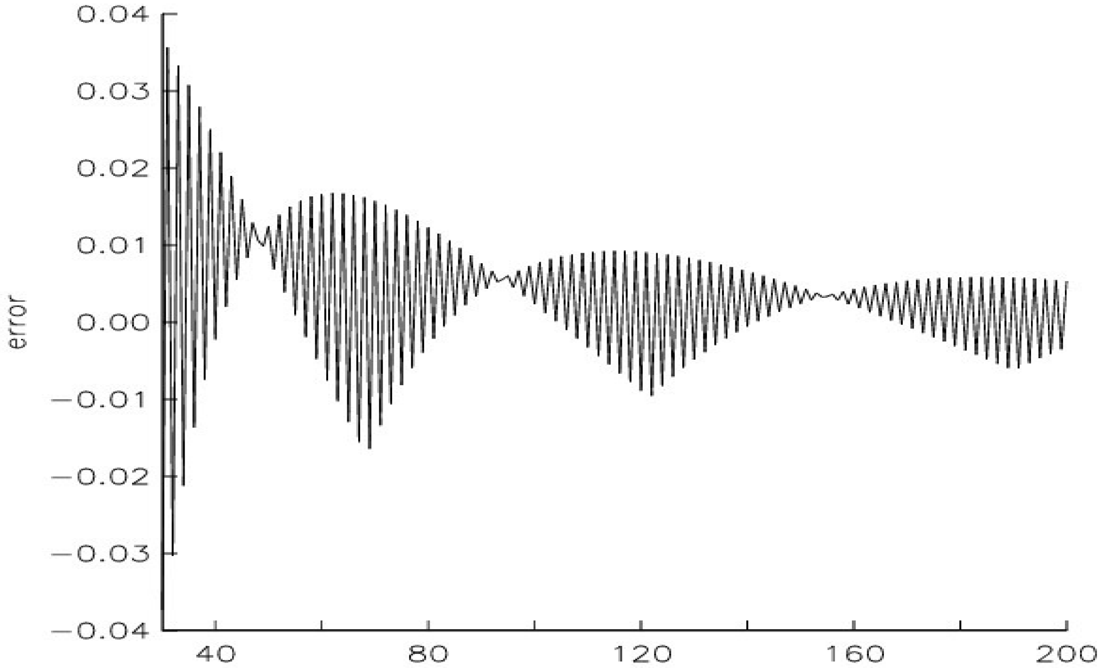

Figure 5.2: Illustration of the sawtooth effect

$$V_{exact} = V_m + \frac{\kappa}{m} + o(\frac{1}{m})$$

We can use these simultaneous equations to determine κ and thus know the first error terms and thus improve the accuracy of the method. The equation becomes

$$V_{exact} = \frac{nV_n - mV_m}{n - m} + o(\frac{1}{m - n})$$

5.2 Errors

Sawtooth Effect

For a European option, when we increase n and plot $Error_n$ against n we see the distinctive shape in figure 5.2. We see two distinct features, the first is a sawtoothing and the second is periodic humps. The sawtoothing is known as the 'odd-even effect' (Omberg, Advances in Futures and Options research, 1987) where as you move from say 25 steps to 26 steps the change in V_n is very large. This happens as the final nodes in the lattice move relative to the exercise price of the option, where there is a discontinuity in the option price. The periodic humps are also a result of this as (unless $S = X$) the position of the nodes relative to the exercise price change as you increase the the number of time steps n. In figure 5.3 an attempt is made to

