MATH39032
Mathematical Modelling of Finance
Worksheet 5

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2020

In the following question tick **all** options that apply.

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1. An option, $V(S,t)$, has the payoff

$$V(S,T) = \max(X_1 - S, 0, S - X_2)$$

and $X_2 > X_1$. Now consider a transformation, such that $S = X_2 e^x$, $V = X_2 v$, and $\zeta = \frac{X_1}{X_2}$. What is the terminal condition for $v(x,T)$?

- ◯ $v(x,T) = \max(\zeta - e^x, 0, e^x - 1)$
- ◯ $v(x,T) = \max(1 - e^x, 0, e^x - 1)$
- ◯ $v(x,T) = \max(1 - e^x, 0, e^x - \zeta)$

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2. Consider the Black-Scholes equation for an option, $V(S,t)$, in the usual notation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Set $S = Xe^{x-f(t)}$, $V = Xv$, and write down the transformed equation.

- ◯
$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial S} - (r + f(t))v = 0$$

- ◯
$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left(f'(t) + r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial S} - rv = 0$$

- ◯
$$(1 + f'(t))\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial S} - rv = 0$$

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3. Consider the problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

with

$$u(x,0) = \begin{cases} 0 & \text{for } x \le 0 \\ x & \text{for } x > 0 \end{cases}.$$

Search for a solution of the form $u(x,\tau) = \tau^{\frac{1}{2}} U(\xi)$ where $\xi = x/\sqrt{\tau}$. Which of these are the correct equation and boundary conditions?

○

$$2U'' + \xi U' - U = 0$$

with $U \to \xi$ as $\xi \to \infty$, $U \to 0$ as $\xi \to -\infty$

○

$$2U'' + \xi U' = 0$$

with $U \to 1$ as $\xi \to \infty$, $U \to 0$ as $\xi \to -\infty$

○

$$2\xi U'' + (4 + \xi^2)U' = 0$$

with $U \to 1$ as $\xi \to \infty$, $U \to 0$ as $\xi \to -\infty$

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4. Consider again the problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

with

$$u(x,0) = \begin{cases} 0 & \text{for } x \le 0 \\ x & \text{for } x > 0 \end{cases}.$$

Using your ODE for $U(\xi)$ from question 3, try seeking a solution of the form $U(\xi) = \xi U_1(\xi)$ and solve the resulting equation.

○

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} e^{-\xi^2/4} + x \int_{-\infty}^{x/\sqrt{\tau}} \frac{1}{s^2} e^{-s^2/4} ds$$

○

$$u(x,\tau) = \sqrt{\frac{\tau}{\pi}} e^{-x^2/(4\tau)} + xN\left(\frac{x}{\sqrt{2\tau}}\right)$$

where N is standard cumulative normal distribution.

○

$$u(x,\tau) = x \int_{-\infty}^{x/\sqrt{\tau}} \frac{1}{s^2} e^{-s^2/4} ds$$

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