MATH39032 (Mathematical modelling of finance) Solutions 5

1. Since $v(0,\tau) = u(0,\tau)$, and $v(0,\tau) = -u(0,\tau)$, then $v(0,\tau) = 0$. Now

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{0} u_0(s)e^{-(x-s)^2/4\tau}ds + \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}ds$$

$$= \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} u_0(-s)e^{-(x+s)^2/4\tau}ds + \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}ds$$

But if we demand $u_0(-x) = -u_0(x)$, then

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} u_0(s)\left[e^{-(x-s)^2/4\tau} - e^{-(x+s)^2/4\tau}\right] ds.$$

2. In all cases we set $u = \tau^{\alpha}\hat{U}(\xi)$, where $\xi = x/\tau^{\frac{1}{2}}$. Then

$$\frac{\partial u}{\partial \tau} = \tau^{\alpha-1}\left[\alpha\hat{U} - \frac{1}{2}\xi\hat{U}_{\xi}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \tau^{\alpha-1}\hat{U}_{\xi\xi}$$

(a) Substitution into the equation yields

$$\alpha\hat{U} - \frac{1}{2}\xi\hat{U}_{\xi} = \hat{U}_{\xi\xi} + x\tau^{1-\alpha}$$

This last term may be written in the form $\xi\tau^{3/2-\alpha}$, and so a similarity solution is possible only if $\alpha = \frac{3}{2}$, the governing equation being

$$\frac{3}{2}\hat{U} - \frac{1}{2}\xi\hat{U}_{\xi} = \hat{U}_{\xi\xi} + \xi$$

with $\hat{U}(0) = \hat{U}(\infty) = 0$. The above ODE has one homogeneous solution of the form $U_1 = \xi^3 + 6\xi$. To obtain the other homogeneous solution, together with the particular integral, set $U_2 = U_1(\xi)G(\xi)$ and substitute this into the ODE. The result is a first order ODE for $G'(\xi)$ (this is the method of variation of parameters). The remainder of the solution is an exercise!

(b) In this case the governing equation can be written

$$\alpha\hat{U} - \frac{1}{2}\xi\hat{U}_{\xi} = \hat{U}_{\xi\xi} + \tau^{1-\alpha}$$

Consequently a similarity solution exists if $\alpha = 1$, the governing equation being

$$\hat{U} - \frac{1}{2}\xi\hat{U}_{\xi} = \hat{U}_{\xi\xi} + 1$$

The boundary conditions for $\hat{U}$ are as for (a) above. One solution of the homogeneous ODE is $U = \xi^2 + 2$; again, the method of variation of parameters can be used and again its solution is an exercise.

For the final part, the problem is exactly as for (b), except $\hat{U}(0) = 1$.

3. Writing $u = e^{\alpha x + \beta\tau} v$, then

$$\frac{\partial u}{\partial x} = e^{\alpha x+\beta\tau}\left[\alpha v + \frac{\partial v}{\partial x}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = e^{\alpha x+\beta\tau}\left[\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right]$$

$$\frac{\partial u}{\partial \tau} = e^{\alpha x+\beta\tau}\left[\beta v + \frac{\partial v}{\partial \tau}\right]$$

Substituting into given equation leads to

$$\beta v + \frac{\partial v}{\partial \tau} = \alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + a\left[\alpha v + \frac{\partial v}{\partial x}\right] + bv$$

Hence diffusion equation obtained if

$$2\alpha + a = 0, \quad -\beta + \alpha^2 + \alpha a + b = 0.$$

If we write $\hat{\tau} = F(\tau)$, then equation becomes

$$c\frac{\partial u}{\partial \hat{\tau}}F'(\tau) = \frac{\partial^2 u}{\partial x^2}$$

Hence if $F'(\tau) = 1/c(\tau)$, the diffusion equation is obtained.

Taking the Black-Scholes equation, and dividing through by σ^2 we obtain

$$\frac{1}{\sigma^2}\frac{\partial V}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \frac{r}{\sigma^2}S\frac{\partial V}{\partial S} - \frac{r}{\sigma^2}V = 0.$$

The transformation $t = T - \tau/\frac{1}{2}\sigma^2$ causes problems. If the volatility depends on time, then

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\left[\frac{d\tau}{dt}\right].$$

But

$$\frac{d\tau}{dt} = -\frac{1}{2}\sigma^2 + \sigma\sigma_t(T-t)$$

and hence in the transformed Black-Scholes equation the coefficient of $\frac{\partial V}{\partial \tau}$ is non-constant, i.e,

$$(-\frac{1}{2} + \frac{2\sigma_t}{\sigma^3}\tau)\frac{\partial V}{\partial \tau} + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \frac{r}{\sigma^2}S\frac{\partial V}{\partial S} - \frac{r}{\sigma^2}V = 0,$$

(other terms are constant, under the transformation), and so the above approach may be adopted to obtain the diffusion equation.