MATH39032
Solutions 4

1. (a)

$$\frac{\partial V}{\partial S} = e^{-x}\frac{\partial v}{\partial x}$$
$$\frac{\partial^2 V}{\partial S^2} = \frac{e^{-2x}}{X}\left(-\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right)$$
$$\frac{\partial V}{\partial t} = -X\frac{\partial v}{\partial t'}.$$

The Black Scholes equation becomes

$$\frac{1}{2}\sigma^2(t')X^2e^{2x}\frac{e^{-2x}}{X}\left(-\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)+r(t')Xe^xe^{-x}\frac{\partial v}{\partial x}-r(t')Xv-X\frac{\partial v}{\partial t'}$$

which simplifies to

$$\frac{\partial v}{\partial t'} = \frac{1}{2}\sigma^2(t')\frac{\partial^2 v}{\partial x^2} + [r(t') - \frac{1}{2}\sigma^2(t')]\frac{\partial v}{\partial x} - r(t')v.$$

(b)

$$\frac{\partial v}{\partial t'} = \frac{\partial v}{\partial \tau}\frac{d\tau}{dt'} = \frac{1}{2}\sigma^2\frac{\partial v}{\partial \tau}$$

which, on substitution gives

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r(\tau)}{\sigma^2(\tau)} - 1\right)\frac{\partial v}{\partial x} - \frac{2r(\tau)}{\sigma^2(\tau)}v = 0,$$

thus

$$a(\tau) = \frac{2r(\tau)}{\sigma^2(\tau)} - 1$$
$$b(\tau) = \frac{2r(\tau)}{\sigma^2(\tau)}$$

(c) If $v = F(x + A(\tau))e^{-B(\tau)}$ then let

$$\hat{x} = x + A(\tau)$$

and then

$$\frac{\partial v}{\partial \tau} = -\frac{dB}{d\tau}e^{-B}F(\hat{x}) + e^{-B}\frac{dF}{d\hat{x}}\frac{dA}{d\tau}$$
$$\frac{\partial v}{\partial x} = e^{-B}\frac{dF}{d\hat{x}}$$

which on substitution gives

$$-\frac{dB}{d\tau}F(\hat{x}) + \frac{dF}{d\hat{x}}\frac{dA}{d\tau} = a(\tau)\frac{dF}{d\hat{x}} - b(\tau)F$$

thus

$$\frac{dB}{d\tau} = b$$

and

$$\frac{dA}{d\tau} = a.$$

(d)

$$\frac{\partial v}{\partial \tau} = -\frac{dB}{d\tau}e^{-B}u + e^{-B}\left(\frac{\partial u}{\partial \hat{x}}\frac{dA}{d\tau} + \frac{\partial u}{\partial \tau}\right)$$

$$\frac{\partial v}{\partial x} = e^{-B}\frac{\partial u}{\partial \hat{x}}$$

$$\frac{\partial^2 v}{\partial x^2} = e^{-B}\frac{\partial^2 u}{\partial \hat{x}^2}$$

thus, on substitution

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \hat{x}^2}.$$

2. As usual on substitution

$$\frac{\partial u}{\partial \tau} = \frac{dU}{d\xi}\frac{d\xi}{d\tau} = -\frac{1}{2}x\tau^{-3/2}\frac{dU}{d\xi}$$

$$\frac{\partial u}{\partial x} = \frac{dU}{d\xi}\frac{d\xi}{dx} = \tau^{-1/2}\frac{dU}{d\xi}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \tau^{-1/2}\frac{d}{d\xi}\left(\tau^{-1/2}\frac{dU}{d\xi}\right) = \tau^{-1}\frac{d^2 U}{d\xi^2}$$

and so, replacing $x/\sqrt{\tau}$ by ξ and multiplying by τ gives the ODE

$$\frac{d^2 U}{d\xi^2} + \frac{1}{2}\xi\frac{dU}{d\xi} = 0.$$

Integrating the ODE once gives

$$\frac{dU}{d\xi} = Ce^{-\xi^2/4}$$

(C constant) and then solving gives

$$U(\xi) = C\int_{-\infty}^{\xi} e^{-s^2/4}ds + D$$

(D constant). The boundary conditions become

$$U(-\infty) = 0$$

which gives

$$0 = 0 + D$$

thus

$$D = 0.$$

The other condition is

$$U(\infty) = 1$$

which gives

$$1 = C\int_{-\infty}^{\infty} e^{-s^2/4}ds$$

and so

$$C = \frac{1}{2\sqrt{\pi}}$$

giving

$$u(x,\tau) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{\tau}} e^{-s^2/4} ds$$

3. As u_1 and u_2 are both solutions then so is any linear combination of the two hence $u_1 - u_2$ is also a solution. The initial condition will be

$$v(x,0) = u_1(x,0) - u_2(x,0) = u_0(x) - u_0(x) = 0$$

As

$$E(\tau) = \int_{-\infty}^{\infty} v^2 dx$$

then as $v^2 > 0$ then so must E and so $E(\tau) \geq 0$ also as $v = 0$ when $\tau = 0$ then $E(0) = 0$. Differentiating E gives

$$\frac{dE}{d\tau} = \int_{-\infty}^{\infty} 2v \frac{\partial v}{\partial \tau} dx$$

but we know that, as v is a solution to the heat-conduction equation,

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}$$

so

$$\frac{dE}{d\tau} = \int_{-\infty}^{\infty} 2v \frac{\partial^2 v}{\partial x^2} dx$$

and on integration by parts

$$\frac{dE}{d\tau} = \left[2v \frac{\partial v}{\partial x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2 \left(\frac{\partial v}{\partial x} \right)^2 dx$$

but by the boundary conditions the first term is zero whilst the integral term ≥ 0 hence

$$\frac{dE}{d\tau} \leq 0$$

If $E(0) = 0$, $E(\tau) \geq 0$ and $dE/d\tau \leq 0$ then $E = 0$ from the definition of E. The only way this is possible is if $v = 0$, but if $v = 0$ then $u_1 \equiv u_2$ and we have proved uniqueness.