MATH39032 (Mathematical modelling of finance) Solutions 8

1. Setting $w = \beta^{\frac{1}{2}}$, $u = \eta - \gamma r$, leads to the PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\beta\frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r - \lambda\beta^{\frac{1}{2}})\frac{\partial V}{\partial r} - rV = 0$$

Setting $V = Ze^{A(t;T) - rB(t;T)}$ leads to

$$A_t - rB_t + \tfrac{1}{2}\beta B^2 - (\eta - \gamma r - \lambda\beta^{\frac{1}{2}})B - r = 0$$

The $O(r)$ and $O(r^0)$ terms can be considered independently.

$$O(r): \quad -B_t + \gamma B - 1 = 0$$

$$O(r^0): \quad A_t + \tfrac{1}{2}\beta B^2 + (\lambda\beta^{\frac{1}{2}} - \eta)B = 0$$

subject to boundary conditions $A(t = T) = B(t = T) = 0$.

The first of the above leads to

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

and the second to

$$A = \frac{1}{\gamma^2}(B - T + t)(\eta\gamma - \tfrac{1}{2}\beta - \lambda\gamma\beta^{\frac{1}{2}}) - \frac{\beta B^2}{4\gamma}$$

2. (i) If $V = S^{1-2r/\sigma^2}V_1(S,t)$, then

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_1}{\partial S^2} + (\sigma^2 - r)S\frac{\partial V_1}{\partial S} - rV_1 = 0,$$

(ii) If $\xi = S_d^2/S$, then

$$\frac{\partial V_1}{\partial S} = -\frac{S_d^2}{S^2}\frac{\partial V_2}{\partial \xi}$$

$$\frac{\partial^2 V_1}{\partial S^2} = \frac{2S_d^2}{S^3}\frac{\partial V_2}{\partial \xi} + \frac{S_d^4}{S^4}\frac{\partial^2 V_2}{\partial \xi^2}$$

Substitution of this into the above PDE leads to

$$\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 V_2}{\partial \xi^2} + r\xi\frac{\partial V_2}{\partial \xi} - rV_2 = 0$$

(iii) C_{DO} is a linear combinarion of two solutions of the BSE, which is linear, and hence C_{DO} is a solution of the BSE.

(iv) We require: (a) $C_{DO} \to S$ as $S \to \infty$; (b) $C_{DO} = 0$ on S_d; (c) $C_{DO} = \max(S - X, 0)$ at $t = T$.

(a) implies $A = 1$, (b) implies $B = -A = -1$ and so

$$C_{DO} = V(S,t) - (\frac{S}{S_d})^{1-2r/\sigma^2} V(\frac{S_d^2}{S}, t)$$

At $t = T$:
If $S > X$, $\frac{S_d^2}{S} < X$ and so $C_{DO} = S - X$
If $S_d < S < X$, $\max(S - X, 0) = 0$ and $\max(\frac{S_d^2}{S} - X, 0) = 0$
Consequently the final condition is satisfied.