1 Convergence and Errors

- Convergence
- Sawtooth Effect
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2. Exotic Options
   - American Options
   - Computational Effort
1. **Convergence and Errors**
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2. **Exotic Options**
   - American Options
   - Computational Effort

3. **Summary**
Developed a multistep binomial lattice which will approximate the value of a European option

Extended the method to allow:
- underlying assets paying out dividends,
- Early exercise.

There is a degree of freedom in choice of parameters when fitting the binomial distribution to the Black-Scholes lognormal distribution.
Developed a multistep binomial lattice which will approximate the value of a European option

- Extended the method to allow:
  - underlying assets paying out dividends,
  - Early exercise.

- There is a degree of freedom in choice of parameters when fitting the binomial distribution to the Black-Scholes lognormal distribution.
  - $ud = 1$ for the CRR model.
  - $q = \frac{1}{2}$ for the Rendleman and Bartter model.

- It is easy to incorporate early exercise.
Investigate the accuracy of the binomial method.

Define two types of error arising:
- ‘distribution error’;
- ‘non-linearity error’.

How to ensure monotonic convergence, to enable extrapolation.
Investigate the accuracy of the binomial method.
Define two types of error arising:
- ‘distribution error’;
- ‘non-linearity error’.
How to ensure monotonic convergence, to enable extrapolation.
Analyse computational effort when valuing an option
Simple method for reducing the computational effort.
When analysing convergence we need to consider the error from a numerical scheme, if $V_{exact}$ is the correct option value and $V_n$ is the value from a binomial tree with $n$ steps then:

$$\text{Error}_n = V_{exact} - V_n$$
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$$Error_n = V_{exact} - V_n$$

**Definition:**

If for all time steps, $n$, where $c$ is the order of convergence, we can write

$$Error_n = O\left(\frac{1}{n^c}\right)$$

then as long as $c > 0$, $V_n$ will converge to $V_{exact}$. 
Empirical Evidence of Convergence

The following slide shows the error from the CRR model relative to $1/n$, where the upper line denotes how the error would reduce with $1/n$ convergence and the sawtooth pattern is the actual error.

We would like this convergence to be monotonic because:

1. Want an increase in steps to give more accuracy,
2. Be able to use extrapolation procedures.

The convergence of binomial lattices is far from monotonic.

Why this is the case?
Empirical evidence

- The below diagram (from Leisen and Reimer, 1996) shows the error from the CRR model relative to $1/n$, where the upper line denotes how the error would reduce with $1/n$ convergence and the sawtooth pattern is the actual error.

![Error Diagram](image)

**Fig. 4.** Graphical representation and examination of the error bound; x-axis and y-axis with log-scale; example with CRR-model and the following selection of parameters: $S = 100$, $K = 110$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $n = 10, \ldots, 1000$. 
Sawtooth

- For a European option, when we increase $n$ and plot $\text{Error}_n$ against $n$ we see the following shape:
We see two distinct features, the first is a sawtothing and the second is periodic humps.

The sawtothing is known as the ‘odd-even effect’ where as you move from say 25 steps to 26 steps the change in $V_n$ is very large.

Caused by nodes in the lattice moving relative to the exercise price of the option

The following slide explains the odd-even effect:

The binomial approximation to the normal is depicted for lattices with 5 and 6 steps. The shading denotes which nodes contribute value to the option price if $X = 100$. 
Explanation of odd-even effect

- The binomial approximation to the normal is depicted here for lattices with 5 and 6 steps. The red shading denotes which nodes contribute value to the option price if $K = 100$. 

![Graph showing odd-even effect](image-url)
Explanation of Periodic Ringing

- Due to the discontinuity in the option payoff, location of the final nodes is important.
- Periodic humps are connected to the position of the binomial nodes.
- Let \( \Lambda \) be denoted by

\[
\Lambda = \frac{S_k - X}{S_k - S_{k-1}}
\]

where \( S_k \) is the closest node above the exercise price and \( S_{k-1} \) below.
- The next slide plots \( \Lambda \) against the error from the binomial lattice.
Explanation of periodic ringing

- The dashed lines here denote the error and the solid lines the corresponding value of $\Lambda$. Only even numbers of steps were considered.
Types of Error

- We can distinguish between the two types of error
  - ‘Distribution error’
    which arises from the binomial approximation to the lognormal distribution.
  - ‘Non-linearity error’
    which arises from not having the nodes in the tree or grid aligned correctly with the features for the option.
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- ‘Distribution error’ which arises from the binomial approximation to the lognormal distribution.
- ‘Non-linearity error’ which arises from not having the nodes in the tree or grid aligned correctly with the features for the option.

Non-linearity error may be larger than the distribution error.

For example, the strike price in a vanilla European.

This can cause serious errors for more exotic options, especially barriers and lookbacks.
REMving Non-Linearity Error

The Leisen and Reimer approach

- Select $q$, $u$ and $d$ so that we centre around the strike
- Ensures that the non-linearity error is removed.
- The choices which do this are as follows where $N$ is the total number of time steps.