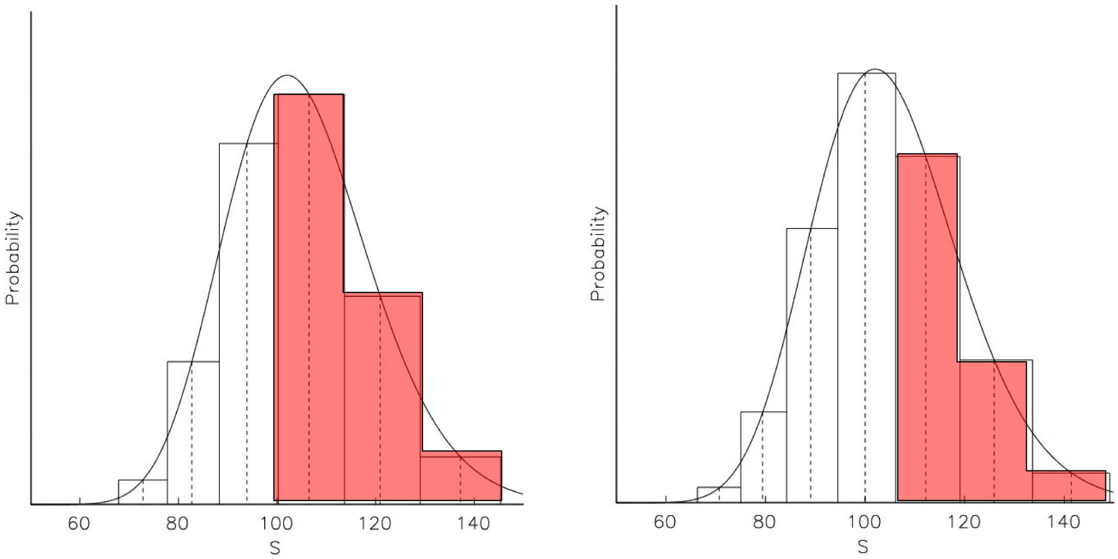

Figure 5.3: Probability density functions for a binomial model with 5 and 6 steps.

graphically explains the odd-even effect. The binomial approximation to the normal is depicted for lattices with 5 and 6 steps. The shading denotes which nodes contribute value to the option price if $X = 100$.

Explanation of periodic ringing

Due to the discontinuity in the option payoff, the location of the final nodes are very important in determining V_n. In the first diagram, the node at 110 contributes an option value of 10 with a large probability and so this lattice overvalues the European option. However, in the second diagram the node at 100, contributes nothing to the option value and so this lattice undervalues the option. The periodic humps can also be demonstrated to be connected to the position of the binomial nodes. If we introduce a measure Λ denoted by

$$\Lambda = \frac{S_k - X}{S_k - S_{k-1}}$$

where S_k is the closest node above the exercise price and S_{k-1} is the node below the exercise price. In figure 5.4 Λ against the error from the binomial lattice (with only even steps to remove the odd-even effect) The dashed lines denote the error and the solid lines the corresponding value of Λ. Only even numbers of steps were considered.

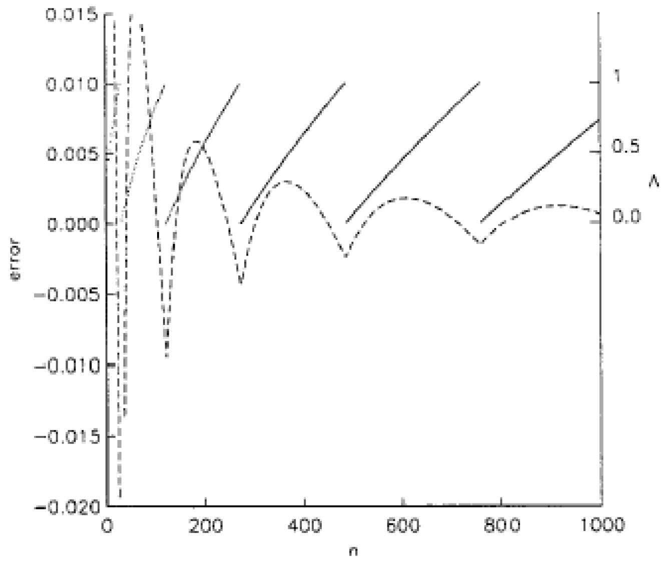

Figure 5.4: The error in the binomial (dashed) versus the distance from the strike Λ (solid)

5.3 Types of Errors

Figlewski (Journal of Financial Economics, 1999) introduced a definition to distinguish between the two types of error that one observes when pricing derivatives using binomial lattices. First there is **'distribution error'** which arises from the binomial distribution only providing an approximation to the lognormal distribution. This is the error that reduces at $1/n$, this can typically be reduced by extrapolation techniques. Second, and more importantly there is **'non-linearity error'** . This arises from not having the nodes in the tree or grid aligned correctly with the features for the option. For example, the strike price in a vanilla European. This can cause serious errors for more exotic options, especially barriers and lookbacks.

