MATH39032 (Mathematical modelling of finance) Solutions 6

1.

$$\frac{\partial V}{\partial t} = 0$$

Then if $V = A(S)$, A satisfies the following ODE (of Euler type):

$$12\sigma^2 S^2 A'' + (r - D)SA' - rA = 0$$

ODEs of this type have solutions of the form $A = A_0 S^\alpha$, A_0 constant. Consequently

$$A' = A_0 \alpha S^{\alpha - 1}$$
$$A'' = A_0 \alpha(\alpha - 1) S^{\alpha - 2}.$$

Substituting these into the ODE (and dividing through by A_0) leads to the following (quadratic) equation for α:-

$$\frac{1}{2}\sigma^2 \alpha(\alpha - 1) + (r - D)\alpha - r = 0$$

on solving for α gives

$$\alpha = \frac{-\left(r - D - \frac{1}{2}\sigma^2\right) \pm \left[\left(r - D - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2\right]^{1/2}}{\sigma^2}$$

simplifying

$$\alpha = \frac{1}{2}\left\{-\frac{2}{\sigma^2}(r - D - \frac{1}{2}\sigma^2) \pm \left[\frac{4}{\sigma^4}(r - D - \frac{1}{2}\sigma^2)^2 + \frac{8r}{\sigma^2}\right]^{\frac{1}{2}}\right\}.$$

In the case of a put option then $P(S, t) = 0$ as $S \to \infty$ and so the positive root must be ignored, and for the call option $C(0, t) = 0$ and so the negative root is ignored.

Thus,

$$C(S, t) = AS^{\frac{1}{2}\left\{-\frac{2}{\sigma^2}(r - D - \frac{1}{2}\sigma^2) + \left[\frac{4}{\sigma^4}(r - D - \frac{1}{2}\sigma^2)^2 + \frac{8r}{\sigma^2}\right]^{\frac{1}{2}}\right\}}$$

and

$$P(S, t) = AS^{\frac{1}{2}\left\{-\frac{2}{\sigma^2}(r - D - \frac{1}{2}\sigma^2) - \left[\frac{4}{\sigma^4}(r - D - \frac{1}{2}\sigma^2)^2 + \frac{8r}{\sigma^2}\right]^{\frac{1}{2}}\right\}}$$

where A is a constant.

2. For $t_{d_2}^+ \leq t \leq T$ there are no dividends, and so the option price is identical to a vanilla call, i.e.

$$C_d(S,t) = C(S,t;X).$$

For $t_{d_1}^+ \leq t \leq t_{d_2}^-$ there is the equivalent of one dividend, and so the call corresponds to a call with a single dividend as derived in the lecture notes, and so

$$C_d(S,t) = (1-d_2)C(S,t;X(1-d_2)^{-1}).$$

For $0 \leq t \leq t_{d_1}^-$ we apply the result for a single dividend payment to the new call price, to yield

$$C_d(S,t) = (1-d_2)(1-d_1)C(S,t;X(1-d_1)^{-1}(1-d_2)^{-1}).$$

3. (i) If stock S before dividend, value $S - d_y S$ afterwards, otherwise arbitrage possibilities.

   (ii) Holder of put option receives NO dividend, so value of option across dividend date continuous:

$$P(t_d^-, S^-) = P(t_d^+, S^+)$$

and so

$$P(t_d^-, S) = P(t_d^+, S(1-d_y))$$

(iii) For $t_d^+ \le t < T$) $P_d(S,t) = P(S,t;X)$

Consider now $0 \le t < t_d^-$: $P_d(S,t_d^-) = P(t_d^+, S(1-d_y);X)$ and so

$$\begin{aligned} P(S(1-d_y),T;X) &= \max(X - S(1-d_y),0) \\ &= (1-d_y)\max(\frac{X}{1-d_y} - S, 0) \\ &= (1-d_y)P(S,T;\frac{X}{1-d_y}) \end{aligned}$$

The dividend has the effect of *increasing* the value of the put option.

(iv) If $d_1, d_2, \ldots, d_n$ dividends, then if $t_{k-1} < t < t_k$

$$P(t,S;X) = (1-d_n)(1-d_{n-1})\ldots(1-d_k)P(S,t;\frac{X}{(1-d_n)(1-d_{n-1})\ldots(1-d_k)})$$