$q = h(d_2)$

$u = e^{(r-\delta)\Delta t} q^*/q$

$d = \frac{e^{(r-\delta)\Delta t} - qu}{1 - q}$

where (next slide)
The convergence is smooth and second order,
Removing non-linearity error

\[ d_{1,2} = \frac{\log(S/X) + (r - \delta \pm \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}} \]

\[ h(x) = \frac{1}{2} + \frac{x}{|x|} \sqrt{\frac{1}{4} - \frac{1}{4} \exp\left[-\left(\frac{x}{N + \frac{1}{3}}\right)^2(N + \frac{1}{6})\right]} \]

\[ q^* = h(d_1) \]

- The convergence is smooth and second order,
- but only works on European options.
- What to do with exotic options?
Extrapolation is a powerful technique to improve the accuracy of your solutions

Convergence **MUST** be monotonic and at a known rate.

Then consider the following equations for lattices with different numbers of time steps

\[
V_{\text{exact}} = V_n + \frac{\kappa}{n} + o\left(\frac{1}{n}\right)
\]

\[
V_{\text{exact}} = V_m + \frac{\kappa}{m} + o\left(\frac{1}{m}\right).
\]
We can use these simultaneous equations to determine $\kappa$ and eliminate errors

$$V_{\text{exact}} = \frac{nV_n - mV_m}{n - m} + o\left(\frac{1}{m - n}\right)$$

Then using say $N = 100$ and $N = 200$ we can generate an accurate estimate for $V_{\text{exact}}$

Convergence of the extrapolated method is at least $c + 1$ (so 2nd order for the Binomial method).
Nonlinearity and the American

- Nonlinearity error is more complex in American options.
- At every time step there is also the early exercise boundary introducing nonlinearity error.
- We do not know where this boundary will be a priori.
- So the binomial lattice cannot be constructed to remove the nonlinearity error from the early exercise boundary.
Nonlinearity and the American

- Nonlinearity error is more complex in American options.
- At every time step there is also the early exercise boundary introducing nonlinearity error.
- We do not know where this boundary will be a priori.
- So the binomial lattice cannot be constructed to remove the nonlinearity error from the early exercise boundary.
- There are many approaches to improving the standard CRR method for valuing American options.
- The Leisen and Reimer approach for European options also works well for American options.
- As the largest nonlinearity error (from the discontinuous payoff) has been entirely removed.
Use a combination of the Black-Scholes formula for a European option and the CRR binomial lattice.

The idea is that between the penultimate timestep and expiry the continuation value of the American option is a European option with time to expiry $\Delta t$.

So you can calculate the American option values at $T - \Delta t$ precisely without having any nonlinearity error from the discontinuous payoff.
The issue of nonlinearity error can become more pronounced for options with exotic features.

A particular example is barrier options, since the discontinuity at payoff in value not just derivative.

Can just use a lattice with enough time steps to overcome the problem.

However, if the problem has multiple stochastic variables (such as stochastic volatility) or an interest rate derivative with a sophisticated term structure model then nonlinearity error can be a real problem.
In the binomial lattice for one underlying asset at each time step $i$ there are $i + 1$ nodes giving $(N + 1)(N + 2)/2$ total calculations in an $N$-step lattice.

As we move to the two underlying model then each step has $(i + 1)^2$ nodes giving $(N + 1)^2(N + 2)^2/4$ total calculations, which is the square of the effort in the one underlying case.

As you introduce $k$ underlying assets the the total number of calculations grows exponentially to $(N + 1)^k(N + 2)^k/2^k$ which can become a very large number.

Typically due to memory constraints it is difficult to get reasonable accuracy with more then 5 underlying assets or sources of uncertainty...
So as to compare the strengths of different numerical methods Broadie and Detemple (Management Science, 2004) introduce the idea of representing the convergence as a function of work which is the computational effort required.

Thus for a lattice with $N$ time steps and $d$ underlying assets the work $w$ is approximately $N^{d+1}$ and the convergence is at the rate of $1/N$ and so the convergence can be seen as $O(w^{-1/(d+1)})$.

With Monte-Carlo methods this is $O(w^{-1/2})$ and finite-difference methods $O(w^{-2/(d+1)})$.
CURTAILED RANGE

For most options (especially American options) in more than one underlying asset a simple way of reducing the computational effort is simply to ignore the vast majority of lattice calculations.

In their curtailed range method Andricopoulos et al., (Journal of Derivatives, 2004) showed that for options on just one underlying with 1000 steps, the time saving was 87%, for options on three underlying assets with 100 steps the time saving was 91%.

The idea is that in large lattices many of the calculations are superfluous as they represent scenarios where the underlying asset price has moved in excess of ten standard deviations and so contribute nothing to the value of the option.
We have analysed the binomial pricing model in detail, in general it converges at the rate of $1/N$ where $N$ is the number of time-steps in the tree.

However, this convergence is often non-monotonic due to nonlinearity error caused by discontinuities in the option price. This can be illustrated by considering the discontinuous payoff from a European call or put option.

There are methods of overcoming this, and it is particularly important for American options where there is no analytic solution.

Finally, we analysed how to construct a lattice for more than one underlying asset and how this effects the computational effort or work.