Removing non-linearity error

In the case of European options the most elegant way of overcoming non-linearity error is Leisen and Reimer (1996), they use the degree of freedom in selecting q, u and d so that the lattice is centred around the exercise price X, to ensure that the non-linearity error is removed (or remains constant). The choices which do this are as follows where N is the total number of time steps.

$$q = h(d_2)$$

$$u = e^{(r-\delta)\Delta t} q^* / q$$

$$d = \frac{e^{(r-\delta)\Delta t} - qu}{1-q}$$

where

$$d_{1,2} = \frac{\log(S/X) + (r - \delta \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$h(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{4}\exp[-(\frac{x}{N+\frac{1}{3}})^2(N+\frac{1}{6})]}$$

$$q^* = h(d_1)$$

This method, although seemingly complex is very simple to program (compared to Widdicks et al., Journal of Futures markets, 2002) and provides very accurate option values. The convergence is smooth and so is amenable to standard extrapolation techniques. Unfortunately, it was designed specifically to price European options, for which we already have analytic solutions, the real test will come when evaluating American options.

5.4 Exotic options

What about Americans?

The issue of nonlinearity error is more complex in American options as we have to worry about more than simply the discontinuous payoff. At every time step there is also the early exercise boundary (which separates exercise from non-exercise). We do not know where this boundary will be a priori and so naturally the binomial lattice will not be constructed to remove the nonlinearity error from the early exercise boundary. There are many approaches to improving the standard CRR method for valuing American options, two of which are detailed here.

The Leisen and Reimer approach for European options also works well for American options as the largest nonlinearity error (from the discontinuous payoff) has been entirely removed. Thus, this method still provides accurate American option values and is simple to program. An alternative method for pricing American options is provided by Broadie and Detemple (Review of Financial Studies, 1996) who avoid the problem of the discontinuous payoff by using a combination of the Black-Scholes formula for a European option and the CRR binomial lattice. The idea is that between the penultimate timestep and expiry the continuation value of the American option is a European option with time to expiry. Δt. So you can calculate the American option values at $T - \Delta t$ precisely without having any nonlinearity error from the discontinuous payoff.

If we have an n step tree with u, d and q as in CRR the Broadie and Detemple method suggests the following algorithm for pricing an American put option with N time steps and time to expiry T (ERROR):

$$S_{i,j} = S_0 u^j d^{i-j}$$

$$V_{N,j} = \max(X - S_{N,j}, 0)$$

$$V_{N-1,j} = \max(V_{h,N-1,j}, V_{x,N-1,j})$$

$$V_{h,N-1,j} = BS(S_{N-1,j}, (N-1)\Delta t)$$

$$V_{x,N-1,j} = X - S_{N-1,j}$$

$$V_{i,j} = \max(V_{h,i,j}, V_{x,i,j}) \quad \text{for} \quad i < N-1$$

$$V_{h,i,j} = e^{-r\Delta t}(qV_{i+1,j+1} + (1-q)V_{i+1,j}) \quad \text{for} \quad i < N-1$$

$$V_{x,i,j} = X - S_{i,j} \quad \text{for} \quad i < N-1$$

$$BS(S,t) = Xe^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(d_2)$$

$$d_{1,2} = \frac{\log(S/X) + (r - \delta \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

The issue of nonlinearity error can become more pronounced for options with exotic features. A particular example is barrier options. Typically as more and more sources of non-linearity error are introduced it becomes increasingly difficult to adapt the binomial lattice to provide monotonic convergence. Often this is not problematic as due to increasing computing power one can just use a lattice with enough time steps to overcome the problem (such as with American options). However, if the problem has multiple stochastic variables (such as stochastic volatility) or an interest rate derivative with a sophisticated term structure model then nonlinearity error can be a real problem.

More than one underlying asset

There are many derivative pricing problems that require modelling more than one stochastic variable. These could be problems where we consider stochastic volatility or when the payoff from the derivative is a function of two or more underlying assets. We have seen exchange options, but there are also basket options, best of options, chooser options and a whole host of exotic derivatives which require such modelling. Here we focus on one lattice approach to valuing such options when there are two underlying assets. This approach can be generalised to any number of assets (see Kamrad and Ritchken, Management Science, 1991 amongst others).

