MATH39032
Mathematical Modelling of Finance
Worksheet 8

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In the following question tick **all** options that apply.

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1. For an American put option P and American call option C, both with the same strike price X written on an asset that doesn't pay dividends, the following (in)equalities hold:-

   ◯ $C - P = S - Xe^{-r(T-t)}$

   ◯ $C - P \leq S - Xe^{-r(T-t)}$

   ◯ $C \geq S - Xe^{-r(T-t)}$

   ◯ $P \leq Xe^{-r(T-t)}$

   ◯ $C - P \geq S - X$

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2. An American put option P is a put option which can be exercised at anytime to receive X – where X is the exercise price. A perpetual American put option is one which has no expiry date, as such its value is described by the equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0.$$

   Given a solution of the form

$$P(S) = AS^{\alpha_1} + BS^{\alpha_2}$$

   derive α_1 and α_2.

   ◯ $\alpha_1 = 1$ and $\alpha_2 = -1$

   ◯ $\alpha_1 = 1 + \frac{\sqrt{\sigma^2 - r}}{\sigma^2}$ and $\alpha_2 = 1 - \frac{\sqrt{\sigma^2 - r}}{\sigma^2}$

   ◯ $\alpha_1 = 1$ and $\alpha_2 = -2r/\sigma^2$

   ◯ $\alpha_1 = 1 + \sqrt{1 + r/\sigma^2}$ and $\alpha_2 = 1 - \sqrt{1 + r/\sigma^2}$

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3. State the boundary conditions for the perpetual American put option $P(S)$:-

- ◯ $P(S_f) = X - S_f$, $P(S) \to 0$ as $S \to \infty$
- ◯ $P \to X$ as $S \to 0$, $P(S) \to 0$ as $S \to \infty$
- ◯ $P(S_f) = X - S_f$, $\partial P/\partial S(S_f) = -1$, $P(S) \to 0$ as $S \to \infty$

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4. Apply the boundary conditions for the perpetual American put option $P(S)$ to derive the correct solution in terms of r, σ, and X:-

- ◯ $P(S) = A^{2r/\sigma^2} S^{-2r/\sigma^2}$
- ◯ $P(S) = \frac{\sigma^2}{2r}\left[\frac{X}{1+\sigma^2/(2r)}\right]^{1+2r/\sigma^2} S^{-2r/\sigma^2}$
- ◯ $P(S) = (1+2r/\sigma^2)\left[\frac{(2r/\sigma^2)X}{2r/\sigma^2-1}\right]^{-2r/\sigma^2} S^{-2r/\sigma^2}$

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5. Given the values $X = 50$, $r = 0.04$, $\sigma = 0.3$, calculate the location of the free boundary for the perpetual American put option $P(S)$:-

- ◯ 13.54
- ◯ 14.31
- ◯ 438.50
- ◯ 23.53

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6. An American call option C is a call option which can be exercised at anytime to receive X – where X is the exercise price. A perpetual American call option is one which has no expiry date, as such its value is described by the equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-D)S\frac{\partial C}{\partial S} - rC = 0$$

when the underlying asset has a continuous dividend yield D. Given a solution of the form

$$C(S) = AS^{\alpha_1} + BS^{\alpha_2}$$

derive α_1 and α_2 when $D = r$.

- ◯ $\alpha_1 = 1$ and $\alpha_2 = -1$
- ◯ $\alpha_1 = \frac{1}{2} + \frac{\sqrt{\sigma^2 - r}}{2\sigma^2}$ and $\alpha_2 = \frac{1}{2} - \frac{\sqrt{\sigma^2 - r}}{2\sigma^2}$
- ◯ $\alpha_1 = 1$ and $\alpha_2 = -1 + 2(r-D)/\sigma^2$
- ◯ $\alpha_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8r/\sigma^2}$ and $\alpha_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + 8r/\sigma^2}$
