2 hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

18 May, 2017

2pm – 4pm

Answer **all** 4 questions in **Section A** (60 marks in all) and
2 of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1. Consider a portfolio comprising European call options on the same underlying asset and same expiration date. In particular, this comprises long two options C_{20} with exercise price \$20, short three options C_{30} with exercise price \$30 and long one option C_{50} with strike \$50. In addition, consider P_{50}, the corresponding put option with exercise price \$50. Show that for all times prior to expiry

$$P_{50} \geq 2C_{20} - 3C_{30} + C_{50}.$$

[15 marks]

A2. This question concerns the calculation of delivery prices:

(i) What is the delivery price for a futures contract on a stock S, currently with value \$10, for delivery in 9 months time; the (constant) risk-free rate is 5% p.a.? Justify your answer.

(ii) What is the delivery price for a futures contract on a stock S, currently with value \$25, for delivery in 1 year's time; the (constant) risk-free rate is 2.5% p.a. for the first 6 months, and 4% p.a. thereafter? Justify your answer.

(iii) What is the delivery price for a futures contract on 1 kilo of gold, currently with value £31500, for delivery in 6 months time; the (constant) risk-free rate is 4% p.a. and the cost of storing/securing (paid by the owner of the gold) the metal is 10% p.a. of its (current) value? Justify your answer.

[15 marks]

A3. Draw the expiry payoff diagrams for each of the following portfolios:

(i) Long two shares, short one put with an exercise price X.

(ii) Long one share, short two puts both with exercise price X_1, long one call with exercise price X_2. Consider each of the cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

[15 marks]

A4. Consider the Black-Scholes equation for an option, V, on a stock S which pays no dividends, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

in the usual notation, where r and σ are both constants.

(i) By neglecting the time derivative in the above equation, seek solutions of the form

$$V(S) = A_1 S^{\alpha_1} + A_2 S^{\alpha_2},$$

where A_1 and A_2 are constants, and $\alpha_1 > \alpha_2$. Determine the values of α_1 and α_2.

(ii) Consider a perpetual American option $V(S)$, i.e. an option with no expiry date but where it is possible at any point in time to exercise, which has a (non-standard) exercise value of $e^{-\lambda S}$, where λ is a constant. This can be valued using the time independent solutions in (i) above. The exercise boundary is denoted by S_f. State and justify the two conditions at $S = S_f$ and the other appropriate boundary condition.

(iii) Determine S_f.

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5. The change in price of an asset S satisfies the stochastic differential equation

$$dS = a(S,t)dt + b(S,t)dX$$

where a and b are given functions of S and t and dX is sampled from a Brownian motion of mean zero and variance dt.

(i) Using Ito's Lemma and delta hedging, derive the equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}b^2(S,t)\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

for the fair price of an option $V(S,t)$, where r is a constant risk-free interest rate.

(ii) If $b = \sigma(t)S$ where $\sigma(t)$ is a known function of time, find $V(S,t)$ for $t < T$ for a payoff $V(S,T) = S^2$. You may assume $V(S,t) = S^2 f(t)$, and you may leave your answer in terms of an integral of $\sigma(t)$.

(iii) If $b = \sigma_1$ where σ_1 is a constant, find $V(S,t)$ for $t < T$ for a payoff $V(S,T) = S^2$. You may assume $V(S,t) = S^2 f_1(t) + f_2(t)$ (where $f_1(t)$ and $f_2(t)$ are to be fully determined).

[20 marks]

B6.

(i) Suppose that an asset of value S pays out a dividend $DSdt$ in time dt (i.e. a constant dividend yield D). Given that an option $V(S,t)$ on this asset satisfies the modified Black-Scholes equation (in the usual notation)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0,$$

show that the substitution

$$V(S,t) = e^{-D(T-t)}V_1(S,t)$$

results in the standard Black-Scholes equation, but with a modified interest rate (which you are to determine).

(ii) You are given that the solution for a vanilla (non-dividend paying) European call option is

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and X is the exercise price and $N(x)$ is the cumulative distribution function. Write down the corresponding result for a call option with a dividend yield of D.

(iii) Consider a stock whose price today (9 months from the expiration of an option) is £50, the exercise price of a European call option on this stock is £51, the risk-free interest rate is 1.5% per annum (fixed) and the volatility (constant) is 20% per $(\text{annum})^{\frac{1}{2}}$. The stock pays a continuous dividend at a rate 5% of the value of the stock (per annum).

What is today's value of a European call option on this asset? Values of the cumulative distribution function may be determined by interpolation of the provided tables.

(iv) Given that the put-call parity relationship for non-dividend, vanilla European options is

$$C + Xe^{-r(T-t)} - P - S = 0,$$

write down the corresponding relationship between call and put (C_d and P_d, respectively) options on a continuous dividend paying asset, with a yield D.

[20 marks]

B7. Consider an option $V(S_1, S_2, t)$ where the two underlyings S_1 and S_2 have constant volatilities σ_1 and σ_2 respectively, and drifts μ_1, μ_2 respectively; the risk-free interest rate is r.

(i) Justify the use of the following stochastic processes to model the movement of stocks/shares:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i$$

where the dW_i are Wiener processes, such that

$$E[dW_i^2] = dt, \quad E[dW_1 dW_2] = \rho dt.$$

(ii) By considering a portfolio (using the dS_i in (i) above)

$$\Pi = V(S_1, S_2, t) - \Delta_1 S_1 - \Delta_2 S_2,$$

find the choices of Δ_1 and Δ_2 for which the portfolio is perfectly hedged.

(iii) Equating the hedged portfolio in (ii) above to the risk-free return on the portfolio, show that the option value is determined from

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0.$$

(iv) Consider now the case of an option whose payoff at expiry ($t = T$) is

$$V(S_1, S_2, t = T) = \max(q_1 S_1 - q_2 S_2, 0),$$

where q_1 and q_2 are constants. Assuming a solution of the form

$$V(S_1, S_2, t) = q_1 S_2 H(\xi, t),$$

where $\xi = \frac{S_1}{S_2}$, write down all the boundary conditions for $H(\xi, t)$.

(v) If $\rho = 0$, show that H satisfies the PDE

$$\frac{\partial H}{\partial t} + \frac{1}{2}\bar{\sigma}^2 \xi^2 \frac{\partial^2 H}{\partial \xi^2} = 0,$$

where $\bar{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2}$.

[20 marks]

END OF EXAMINATION PAPER