# Lecture 4

# The binomial model

A fundamental theorem of finance (in discrete time), also commonly known as *The fundamental theorem of asset pricing*, or *The fundamental theorem of arbitrage pricing* or *The fundamental theorem of arbitrage-free pricing*: if there are no arbitrage opportunities and markets are complete (i.e. all assets are replicable) then there exists a unique, risk-neutral, pricing measure. As such we can write the value of any asset, in particular an option at time $t$, $V_t$, as

$$V_0 = \frac{1}{1+R} E^Q[V_T]$$

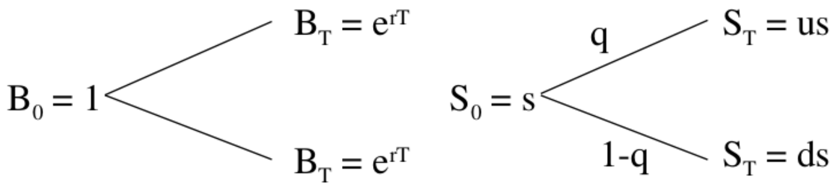
where $R$ is the risk-free interest rate over time $T$. We can also write this in continuously compounded terms as

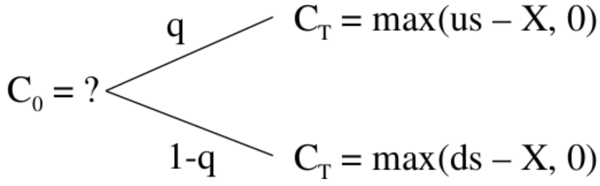
$$V_0 = e^{-rT} E^Q[V_T]$$

We can apply the same argument from time period to time period and so it is possible to have binomial trees with multiple time steps to simulate the movement of the underlying asset more accurately. As we have more steps in our tree we essentially have a binomial distribution with more and more possible outcomes which should eventually approximate to the continuous, lognormal distribution we saw in our continuous time pricing models such as Black-Scholes.

## 4.1 Basic binomial Trees

If we have a three asset world with a Bond, $B_t$, a Stock, $S_t$ and a call option $C_t$, where interest rates are continuously compounded and the risk neutral probability of the up and down states occurring are $q$ and $(1-q)$ then we have




### 4.1.1 Determining $q$, $u$ and $d$.

As the probabilities are risk neutral we require that the expected return on the stock is the same as that of the risk-free bond, thus

$$qsu + (1-q)sd = se^{rT}$$

$$qu + (1-q)d = e^{rT}$$

We would also like to match the variance of our returns to that from the data. From our continuous model we know that under the risk-neutral model

$$dS = rSdt + \sigma SdX$$

and we can solve this SDE to obtain

$$S_T = s\exp[(r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\phi]$$

Taking expectations we have for the continuous case:

$$E[S_T] = s\exp(rT)$$

$$E[(S_T)^2] = s^2\exp[(2r+\sigma^2)T]$$

and in the binomial case

$$E[S_T] = s(qu + (1-q)d)$$

$$E[S_T^2] = s^2(qu^2 + (1-q)d^2)$$

thus

$$e^{rT} = qu + (1-q)d$$

$$e^{(2r+\sigma^2)T} = qu^2 + (1-q)d^2$$

However, this still leaves us with one degree of freedom to determine all of $q$, $u$ and $d$ since there are 3 unknowns and only 2 equations.

The two most popular models for binomial pricing are Cox, Ross and Rubinstein (1979) (CRR for short) whose extra degree of freedom is to set

$$ud = 1$$

thus

$$u = e^{\sigma\sqrt{T}}, \quad d = e^{-\sigma\sqrt{T}}, \quad q = \frac{e^{rT} - d}{u - d}$$

The other is Rendleman and Bartter (1979) who choose:

$$q = \frac{1}{2}$$

and so

$$u = e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}}, \quad d = e^{(r-\frac{1}{2}\sigma^2)T-\sigma\sqrt{T}}.$$

### 4.1.2 Constructing the tree

So now we have expressions for $u$ and $d$ which will ensure that the binomial tree will approximate the continuous lognormal distribution which arises from the geometric Brownian motion assumptions. There are other ways to construct the tree if the underlying asset follows a different stochastic process but we do not consider those here.

