Two hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

4 June 2019

14:00 - 16:00

Answer **ALL SIX** questions.

Electronic calculators may be used in accordance with the University regulations

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1. By using no arbitrage argument, derive *put-call parity* for the European put option, P, and the European call option, C, with the same exercise price X and expiry date T. Assume the constant risk-free interest rate r.

[6 marks]

2. Draw the expiry payoff diagrams for each of the following portfolios:

(i) Short two shares, long three calls with an exercise price X.

(ii) Short two shares, long three puts with exercise price X_1, long four calls with exercise price X_2. Consider only the case $X_1 < X_2$.

[12 marks]

3. The change in price of an asset S satisfies the stochastic differential equation

$$dS = 0.5S dt + (1+t)\sqrt{S} dW,$$

where $W(t)$ is a Wiener process. Using Ito's Lemma along with the delta-hedge argument, derive the partial differential equation satisfied by the price $V(S,t)$ of an option on the underlying asset.

[12 marks]

4. Consider the Black-Scholes equation for the price for an option, V of an option, on a stock S that pays no dividends, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where r and σ are both constants.

(i) Show by substitution that

$$V(S,t) = Ae^{rt+\beta \ln S},$$

where A and β are constants, is a solution of the Black-Scholes equation.

(ii) Determine the values of the constant β.

(iii) Find the partial differential equation satisfied by $V(Z,t)$, where $Z = -\ln S$.

[20 marks]

5. You are given that the solution for a vanilla (non-dividend paying) European call option is

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and X is the exercise price and $N(x)$ is the cumulative distribution function, namely

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy.$$

(i) Find the present value of a three-month European call option on a stock with the exercise price \$100, when the current stock price is \$100 and $\sigma = 1$. The risk-free interest rate is 0% p.a.

(ii) Find the limits

$$\lim_{X\to 0} C(S,t), \qquad \lim_{\sigma\to\infty} C(S,t), \qquad \lim_{\sigma\to 0} C(S,t).$$

(iii) By using the formula

$$\Delta = \frac{\partial C}{\partial S} = N(d_1),$$

find the explicit expression for

$$\Gamma = \frac{\partial^2 C}{\partial S^2}$$

and sketch the graph of Δ as a function of S in the limit $\sigma \to 0$.

(iv) Show that

$$\frac{\partial C}{\partial X} = f(t)N(d_2)$$

and find the function $f(t)$.

[25 marks]

6. Suppose that an asset of value S pays out a dividend $DSdt$ over the time period dt (i.e. a constant dividend yield D). A put option $P(S,t)$ on this asset satisfies the modified Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-D)S\frac{\partial P}{\partial S} - rP = 0,$$

where r and σ are constants.

(i) Let $P(S,t)$ be a solution of the modified Black-Scholes equation for $D = r$. We set $S = e^z$, $\tau = T - t$ (T is the expiry date), $P = v(z,\tau)$. Find the partial differential equation satisfied by $v(z,\tau)$.

(ii) Find the solution of the modified Black-Scholes equation for $D = r$ in the form

$$P(S) = AS^\beta,$$

where A and β are constants. Find the two possible values of β.

(iii) Consider a *perpetual* American put option on an asset paying a continuous constant dividend yield, which is given by $D = r$. If the exercise boundary is denoted by S_f, write down the two conditions to be imposed on S_f. State why only one of the solutions in (ii) is retained, and which one it is.

(iv) Use the conditions at $S = S_f$ to determine both the value of the constant A and S_f itself. Sketch the graph of S_f as a function of β. What happens to S_f as $D \to 0$?

[25 marks]

END OF EXAMINATION PAPER