# MATH39032
# Mathematical Modelling of Finance
# Worksheet 6

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In the following question tick **all** options that apply.

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1. Suppose that $u(x,\tau)$ satisfies the following initial value problem on a semi-infinite interval:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad \tau > 0,$$

with boundary conditions

$$u(x,0) = u_0(x), \quad x > 0, \quad u(0,\tau) = 0, \quad \tau > 0.$$

A general solution to the problem on an infinite interval:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2},$$

with initial condition

$$v(x,0) = v_0(x),$$

is given by

$$v(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} v_0(s) e^{-(x-s)^2/4\tau} ds.$$

How do we find the solution for $u$?

- ◯ Set the initial condition $v_0$ in terms of $u_0$ such that all boundary conditions are satisfied.
- ◯ Set the initial condition $v_0$ in terms of $u_0$ such that initial conditions are satisfied.

Consider a problem where

$$u(x,0) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{if} \quad x > 1 \end{cases},$$

and

$$u(0,\tau) = 0, \quad \tau > 0.$$

The solutions is (with $N$ standard cumulative normal distribution)

- ◯ $u(x,\tau) = \left[ N\left(\frac{x+1}{\sqrt{2\tau}}\right) - N\left(\frac{x-1}{\sqrt{2\tau}}\right) \right]$
- ◯ $u(x,\tau) = 2\sqrt{\tau\pi}\left[ N(x+1) - N(x-1) \right]$
- ◯ $u(x,\tau) = \left[ 2N\left(\frac{x}{\sqrt{2\tau}}\right) - N\left(\frac{x+1}{\sqrt{2\tau}}\right) - N\left(\frac{x-1}{\sqrt{2\tau}}\right) \right]$

2. Consider the problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + 1, \qquad x > 0, \quad \tau > 0,$$

with

$$u(x,0) = 0, \quad x > 0, \quad u(0,\tau) = \tau, \quad \tau > 0.$$

Search for a solution of the form $u(x,\tau) = \tau^\alpha U(\xi)$ where $\xi = x/\sqrt{\tau}$. Determine an appropriate $\alpha$ and find the ODE and boundary conditions for $U$.

- ○ $U - \frac{1}{2}\xi U' = U'' + x$ with $U(0) = U(\infty) = 0$.
- ○ $U - \frac{1}{2}\xi U' = U'' + 1$ with $U(0) = 1$ and $U(\infty) = 0$.
- ○ $-\frac{1}{2}\xi U' = U'' + x$ with $U(0) = 0$ and $U(\infty) = 0$.

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3. Using your ODE from question 2, try a solution of the form

$$U(\xi) = \left(\xi^2 + 2\right) G(\xi)$$

to determine an ODE for $G'$.

- ○ $$\left(\xi^3 + 6\xi\right) G'' + \left(\xi^4 + 12\xi^2 + 12\right) G' + 12G + \xi = 0$$
- ○ $$\left(\xi^2 + 2\right) G'' + \left(\frac{1}{2}\xi^3 + 5\xi\right) G' + 1 = 0$$
- ○ $$\left(\xi^2 + 2\right) G'' + \left(\xi^4 + 5\xi^2 + 5\right) G' + 1 = 0$$

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4. What are the boundary conditions on $G$?

- ○ $$G(0) = 0 \qquad \text{and } G(\infty) = 0$$
- ○ $$G(0) = \frac{1}{2} \qquad \text{and } G \sim O\left(\frac{1}{\xi^3}\right) \text{ as } \xi \to \infty$$
- ○ $$G(0) = \frac{1}{2} \qquad \text{and } G \sim O\left(\frac{1}{\xi}\right) \text{ as } \xi \to \infty$$