Now we turn our attention to valuing a European call option before considering natural extensions. The current value of the underlying asset is $S_0$, the time to expiry is $T$, we have $N$ time steps, the continuously compounded risk-free rate is $r$, the volatility of the underlying asset is $\sigma$ and the exercise price of the option is $X$. The size of the time-step $\Delta t = T/N$

If we denote the value of the underlying asset after timestep $i$ and upstate $j$ by $S_{ij}$ and the option price by $V_{ij}$ then we have that:

$$\begin{aligned} S_{ij} &= S_0 u^j d^{i-j} \\ V_{Nj} &= \max(S_0 u^j d^{N-j} - X, 0) \\ V_{ij} &= e^{-r\Delta t}(qV_{i+1,j+1} + (1-q)V_{i+1,j}) \quad \text{for} \quad i < N \end{aligned}$$

where $q$, $u$ and $d$ are selected according to your preferred model (CRR or alternative).

**Example 4.1** (Psuedo Code)**.** Outline the easiest way to code up a binomial tree. . .

**Solution 4.1.**

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### 4.1.3 Example

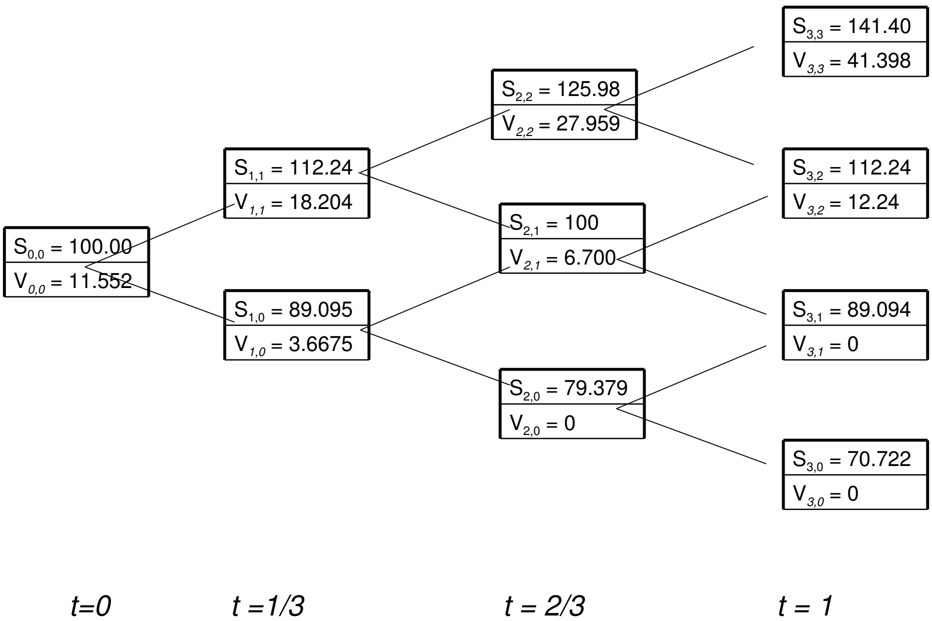
Consider a European call option where $S_0 = 100$, $X = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.2$. Choosing the CRR tree we have

$$u = e^{\sigma\sqrt{\Delta t}} = 1.1224$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 0.8909$$

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.5584$$

Next we show the calculation of the European call option price using 3 time steps, where we end up with an option value of \$11.55.




## 4.2 American option valuation

American options are call (put) options where it is possible to exercise early at time $t$ to receive $S_t - X$ ($(X - S_t)$ for a put) We will consider the American option pricing problem from different perspectives for the three types of numerical methods. As noted already, it can be problematic to solve using forward induction methods such as Monte Carlo techniques. The problem is a free boundary or optimal stopping problem where the current option value $V_t$ is given by

$$V_t = \max_{\tau} E_t^Q[e^{-r(\tau-t)} Payoff(S_\tau)]$$

where $\tau$ denotes the continuum of possible stopping times. This representation is not particularly useful when attempting to value the option.

Obviously any rational investor would only exercise if the value from exercising is greater than the value from not exercising, i.e. holding the option for one more period. The nice thing about binomial lattices is that as we calculate backward we already know the value of holding the option until the next period (the continuation value) and we know the early exercise value (the payoff from the option) and so it is straightforward to adapt our European option pricing model to deal with American options.

**Example 4.2.** The Option to Hold How do we use variational inequalities to adapt our binomial tree to work with American options?

**Solution 4.2.**

Thus at each node in the tree we need to compare two values, the continuation (or hold) value, $V_{hij}$ and the early exercise value $V_{xij}$ to determine $V_{ij}$ where

$$V_{ij} = Payoff(S_{Nj}) \quad \text{for} \quad t = T$$

$$V_{hij} = e^{-r\Delta t}(qV_{i+1,j+1} + (1-q)V_{i+1,j}) \quad \text{for} \quad t < T$$

$$V_{xij} = Payoff(S_{ij}) \quad \text{for} \quad t < T$$

$$V_{ij} = \max(V_{hij}, V_{xij}) \quad \text{for} \quad t < T$$

and Payoff is the appropriate payoff function for each option.

