MATH39032

Solutions 3

1. (a)

$$\Pi = -S + 2C = -S + 2\max(S - X, 0)$$

$$\begin{array}{ll} S < X & \Pi = -S \\ S > X & \Pi = S - 2X. \end{array}$$

Therefore

$$\Pi \to 0 \text{ as } S \to 0$$

and

$$\Pi \to S - 2Xe^{-r(T-t)} \text{ as } S \to \infty$$

(b)

$$\Pi = C + 2P = \max(S - X, 0) + 2\max(X - S, 0)$$

$$\begin{array}{ll} S < X & \Pi = 2(X - S) \\ S > X & \Pi = S - X. \end{array}$$

Therefore

$$\Pi \to 2Xe^{-r(T-t)} \text{ as } S \to 0$$

and

$$\Pi \to S - Xe^{-r(T-t)} \text{ as } S \to \infty$$

(c)

$$\Pi = 2C + P = 2\max(S - X, 0) + \max(X - S, 0)$$

$$\begin{array}{ll} S < X & \Pi = X - S \\ S > X & \Pi = 2(S - X). \end{array}$$

Therefore

$$\Pi \to Xe^{-r(T-t)} \text{ as } S \to 0$$

and

$$\Pi \to 2S - 2Xe^{-r(T-t)} \text{ as } S \to \infty$$

(d) Assuming $X = \frac{1}{2}(X_1 + X_2)$ we have

$$\begin{aligned} \Pi = C(X_1) + P(X_2) - C(X) - P(X) &= \max(S - X_1, 0) + \max(X_2 - S, 0) \\ &\quad - \max(S - X, 0) - \max(X - S, 0) \end{aligned}$$

$$\begin{array}{rl} S < X_1 & \Pi = X_2 - X \\ X_1 < S < X & \Pi = S - X_1 + X_2 - X \\ X < S < X_2 & \Pi = -S - X_1 + X_2 + X \\ S > X_2 & \Pi = X - X_1 \end{array}$$

Therefore

$$\Pi \to (X_2 - X)e^{-r(T-t)} \text{ as } S \to 0$$

and

$$\Pi \to (X - X_1)e^{-r(T-t)} \text{ as } S \to \infty$$

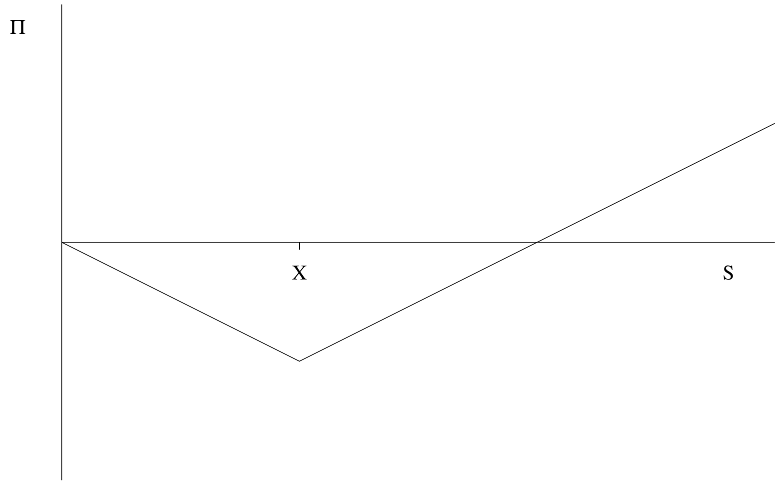

Figure 1: The payoff diagram for 1(a)

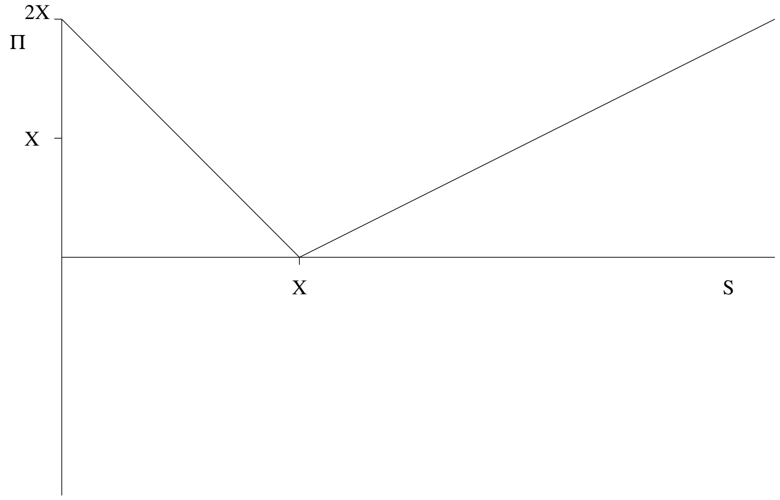

Figure 2: The payoff diagram for 1(b)

