MATH39032 (Mathematical modelling of finance) Solutions 7

1. (a) The solution to the ODE is

$$P(S) = AS^{-\frac{2r}{\sigma^2}} + BS$$

but we require $P(S) = 0$ as $S \to \infty$ and so B must be zero as other wise $P(S) \to \infty$ as $S \to \infty$. Thus

$$P(S) = AS^{-\frac{2r}{\sigma^2}}.$$

(b) The two conditions at $S = S_f$ give

$$AS_f^{\alpha} = X - S_f \tag{1}$$

and

$$A\alpha S_f^{\alpha-1} = -1. \tag{2}$$

Equation (1) gives

$$A = \frac{X - S_f}{S_f^{\alpha}}$$

substituting this into equation (2) gives

$$\alpha\left(\frac{X}{S_f} - 1\right) = -1$$

which, on rearranging gives

$$S_f = \frac{\alpha X}{\alpha - 1}$$

as required. Thus, from equation (2)

$$A = \frac{-1}{\alpha}\left[\frac{\alpha X}{\alpha - 1}\right]^{1-\alpha}$$

and so

$$P(S) = -\frac{1}{\alpha}\left[\frac{\alpha X}{\alpha - 1}\right]^{1-\alpha} S^{\alpha}$$

2. (i) The first inequality says that the put option value must be above its payoff, which is one of the conditions for an American put.

(ii) For the second, consider the portfolio $C - S + Xe^{-r(T-t)}$ (long one call, short an asset, with a bank deposit that will equal X at expiry). If we exercise the option at any time before expiry, paying X and receiving S, the result is negative. But at expiry the portfolio has zero value if $S \geq X$ and positive value if $S < X$. It is therefore not optimal to exercise the option before expiry, since by waiting you can obtain a better outcome. Therefore the American call (without dividends) has the same value as is European counterpart, while the American put is more valuable than the European put.

(iii) The first inequality follows by considering $C - P + X - S$. The call will never be early exercised. If the put is exercised early the result is positive ($C \geq 0$), while if the portfolio is left to expiry it is worthless. (Note the effect of interest rates has been ignored). The confirmation of the second inequality follows from put-call parity for Europeans (where P_E denotes the value of a European put):

$$C - P_E = S - Xe^{-r(T-t)}$$

(where we are assuming the American call has the same price as a European call). Since $P \geq P_E$, then

$$C - P \leq S - Xe^{-r(T-t)}$$