2 hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

05 June 2018

14.00 - 16.00

Answer **all** 4 questions in **Section A** (60 marks in all) and
2 of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1. An asset pays a dividend of £1 at the end of each of the next 3 years. The current asset price is £200 and the constant risk-free interest rate is 5% per annum with continuous compounding. Suppose that a zero value forward contract on this asset with delivery date in 4 years and delivery price of £245 is available. Does this situation create any arbitrage opportunities, and if so, how could an investor exploit this?

[15 marks]

A2.

(i) Consider a contract with value function $V(S,t)$ which satisfies the Black-Scholes equation, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

in the usual notation, with constant r and σ, with a payoff at expiry $t = T$ given by

$$V(S,T) = S^n,$$

where $n \geq 0$. By seeking a solution of the form $V(S,t) = S^n f(t)$, determine $V(S,t)$.

(ii) By using (i) above, find (in the form of an infinite series), the value for a contract with payoff

$$V(S,T) = \exp(S)$$

[15 marks]

A3. Draw the expiry payoff diagrams for each of the following portfolios (be sure to label axes and indicate gradients on the diagrams):

(i) Short two shares, short two calls, both with an exercise price X.

(ii) Short one share, long three puts all with exercise price X_1, and short one call with exercise price X_2. Consider each of the cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

[15 marks]

A4. Consider a general stochastic process with the dynamics

$$dS = A(S,t)dt + B(S,t)dW,$$

where W is a Brownian motion.

(i) Let the function $f(S,t)$ have a continuous second derivative with respect to S, and a continuous first derivative with respect to t. Show that

$$df = \left[A(S,t)\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}B^2(S,t)\frac{\partial^2 f}{\partial S^2}\right] dt + B(S,t)\frac{\partial f}{\partial S}dW.$$

(ii) Increments in S, a share price, are described by the following variant on geometric Brownian motion:

$$dS = \mu S dt + \sigma S^\beta dW,$$

where μ is the drift (constant) and σ is the volatility (constant), and β is also constant. Using the result from part (i), find the dynamics dV followed by the option price $V(S,t)$.

(iii) Consider a portfolio comprising one option $V(S,t)$ and $-\Delta$ of the underlying asset, S, which follows the dynamics given in (ii) above. Show that by choosing Δ such that

$$\Delta = \frac{\partial V}{\partial S},$$

and using arbitrage arguments, leads to the following variant of the Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where r is the risk-free interest rate, assumed constant.

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5.

(i) Starting with the Black-Scholes equation for an option on a non-dividend paying asset, in the usual notation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where the volatility σ and interest rate r are constants, if $S = \exp F$, show that $V(F, t)$ satisfies

$$\frac{\partial V}{\partial t} + c\frac{\partial^2 V}{\partial F^2} + d\frac{\partial V}{\partial F} - rV = 0,$$

where you are to determine c and d.

(ii) By writing $V = g_1(t) + g_2(t)F$, determine the value of an option whose payoff at time $t = T$ is $V(S, T) = A \log S + B$, where A and B are constants.

[20 marks]

B6. You may assume that the Black-Scholes equation (in the usual notation) for a put option on a non-dividend paying asset is

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0,$$

where the interest rate r and the volatility σ are both constant.

(i) A binary put option $P_b(S,t)$ has the payoff

$$P_b(S,T) = \begin{cases} 0 & \text{if} \quad S > X \\ K & \text{if} \quad \text{S} < \text{X} \end{cases} .$$

Show that its value is given by

$$P_b(S,t) = K\frac{\partial P}{\partial X},$$

where $P(S,t;X)$ is the value of a European put option with strike X.

(ii) Given that the value of a European put option is

$$P(S,t;X) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where $N(x)$ is the cumulative distribution function of the standard normal distribution, namely

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy,$$

$$d_1 = \frac{\log\frac{S}{X} + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

show that the price of a European binary put option is

$$P_b(S,t;X) = Ke^{-r(T-t)}N(-d_2).$$

(iii) Consider a stock whose price (6 months from the expiration of a binary put option) today is £10.50. The option pays £0.5 if the stock value at expiry is less than £11, and nothing if the stock value at expiry is greater than £11. The risk-free interest rate is 4% per annum (fixed) and the volatility (constant) is 15% per $(\text{annum})^{\frac{1}{2}}$. Using the formula above for $P_b(S,t;X)$, determine its value today. The value of $N(x)$ may be determined by interpolation using the tables of the cumulative distribution function provided.

[20 marks]

B7.

Consider the Black-Scholes equation for a call option, C, where the underlying asset pays a continuous known constant dividend yield, D, namely

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC = 0,$$

in the usual notation, where r and σ are both constants.

(i) By neglecting the time derivative in the above equation, seek solutions of the form

$$C(S) = AS^\alpha,$$

where A is a constant. Show that the two possible values of α are

$$\alpha = \frac{1}{2}\left\{-\frac{2}{\sigma^2}(r - D - \frac{1}{2}\sigma^2) \pm \left[\frac{4}{\sigma^4}(r - D - \frac{1}{2}\sigma^2)^2 + \frac{8r}{\sigma^2}\right]^{\frac{1}{2}}\right\}.$$

(ii) Consider a *perpetual* American call option (on an asset paying a continuous known constant dividend yield, D), i.e. an option with no expiry date but where it is possible at any point in time to exercise and receive $S - X$, where X is the exercise price. This can be valued using the time-independent solutions in (i) above. If the exercise boundary is denoted by S_f, write down the two conditions to be imposed on S_f; justify your answer.

(iii) State why only one of the solutions in (i) is retained, and which one it is.

(iv) Use the conditions at $S = S_f$ to determine both the value of the constant A and S_f itself. Hence determine the value of the perpetual call option $C(S)$.

(v) What happens to S_f as $D \to 0$; explain your answer.

[20 marks]

END OF EXAMINATION PAPER