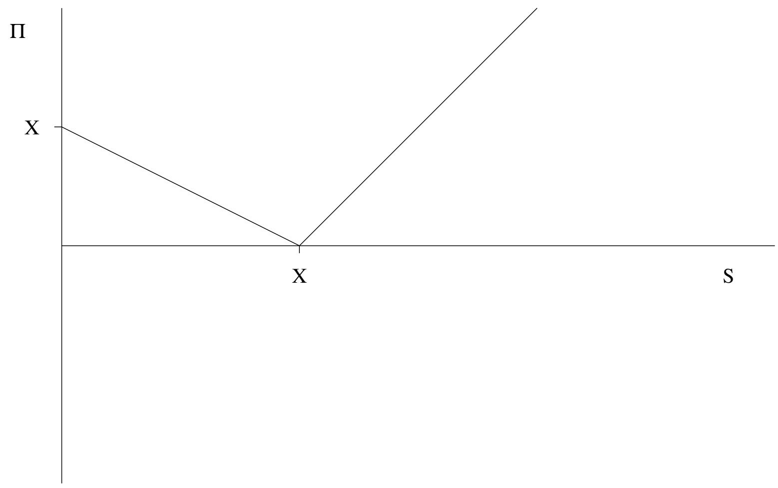

Figure 3: The payoff diagram for 1(c)

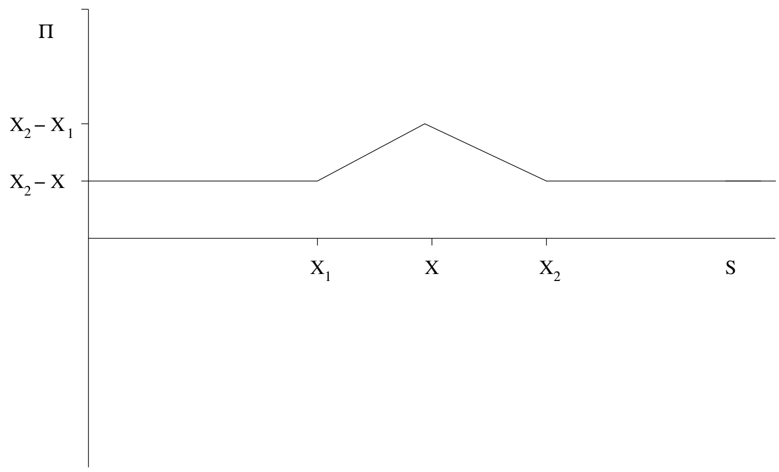

Figure 4: The payoff diagram for 1(d)

2. Consider the portfolio

$$\Pi = V - \Delta_1 S_1 - \Delta_2 S_2. \tag{1}$$

Then change in value dV according to Itô's lemma is

$$\begin{aligned} dV = &\left[\mu_1 S_1 \frac{\partial V}{\partial S_1} + \mu_2 S_2 \frac{\partial V}{\partial S_2} + \frac{\partial V}{\partial t} \right. \\ &\left. + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}\right] dt \\ &+ \sigma_1 S_1 \frac{\partial V}{\partial S_1} dW_1 + \sigma_2 S_2 \frac{\partial V}{\partial S_2} dW_2. \end{aligned}$$