### 4.2.1 Example

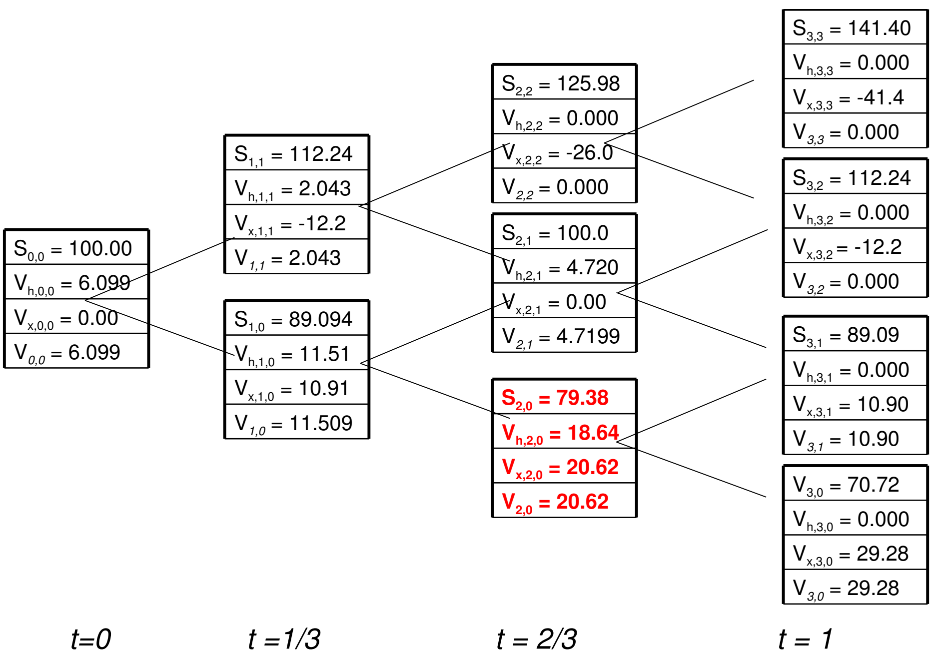
Consider an American put option where $S_0 = 100$, $X = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.2$. Choosing the CRR tree we have

$$u = e^{\sigma\sqrt{\Delta t}} = 1.1224$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 0.8909$$

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.5584$$

Next we show the calculation of the American put option price where we end up with an option value of \$6.099. The nodes in red denote that the holder of the option exercised early.



For American call options with no dividends it is never optimal to exercise early. From the lattice we can determine the early exercise region, the values of $S$ and $t$ for which you would exercise and the early exercise boundary which separates the exercise and non-exercise regions. Technically what we have evaluated here is a Bermudan option, which is an American option that can only be exercised on certain specified dates. However, as we have more and more observation dates then this value will approach the American option price.

**Example 4.3.** The Early Exercise Region How do we know when to exercise the option early?

**Solution 4.3.**

## 4.3 Cash Payments and Dividends

**Example 4.4** (Cash Payments)**.** How do we include cash payments made to the holder of the financial contract we are valuing into the model?

**Solution 4.4.**

**Example 4.5** (Dividend Payments)**.** How do we include continuous (proportional) dividend payments on the underlying asset into the model?

**Solution 4.5.**

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### 4.3.1 Lattices with continuous dividends

Now given this slightly different calculation we have new values of $u$ and $d$ where

