MATH39032

Solutions 2

1.

$$\begin{aligned} E[dS] &= E[\mu dt] + E[\sigma dW] \\ &= \mu dt + \sigma E[dW] \\ &= \mu dt + \sigma \times 0 \end{aligned}$$

Recall $E[dW] = 0$, $E[dW^2] \to dt$ as $dt \to 0$. Then

$$Var[dS] = E[dS^2] - E[dS]^2$$
$$= E[\mu^2 dt^2 + 2\mu\sigma dt dW + \sigma^2 dW^2] - (E[\mu dt + \sigma dW])^2$$
$$= \mu^2 dt^2 + \sigma^2 dt - \mu^2 dt$$
$$= \sigma^2 dt$$

$$\begin{aligned} E[dS^2] &= \mu^2 dt^2 + 2\mu\sigma dt E[dW] + \sigma^2 E[dW^2] \\ &= \mu^2 dt^2 + 0 + \sigma^2 dt \end{aligned}$$

Thus,

$$Var[dS] = \mu^2 dt^2 + \sigma^2 dt - \mu^2 dt^2 = \sigma^2 dt$$

as required.

2. Use Itô's Lemma.

(a)

$$\frac{\partial f}{\partial S} = A, \quad \frac{\partial^2 f}{\partial S^2} = 0 \quad \frac{\partial f}{\partial t} = 0$$

and so, from Itô's lemma

$$df = A\mu S dt + A\sigma S dW.$$

(b)

$$\frac{\partial f}{\partial S} = nS^{n-1}, \quad \frac{\partial^2 f}{\partial S^2} = n(n-1)S^{n-2} \quad \frac{\partial f}{\partial t} = 0$$

and so, from Itô's lemma

$$df = nS^n(\mu + \frac{1}{2}(n-1)\sigma^2)dt + \sigma n S^n dW$$

and

$$df = nf\left[(\mu + \frac{1}{2}(n-1)\sigma^2)dt + \sigma dW\right]$$

3. In this case $f(S) = \log S$. In which case

$$\frac{\partial f}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} \quad \frac{\partial f}{\partial t} = 0$$

and so by Itô's lemma

$$df = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW.$$

This is just a generalised Brownian motion as in question 2 and as such it is Normally distributed with

$$E[df](= E[d(\log S)]) = (\mu - \frac{1}{2}\sigma^2)dt$$

and

$$Var[df](= Var[d(\log S)]) = \sigma^2 dt$$

Thus the log of S is Normally distributed, or S is *lognormally* distributed, with mean $(\mu - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$.

4. Performing a 'Taylor' expansion in both S and t we have

$$f(S + dS, t + dt) - f(S, t) = df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial^2 f}{\partial S \partial t}dSdt + O(dS^3) + O(dt^3)$$

substituting in the expression for dS (writing $A = A(S,t)$ and $B = B(S,t)$) we obtain

$$\begin{aligned} df &= \frac{\partial f}{\partial S}[Adt + BdW] + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}[A^2dt^2 + 2ABdtdW + B^2dW^2] \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial^2 f}{\partial S \partial t}[Adt^2 + BdWdt] + O(dt^3) + O(dS^3) \end{aligned}$$

Now, as $dt \to 0$ then $dW^2 \to dt$ and $dWdt = o(dt)$ leaving

$$df = \left[A(S,t)\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}B^2(S,t)\frac{\partial^2 f}{\partial S^2}\right]dt + B(S,t)\frac{\partial f}{\partial S}dW$$

as required.

5. Performing a 'Taylor' expansion in $S_1, \ldots, S_n$ gives

$$\begin{aligned} df &= \frac{\partial f}{\partial S_1}dS_1 + \ldots \frac{\partial f}{\partial S_n}dS_n + \frac{1}{2}\frac{\partial^2 f}{\partial S_1^2}dS_1^2 + \cdots + \frac{1}{2}\frac{\partial^2 f}{\partial S_n^2}dS_n^2 + \frac{\partial^2 f}{\partial S_1 \partial S_2}dS_1dS_2 \\ &+ \frac{\partial^2 f}{\partial S_1 \partial S_3}dS_1dS_3 + \cdots + \frac{\partial^2 f}{\partial S_{n-1}\partial S_n}dS_{n-1}dS_n + O(S_1^3) + \ldots O(S_n^3) \end{aligned}$$

Now using the rules given and the expressions for dS_i this reduces, eventually, to

$$df = \left[\sum_i \mu_i S_i \frac{\partial f}{\partial S_i} + \frac{1}{2}\sum_i \sigma_i^2 S_i^2 \frac{\partial^2 f}{\partial S_i^2} + \frac{1}{2}\sum_i \sum_{j \neq i} \frac{\partial^2 f}{\partial S_i \partial S_j}\sigma_i \sigma_j S_i S_j \rho_{ij}\right]dt + \sum_i \sigma_i S_i \frac{\partial f}{\partial S_i}dW_i$$

6. (a) The process Y is given by

$$f(X,t) = e^{\kappa t} X$$

then

$$\frac{\partial f}{\partial t} = \kappa Y$$
$$\frac{\partial f}{\partial X} = e^{\kappa t}$$
$$\frac{\partial^2 f}{\partial X^2} = 0$$

so

$$df = \left[\kappa Y + \kappa(\theta - X)e^{\kappa t}\right]dt + \sigma e^{\kappa t} dW$$

(b)

$$df = \kappa \theta e^{\kappa t} dt + \sigma e^{\kappa t} dW$$

(c)

$$\int_0^T df = \int_0^T \kappa \theta e^{\kappa t} dt + \int_0^T \sigma e^{\kappa t} dW$$

we get

$$f_T - f_0 = \left[\theta(e^{\kappa T} - 1)\right] + \sigma \int_0^T e^{\kappa t} dW$$

and

$$e^{\kappa T} X_T - X_0 = \left[\theta(e^{\kappa T} - 1)\right] + \int_0^T \sigma e^{\kappa t} dW$$
$$X_T = X_0 e^{-\kappa T} + \left[\theta(1 - e^{-\kappa T})\right] + \int_0^T \sigma e^{-\kappa(T-t)} dW_t$$

(d)

$$E[X_T] = X_0 e^{-\kappa T} + \left[\theta(1 - e^{-\kappa T})\right]$$

$$\begin{aligned} var[X_T] &= var\left[\int_0^T \sigma e^{-\kappa(T-t)} dW_t\right] \\ &= E\left[\left(\int_0^T \sigma e^{-\kappa(T-t)} dW_t\right)^2\right] \\ &= \int_0^T \sigma^2 e^{-2\kappa(T-t)} dt \\ &= \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa T}\right] \end{aligned}$$