The model presented here is from Boyle, Evnine and Gibbs (Review of Financial Studies, 1989). They assume that we have two assets both of which follow geometric Brownian motion as before:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dX_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dX_2$$

where

$$X_2 = \rho_2 X_1 + \sqrt{1-\rho^2} X_3$$

To discretise this problem they consider a tree where at each time step the underlying asset prices (S_1, S_2) can both move up or down, giving four possible states as in the table:

Nature of jumps	Probability	Asset prices
Up, up	$p_1 = p_{uu}$	$S_1u_1,\ S_2u_2$
Up, down	$p_2 = p_{ud}$	$S_1u_1,\ S_2d_2$
Down, up	$p_3 = p_{du}$	$S_1d_1,\ S_2u_2$
Down, down	$p_4 = p_{dd}$	$S_1d_1,\ S_2d_2$

In a similar way as for one underlying asset, they ensure that the first two moments of the distributions match and use a CRR analogy which states:

$$u_i d_i = 1$$

$$u_i = e^{\sigma_i \sqrt{\Delta t}}$$

This gives rise to the following probabilities:

$$p_1 = \frac{1}{4}[1 + \rho + \sqrt{\Delta t}(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1} + \frac{r - \frac{1}{2}\sigma_2^2}{\sigma_2})]$$

$$p_2 = \frac{1}{4}[1 - \rho + \sqrt{\Delta t}(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1} - \frac{r - \frac{1}{2}\sigma_2^2}{\sigma_2})]$$

$$p_3 = \frac{1}{4}[1 - \rho + \sqrt{\Delta t}(-\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1} + \frac{r - \frac{1}{2}\sigma_2^2}{\sigma_2})]$$

$$p_4 = \frac{1}{4}[1 + \rho + \sqrt{\Delta t}(-\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1} - \frac{r - \frac{1}{2}\sigma_2^2}{\sigma_2})]$$

5.5 Computational Effort

In the binomial lattice for one underlying asset at each time step i there are $i + 1$ nodes giving $(N + 1)(N + 2)/2$ total calculations in an N-step lattice. As we move to the two underlying model then each step has $(i + 1)^2$ nodes giving $(N + 1)^2(N + 2)^2/4$ total calculations, which is the square of the effort in the one underlying case. As you introduce k underlying assets the the total number of calculations grows exponentially to $(N + 1)^k(N + 2)^k/2^k$ which can become a very large number. Typically due to memory constraints it is difficult to get reasonable accuracy with more then 5 underlying assets or sources of uncertainty...

'Work'

So as to compare the strengths of different numerical methods Broadie and Detemple (Management Science, 2004) introduce the idea of representing the convergence as a function of work which is the computational effort required. Thus for a lattice with N time steps and d underlying assets the work w is approximately N^{d+1} and the convergence is at the rate of $1/N$ and so the convergence can be seen as $O(w^{-1/d+1})$. With Monte-Carlo methods this is $(O(w^{-1/2}))$ and finite-difference methods $(O(w^{-2/d+1}))$

Curtailed range

For most options (especially American options) in more than one underlying asset a simple way of reducing the computational effort is simply to ignore the vast majority of lattice calculations. In their curtailed range method Andricopoulos et al., (Journal of Derivatives, 2004) showed that for options on just one underlying with 1000 steps, the time saving was 87%, for options on three underlying assets with 100 steps the time saving was 91%. The idea is that in large lattices many of the calculations are superfluous as they represent scenarios where the underlying asset price has moved in excess of ten standard deviations and so contribute nothing to the value of the option.

Summary

We have analysed the binomial pricing model in detail, in general it converges at the rate of $1/N$ where N is the number of time-steps in the tree. However, this convergence is often non-monotonic due to nonlinearity error caused by discontinuities in the option price. This can be illustrated by considering the discontinuous payoff from a European call or put option. There are methods of overcoming this, and it is particularly important for American options where there is no analytic solution. Finally, we analysed how to construct a lattice for more than one underlying asset and how this effects the computational effort or work.