$$E[S_T] = s\exp[(r-\delta)T]$$

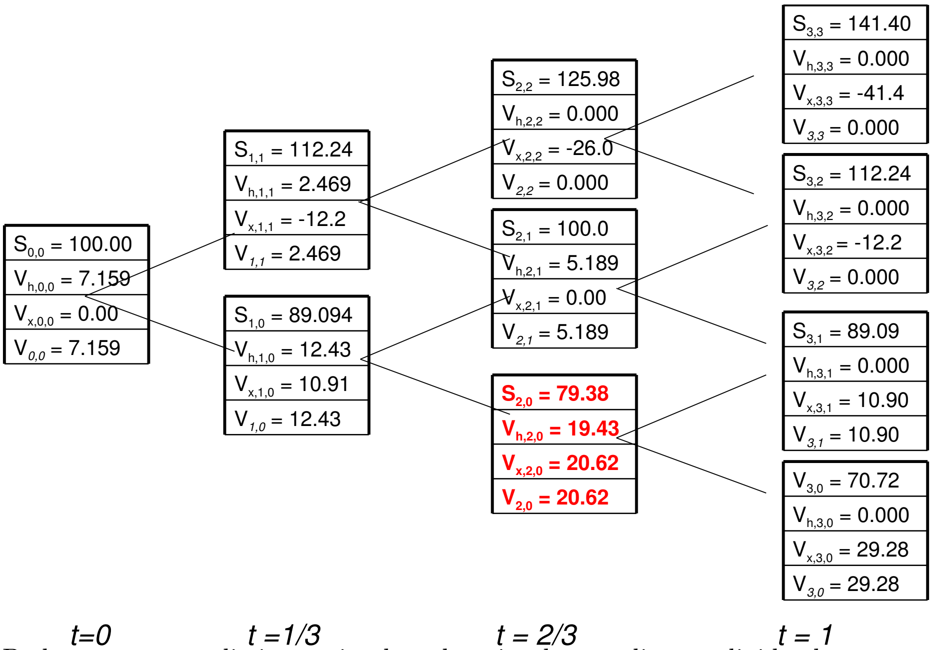
$$E[(S_T)^2] = s^2\exp[(2(r-\delta)+\sigma^2)T]$$

and again you have another choice of a degree of freedom and the CRR approach gives:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{e^{(r-\delta)\Delta t} - d}{u-d}$$

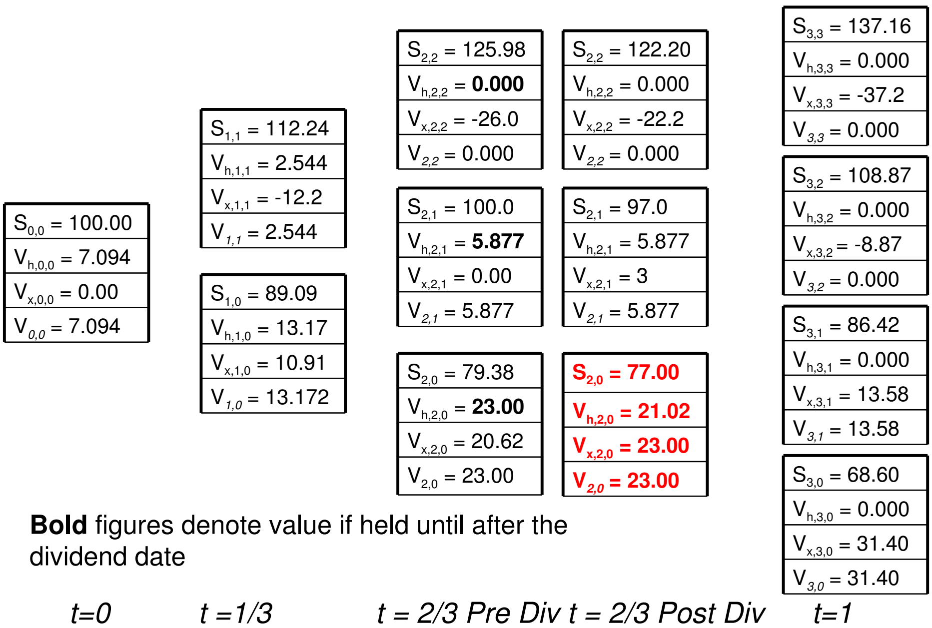
in a tree with step size $\Delta t$

Thus if we consider the American put option only now when $\delta = 0.03$ we see that the theoretical price is now \$7.32




Perhaps a more realistic case is when there is a known discrete dividend payment at a certain point in time. In our example, imagine there is a known dividend, payable after 2/3 of a year which is 3% of the share price. Here the fundamental theorem will hold from period to period and our values of $u$, $d$ and $q$ will remain the same as for the no dividend case but at $t = 2/3$, $S_{2j} \to 0.97 \times S_{2j}$.

This is depicted in the worked example next:



### 4.3.2 Cash dividends

Non-proportional cash dividends can be problematic as this leads to a non-recombining tree as if the dividend is \$2 regardless of the share price then an up move followed by a down move will not be the same as a down move followed by an up move. This leads to a large increase in the computational effort, which will increase exponentially after the cash dividend payment date.

There is an adjustment for European options but this is not of great practical use as Black-Scholes can be used quite simply in the European case. In American option cases the simplest approach is to use interpolation, although this is more naturally done in a finite difference setting as we will see later.

## 4.4 Overview

We have developed a multistep binomial lattice which will approximate the value of a European or American call option when the underlying asset pays out dividends. The construction comes from an extension to the fundamental theorem of finance and you have a choice of parameters which are typically chosen to fit the binomial distribution to the Black-Scholes lognormal distribution. The most useful outcome is the ability to price American options easily.