Assume, across a time period dt, that the value of Δ_1 and Δ_2 are held fixed giving

$$d\Pi = dV - \Delta_1 dS_1 - \Delta_2 dS_2. \tag{2}$$

and substituting Itô gives

$$\begin{aligned} d\Pi = &\left[\mu_1 S_1 \left(\frac{\partial V}{\partial S_1} - \Delta_1\right) + \mu_2 S_2 \left(\frac{\partial V}{\partial S_2} - \Delta_2\right) + \frac{\partial V}{\partial t} \right. \\ &\left. + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}\right] dt \\ &+ \sigma_1 S_1 \left(\frac{\partial V}{\partial S_1} - \Delta_1\right) dW_1 + \sigma_2 S_2 \left(\frac{\partial V}{\partial S_2} - \Delta_2\right) dW_2. \end{aligned}$$

so set

$$\Delta_1 = \frac{\partial V}{\partial S_1}$$

and

$$\Delta_2 = \frac{\partial V}{\partial S_2}.$$

Therefore we have

$$d\Pi = \left[\frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}\right] dt.$$

and using no arbitrage arguments carry on to derive

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho_{12}\sigma_1\sigma_1 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0.$$

3.

$$S = 75, X = 69, r = 0.04, \sigma = 0.2, T - t = 0.75$$

Substitute these values in the formulae for d_1 and d_2 to get

$$d_1 = 0.74121$$

$$d_2 = 0.56800$$

From the tables

$$N(d_1) = 0.77071$$

$$N(d_2) = 0.71498$$

$$C(75, 0.75) = 9.9279$$

From the put-call parity

$$P = C - S + Xe^{-r(T-t)} = 9.9279 - 75 + 69e^{-0.04.0.75} = 1.89$$

4. Black-Scholes equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

(a)

$$\frac{\partial V}{\partial t} = 0, \quad \frac{\partial V}{\partial S} = A, \quad \frac{\partial^2 V}{\partial S^2} = 0$$

Substituting into the Black-Scholes equation leads to a cancellation between the last two terms (the middle two terms are identically zero). Here the value of the option is directly proportional to the value of the underlying asset itself.

$$\Delta = \frac{\partial V}{\partial S} = A$$

(b)

$$\frac{\partial V}{\partial S} = 0, \quad \frac{\partial V}{\partial t} = rAe^{rt}.$$

Hence first two terms in the Black-Scholes are identically zero; the other two terms cancel. This represents a risk-free investment.

$$\Delta = \frac{\partial V}{\partial S} = 0.$$

5. (a) With V independent of time the Black-Scholes equation becomes

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

Search for a solution of the form, $V = AS^\alpha$ to get

$$\begin{aligned} \frac{dV}{dS} &= A\alpha S^{\alpha-1} \\ \frac{d^2V}{dS^2} &= A\alpha(\alpha-1)S^{\alpha-2}. \end{aligned}$$

On substitution into the above ODE we get

$$\frac{1}{2}\sigma^2\alpha(\alpha - 1) + r\alpha - r = 0$$

which gives

$$\alpha = 1, -\frac{2r}{\sigma^2}$$

thus

$$V(S) = AS + BS^{-\frac{2r}{\sigma^2}}$$

where A and B are constants.

(b) We get

$$\begin{aligned}\frac{\partial V}{\partial S} &= = AB' \\ \frac{\partial^2 V}{\partial S^2} &= = AB'' \\ \frac{\partial V}{\partial t} &= A'B\end{aligned}$$

and on substitution into the Black-Scholes equation this gives

$$A'B = -\frac{1}{2}\sigma^2 S^2 B'' A - rSB'A + rAB$$

dividing by AB gives

$$\frac{A'}{A} = -\left[\frac{1}{2}\sigma^2 S^2 \frac{B''}{B} + \frac{rSB'}{B} - r\right].$$

As required, the left hand side is solely a function of t while the right hand side is solely a function of S, for the equality to hold both sides must be equal to a constant, λ say. Thus

$$\frac{A'}{A} = \lambda$$

So,

$$A = A_0 e^{\lambda t}$$

where A_0 is a constant.
For B we have

$$\frac{1}{2}\sigma^2 S^2 B'' + rSB' - rB + \lambda B = 0$$

Again seeking a solution of the form $B_0 S^\alpha$ gives a quadratic equation in α

$$\frac{1}{2}\sigma^2\alpha^2 + \left(r - \frac{1}{2}\sigma^2\right)\alpha - (r - \lambda) = 0$$

which gives

$$\alpha = \frac{-(r - \frac{1}{2}\sigma^2) \pm \left[(r + \frac{1}{2}\sigma^2)^2 - 2\sigma^2\lambda\right]^{1/2}}{\sigma^2}$$

and so

$$B = B_1 S^{\alpha_1} + B_2 S^{\alpha_2}$$

where the values of α are the solutions to the above equation.